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Families of language uniform theories and their generating sets *

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Abstract. We introduce the notion of language uniform theory and study topological properties related to families of language uniform theory and their E-combinations. It is shown that the class of language uniform theories is broad enough. Sufficient conditions for the language similarity of language uniform theories are found. Properties of language domination and of infinite language domination are studied. A characterization for E-closure of a family of language uniform theories in terms of index sets is found. We consider the class of linearly ordered families of language uniform theories and apply that characterization for this special case. The properties of discrete and dense index sets are investigated: it is shown that a discrete index set produces a least generating set whereas a dense index set implies at least continuum many accumulation points and the closure without the least generating set. In particular, having a dense index set the theory of the E-combination does not have e-least models and it is not small. Using the dichotomy for discrete and dense index sets we solve the problem of the existence of least generating set with respect to E-combinations and characterize that existence in terms of orders.

Values for e-spectra of families of language uniform theories are obtained. It is shown that any e-spectrum can be realized by E-combination of language uniform theories. Low estimations for e-spectra relative to cardinalities of language are found.

It is shown that families of language uniform theories produce an arbitrary given Cantor-Bendixson rank and given degree with respect to this rank.

Keywords: E-combination, P-combination, closure operator, generating set, language uniform theory.

We continue to study structural properties of E-combinations of structures and their theories [5; 6]. The notion of language uniform theory is introduced. For the class of linearly ordered language uniform theories we

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solve the problem of the existence of least generating set with respect to E-combinations and characterize that existence in terms of orders. Values for e-spectra of families of language uniform theories are obtained. It is shown that families of language uniform theories produce an arbitrary given Cantor-Bendixson rank and given degree.

1. Preliminaries

Throughout the paper we use the following terminology in [5; 6].

Definition 1. [5]. Let $P = (P_i)_{i\in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i\in I}$ be a family of structures such that P_i is the universe of $\mathcal{A}_i$, $i \in I$, and the symbols P_i are disjoint with languages for the structures $\mathcal{A}_j$, $j \in I$. The structure $\mathcal{A}_P \rightleftharpoons \bigcup\limits_{i\in I} \mathcal{A}_i$ expanded by the predicates P_i is the *P-union* of the structures $\mathcal{A}_i$, and the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_P$ is the *P-operator*. The structure $\mathcal{A}_P$ is called the *P-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ if $\mathcal{A}_i = (\mathcal{A}_P \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Structures $\mathcal{A}'$, which are elementary equivalent to $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$, will be also considered as P-combinations.

Clearly, all structures $\mathcal{A}' \equiv \mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}' \upharpoonright P_i) \upharpoonright \Sigma(\mathcal{A}_i)$ if and only if the set $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \neq \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I}$, we write $\mathcal{A}' = \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I\cup\{\infty\}}$, where $\mathcal{A}'_\infty = \mathcal{A}' \upharpoonright \bigcap\limits_{i\in I} \overline{P_i}$, maybe applying Morleyzation. Moreover, we write $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I\cup\{\infty\}}$ for $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ with the empty structure $\mathcal{A}_\infty$.

Note that if all predicates P_i are disjoint, a structure $\mathcal{A}_P$ is a P-combination and a disjoint union of structures $\mathcal{A}_i$. In this case the P-combination $\mathcal{A}_P$ is called *disjoint*. Clearly, for any disjoint P-combination $\mathcal{A}_P$, $\mathrm{Th}(\mathcal{A}_P) = \mathrm{Th}(\mathcal{A}'_P)$, where $\mathcal{A}'_P$ is obtained from $\mathcal{A}_P$ replacing $\mathcal{A}_i$ by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_P = \mathrm{Th}(\mathcal{A}_P)$, being *$P$-combination* of T_i, which is denoted by $\mathrm{Comb}_P(T_i)_{i\in I}$.

For an equivalence relation E replacing disjoint predicates P_i by E-classes we get the structure $\mathcal{A}_E$ being the *E-union* of the structures $\mathcal{A}_i$. In this case the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_E$ is the *E-operator*. The structure $\mathcal{A}_E$ is also called the *E-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$; here $\mathcal{A}_i = (\mathcal{A}_E \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Similar above, structures $\mathcal{A}'$, which are elementary equivalent to $\mathcal{A}_E$, are denoted by $\mathrm{Comb}_E(\mathcal{A}'_j)_{j\in J}$, where $\mathcal{A}'_j$ are restrictions of $\mathcal{A}'$ to its E-classes. The E-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_E = \mathrm{Th}(\mathcal{A}_E)$, being *$E$-combination* of T_i, which is denoted by $\mathrm{Comb}_E(T_i)_{i\in I}$ or by $\mathrm{Comb}_E(\mathcal{T})$, where $\mathcal{T} = \{T_i \mid i \in I\}$.

Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into $\mathcal{A}_P$ and can not be represented as a disjoint P-combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are E-combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as E-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of $\mathcal{A}'$ to be the *E-representability*.

If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, we have the E'-representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T, where T is a theory of a restriction $\mathcal{B}$ of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and $\mathcal{B}$ is not elementary equivalent to the structures $\mathcal{A}_i$. The resulting structure $\mathcal{A}_{E'}$ (with the E'-representability) is a *e-completion*, or a *e-saturation*, of $\mathcal{A}_E$. The structure $\mathcal{A}_{E'}$ itself is called *e-complete*, or *e-saturated*, or *e-universal*, or *e-largest*.

For a structure $\mathcal{A}_E$ the number of *new* structures with respect to the structures $\mathcal{A}_i$, i. e., of the structures $\mathcal{B}$ which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the *e-spectrum* of $\mathcal{A}_E$ and denoted by e-$\mathrm{Sp}(\mathcal{A}_E)$. The value $\sup\{e\text{-}\mathrm{Sp}(\mathcal{A}')) \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the *e-spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ and denoted by e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$.

If $\mathcal{A}_E$ does not have E-classes $\mathcal{A}_i$, which can be removed, with all E-classes $\mathcal{A}_j \equiv \mathcal{A}_i$, preserving the theory $\mathrm{Th}(\mathcal{A}_E)$, then $\mathcal{A}_E$ is called *e-prime*, or *e-minimal*.

For a structure $\mathcal{A}' \equiv \mathcal{A}_E$ we denote by $\mathrm{TH}(\mathcal{A}')$ the set of all theories $\mathrm{Th}(\mathcal{A}_i)$ of E-classes $\mathcal{A}_i$ in $\mathcal{A}'$.

By the definition, an e-minimal structure $\mathcal{A}'$ consists of E-classes with a minimal set $\mathrm{TH}(\mathcal{A}')$. If $\mathrm{TH}(\mathcal{A}')$ is the least for models of $\mathrm{Th}(\mathcal{A}')$ then $\mathcal{A}'$ is called *e-least*.

Definition 2. [6]. Let $\overline{\mathcal{T}}$ be the class of all complete elementary theories of relational languages. For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ we denote by $\mathrm{Cl}_E(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$, where $\mathcal{A}$ is a structure of some E-class in $\mathcal{A}' \equiv \mathcal{A}_E$, $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, $\mathrm{Th}(\mathcal{A}_i) \in \mathcal{T}$. As usual, if $\mathcal{T} = \mathrm{Cl}_E(\mathcal{T})$ then $\mathcal{T}$ is said to be *E-closed*.

The operator Cl_E of E-closure can be naturally extended to the classes $\mathcal{T} \subset \overline{\mathcal{T}}$ as follows: $\mathrm{Cl}_E(\mathcal{T})$ is the union of all $\mathrm{Cl}_E(\mathcal{T}_0)$ for subsets $\mathcal{T}_0 \subseteq \mathcal{T}$.

For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ of theories in a language Σ and for a sentence φ with $\Sigma(\varphi) \subseteq \Sigma$ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$.

Proposition 1. [6]. *If $\mathcal{T} \subset \overline{\mathcal{T}}$ is an infinite set and $T \in \overline{\mathcal{T}} \setminus \mathcal{T}$ then $T \in \mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an accumulation point for $\mathcal{T}$ with respect to E-closure Cl_E) if and only if for any formula $\varphi \in T$ the set $\mathcal{T}_\varphi$ is infinite.*

Definition 3. [6]. Let $\mathcal{T}_0$ be a closed set in a topological space $(\mathcal{T}, \mathcal{O}_E(\mathcal{T}))$, where $\mathcal{O}_E(\mathcal{T}) = \{\mathcal{T} \setminus \mathrm{Cl}_E(\mathcal{T}') \mid \mathcal{T}' \subseteq \mathcal{T}\}$. A subset $\mathcal{T}'_0 \subseteq \mathcal{T}_0$ is said to be *generating* if $\mathcal{T}_0 = \mathrm{Cl}_E(\mathcal{T}'_0)$. The generating set $\mathcal{T}'_0$ (for $\mathcal{T}_0$) is *minimal* if $\mathcal{T}'_0$ does not contain proper generating subsets. A minimal generating set $\mathcal{T}'_0$ is *least* if $\mathcal{T}'_0$ is contained in each generating set for $\mathcal{T}_0$.

Theorem 1. [6]. *If $\mathcal{T}'_0$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:*

(1) *$\mathcal{T}'_0$ is the least generating set for $\mathcal{T}_0$;*

(2) *$\mathcal{T}'_0$ is a minimal generating set for $\mathcal{T}_0$;*

(3) *any theory in $\mathcal{T}'_0$ is isolated by some set $(\mathcal{T}'_0)_\varphi$, i.e., for any $T \in \mathcal{T}'_0$ there is $\varphi \in T$ such that $(\mathcal{T}'_0)_\varphi = \{T\}$;*

(4) *any theory in $\mathcal{T}'_0$ is isolated by some set $(\mathcal{T}_0)_\varphi$, i.e., for any $T \in \mathcal{T}'_0$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_\varphi = \{T\}$.*

Proposition 2. [6]. *For any closed nonempty set $\mathcal{T}_0$ in a topological space $(\mathcal{T}, \mathcal{O}_E(\mathcal{T}))$ and for any $\mathcal{T}'_0 \subseteq \mathcal{T}_0$, the following conditions are equivalent:*

(1) *$\mathcal{T}'_0$ is the least generating set for $\mathcal{T}_0$;*

(2) *any/some structure $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, where $\{\mathrm{Th}(\mathcal{A}_i) \mid i \in I\} = \mathcal{T}'_0$, is an e-least model of the theory $\mathrm{Th}(\mathcal{A}_E)$ and E-classes of each/some e-largest model of $\mathrm{Th}(\mathcal{A}_E)$ form models of all theories in $\mathcal{T}_0$;*

(3) *any/some structure $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, where $\{\mathrm{Th}(\mathcal{A}_i) \mid i \in I\} = \mathcal{T}'_0$, $\mathcal{A}_i \not\equiv \mathcal{A}_j$ for $i \neq j$, is an e-least model of the theory $\mathrm{Th}(\mathcal{A}_E)$, where E-classes of $\mathcal{A}_E$ form models of the least set of theories and E-classes of each/some e-largest model of $\mathrm{Th}(\mathcal{A}_E)$ form models of all theories in $\mathcal{T}_0$.*

Theorem 1 and Proposition 2 answer Question 1 in [6] characterizing the existence of the least generating set. The following question also has been formulated in [6].

Question. *Is there exists a theory $\mathrm{Th}(\mathcal{A}_E)$ without the least generating set?*

Below we will consider a class of special theories with respect to their languages and answer the question characterizing the existence of the least generating set in these special cases.

2. Language uniform theories and related E-closures

Definition 4. A theory T in a predicate language Σ is called *language uniform*, or a LU-*theory* if for each arity n any substitution on the set of non-empty n-ary predicates preserves T. The LU-theory T is called IILU-*theory* if it has non-empty predicates and as soon as there is a non-empty n-ary predicate then there are infinitely many non-empty n-ary predicates and there are infinitely many empty n-ary predicates.

Below we point out some basic examples of LU-theories:

• Any theory T_0 of infinitely many independent unary predicates R_k is a LU-theory; expanding T_0 by infinitely many empty predicates R_l we get a IILU-theory T_1.

• Replacing independent predicates R_k for T_0 and T_1 by disjoint unary predicates R'_k with a cardinality $\lambda \in (\omega + 1) \setminus \{0\}$ such that each R'_k has λ elements; the obtained theories are denoted by T_0^λ and T_1^λ respectively; here, T_0^λ and T_1^λ are LU-theories, and, moreover, T_1^λ is a IILU-theory; we denote T_0^1 and T_1^1 by T_0^c and T_1^c; in this case nonempty predicates R'_k are singletons symbolizing constants which are replaced by the predicate languages.

• Any theory T of equal nonempty unary predicates R_k is a LU-theory;

• Similarly, LU-theories and IILU-theories can be constructed using n-ary predicate symbols of arbitrary arity n.

• The notion of language uniform theory can be extended for an arbitrary language taking graphs for language functions; for instance, theories of free algebras can be considered as LU-theories.

• Acyclic graphs with colored edges (arcs), for which all vertices have same degree with respect to each color, has LU-theories. If there are infinitely many colors and infinitely many empty binary relations then the colored graph has a IILU-theory.

• Generic arc-colored graphs without colors for vertices [1; 4], free polygonometries of free groups [2], and cube graphs with coordinated colorings of edges [2; 3] have LU-theories.

The simplest example of a theory, which is not language uniform, can be constructed taking two nonempty unary predicates R_1 and R_2, where $R_1 \subset R_2$. More generally, if a theory T, with nonempty predicates R_i, $i \in I$, of a fixed arity, is language uniform then cardinalities of $R_{i_1}^{\delta_1}(\bar{x})\wedge\ldots\wedge R_{i_j}^{\delta_1}(\bar{x})$ do not depend on pairwise distinct $i_1, \ldots, i_j$.

Remark 1. Any countable theory T of a predicate language Σ can be transformed to a LU-theory T'. Indeed, since without loss of generality Σ is countable consisting of predicate symbols $R_n^{(k_n)}$, $n \in \omega$, then we can step-by-step replace predicates R_n by predicates R'_n in the following way. We put $R'_0 \rightleftharpoons R_0$. If predicates $R'_0, \ldots, R'_n$ of arities $r_0 < \ldots < r_n$, respectively, are already defined, we take for R'_{n+1} a predicate of an arity $r_{n+1} > \max\{r_n, k_{n+1}\}$, which is obtained from R'_{n+1} adding $r_{n+1} - k_{n+1}$ fictitious variables corresponding to the formula

$$R'(x_1, \ldots, x_{k_{n+1}}) \wedge (x_{k_{n+2}} \approx x_{k_{n+2}}) \wedge (x_{r_{n+1}} \approx x_{r_{n+1}}).$$

If the resulted LU-theory T' has non-empty predicates, it can be transformed to a countable IILU-theory T'' copying these non-empty predicated with same domains countably many times and adding countably many empty predicates for each arity r_n.

Clearly, the process of the transformation of T to T' do not hold for uncountable languages, whereas any LU-theory can be transformed to an IILU-theory as above.

Definition 5. Recall that theories T_0 and T_1 of languages Σ_0 and Σ_1 respectively are said to be *similar* if for any models $\mathcal{M}_i \models T_i$, $i = 0, 1$, there are formulas of T_i, defining in $\mathcal{M}_i$ predicates, functions and constants of language Σ_{1-i} such that the corresponding structure of Σ_{1-i} is a model of T_{1-i}.

Theories T_0 and T_1 of languages Σ_0 and Σ_1 respectively are said to be *language similar* if T_0 can be obtained from T_1 by some bijective replacement of language symbols in Σ_1 by language symbols in Σ_0 (and vice versa).

Clearly, any language similar theories are similar, but not vice versa. Note also that, by the definition, any LU-theory T is language similar to any theory T^σ which is obtained from T replacing predicate symbols R by $\sigma(R)$, where σ is a substitution on the set of predicate symbols in $\Sigma(T)$ corresponding to nonempty predicates for T as well as a substitution on the set of predicate symbols in $\Sigma(T)$ corresponding to empty predicates for T. Thus we have

Proposition 3. *Let T_1 and T_2 be* LU*-theories of same language such that T_2 is obtained from T_1 by a bijection f_1 (respectively f_2) mapping (non)empty predicates for T_1 to (non)empty predicates for T_2. Then T_1 and T_2 are language similar.*

Corollary 1. *Let T_1 and T_2 be countable* IILU*-theories of same language such that the restriction T_1' of T_1 to non-empty predicates is language similar to the restriction T_2' of T_2 to non-empty predicates. Then T_1 and T_2 are language similar.*

Proof. By the hypothesis, there is a bijection f_2 for non-empty predicates of T_1 and T_2. Since T_1 and T_2 be countable IILU-theories then T_1 and T_2 have countably many empty predicates of each arity with non-empty predicates, there is a bijection f_1 for empty predicates of T_1 and T_2. Now Corollary is implied by Proposition 3. $\square$

Definition 6. For a theory T in a predicate language Σ, we denote by $\mathrm{Supp}_\Sigma(T)$ the *support* of Σ for T, i. e., the set of all arities n such that some n-ary predicate R for T is not empty.

Clearly, if T_1 and T_2 are language similar theories, in predicate languages Σ_1 and Σ_2 respectively, then $\mathrm{Supp}_{\Sigma_1}(T_1) = \mathrm{Supp}_{\Sigma_2}(T_2)$.

Definition 7. Let T_1 and T_2 be language similar theories of same language Σ. We say that T_2 *language dominates* T_1 and write $T_1 \sqsubseteq^L T_2$ if for any symbol $R \in \Sigma$, if $T_1 \vdash \exists \bar{x} R(\bar{x})$ then $T_2 \vdash \exists \bar{x} R(\bar{x})$, i. e., all predicates, which are non-empty for T_1, are nonempty for T_2. If $T_1 \sqsubseteq^L T_2$ and $T_2 \sqsubseteq^L T_1$, we say that T_1 and T_2 are *language domination-equivalent* and write $T_1 \sim^L T_2$.

Proposition 4. *The relation $\sqsubseteq^L$ is a partial order on any set of* LU-*theories.*

Proof. Since $\sqsubseteq^L$ is always reflexive and transitive, it suffices to note that if $T_1 \sqsubseteq^L T_2$ and $T_2 \sqsubseteq^L T_1$ then $T_1 = T_2$. It follows as language similar LU-theories coincide having the same set of nonempty predicates. □

Definition 8. We say that T_2 *infinitely language dominates* T_1 and write $T_1 \sqsubset^L_\infty T_2$ if $T_1 \sqsubseteq^L T_2$ and for some n, there are infinitely many new nonempty predicates for T_2 with respect to T_1

Since there are infinitely many elements between any distinct comparable elements in a dense order, we have

Proposition 5. *If a class of theories $\mathcal{T}$ has a dense order $\sqsubseteq^L$ then $T_1 \sqsubset^L_\infty T_2$ for any distinct $T_1, T_2 \in \mathcal{T}$ with $T_1 \sqsubseteq^L T_2$.*

Clearly, if $T_1 \sqsubseteq^L T_2$ then $\mathrm{Supp}_\Sigma(T_1) \subseteq \mathrm{Supp}_\Sigma(T_2)$ but not vice versa. In particular, there are theories T_1 and T_2 with $T_1 \sqsubset^L_\infty T_2$ and $\mathrm{Supp}_\Sigma(T_1) = \mathrm{Supp}_\Sigma(T_2)$.

Let T_0 be a LU-theory with infinitely many nonempty predicate of some arity n, and I_0 be the set of indexes for the symbols of these predicates.

Now for each infinite $I \subseteq I_0$ with $|I| = |I_0|$, we denote by T_I the theory which is obtained from the complete subtheory of T_0 in the language $\{R_k \mid k \in I\}$ united with symbols of all arities $m \neq n$ and expanded by empty predicates R_l for $l \in I_0 \setminus I$, where $|I_0 \setminus I|$ is equal to the cardinality of the set empty predicates for T_0, of the arity n.

By the definition, each T_I is language similar to T_0: it suffices to take a bijection f between languages of T_I and T_0 such that (non)empty predicates of T_I in the arity n correspond to (non)empty predicates of T_0 in the arity n, and f is identical for predicate symbols of the arities $m \neq n$. In particular,

Let $\mathcal{T}$ be an infinite family of theories T_I, and T_J be a theory of the form above (with infinite $J \subseteq I_0$ such that $|J| = |I_0|$). The following proposition modifies Proposition 1 for the E-closure $\mathrm{Cl}_E(\mathcal{T})$.

Proposition 6. *If $T_J \notin \mathcal{T}$ then $T_J \in \mathrm{Cl}_E(\mathcal{T})$ if and only if for any finite set $J_0 \subset I_0$ there are infinitely many T_I with $J \cap J_0 = I \cap J_0$.*

Proof. By the definition each theory T_J is defined by formulas describing $P_k \neq \varnothing \Leftrightarrow k \in J$. Each such a formula φ asserts for a finite set $J_0 \subset I_0$ that if $k \in J_0$ then $R_k \neq \varnothing \Leftrightarrow k \in J$. It means that $\{k \in J_0 \mid P_k \neq \varnothing\} = J \cap J_0$. On the other hand, by Proposition 1, $T_J \in \mathrm{Cl}_E(\mathcal{T})$ if and only if each such formula φ belongs to infinitely many theories T_I in $\mathcal{T}$, i.e., for infinitely many indexes I we have $I \cap J_0 = J \cap J_0$. $\square$

Now we take an infinite family F of infinite indexes I such that F is linearly ordered by $\subseteq$ and if $I_1 \subset I_2$ then $I_2 \setminus I_1$ is infinite. The set $\{T_I \mid I \in F\}$ is denoted by $\mathcal{T}_F$.

For any infinite $F' \subseteq F$ we denote by $\overline{\lim} F'$ the union-set $\bigcup F'$ and by $\underline{\lim} F'$ intersection-set $\bigcap F'$. If $\overline{\lim} F'$ (respectively $\underline{\lim} F'$) does not belong to F' then it is called the *upper* (*lower*) *accumulation point* (for F'). If J is an upper or lower accumulation point we simply say that J is an *accumulation point.*

Corollary 2. *If $T_J \notin \mathcal{T}_F$ then $T_J \in \mathrm{Cl}_E(\mathcal{T}_F)$ if and only if J is an* (*upper or lower*) *accumulation point for some infinite $F' \subseteq F$.*

Proof. If $J = \overline{\lim} F'$ or $J = \underline{\lim} F'$ then for any finite set $J_0 \subset I_0$ there are infinitely many T_I with $J \cap J_0 = I \cap J_0$. Indeed, if $J = \bigcup F'$ then for any finite $J_0 \subset I_0$ there are infinitely many $I \in F'$ such that $I \cap J_0$ contains exactly same elements as $J \cap J_0$ since otherwise we have $J \subset \bigcup F'$. Similarly the assertion holds for $J = \bigcap F'$. By Proposition 6 we have $T_J \in \mathrm{Cl}_E(\mathcal{T}_F)$.

Now let $J \neq \overline{\lim} F'$ and $J \neq \underline{\lim} F'$ for any infinite $F' \subseteq F$. In this case for each $F' \subseteq F$, either J contains new index j for a nonempty predicate with respect to $\bigcup F'$ for each $F' \subseteq F$ with $\bigcup F' \subseteq J$ or $\bigcap F'$ contains new index j' for a nonempty predicate with respect to J for each $F' \subseteq F$ with $\bigcap F' \supseteq J$. In the first case, for $J_0 = \{j\}$ there are no $I \in F'$ such that $I \cap J_0 = J \cap J_0$. In the second case, for $J_0 = \{j'\}$ there are no $I \in F'$ such that $I \cap J_0 = J \cap J_0$. By Proposition 6 we get $T_J \notin \mathrm{Cl}_E(\mathcal{T}_F)$. $\square$

By Corollary 2 the action of the operator Cl_E for the families $\mathcal{T}_F$ is reduced to unions and intersections of *index* subsets of F.

Now we consider possibilities for the linearly ordered sets $\mathcal{F} = \langle F; \subseteq \rangle$ and their closures $\overline{\mathcal{F}} = \langle \overline{F}; \subseteq \rangle$ related to Cl_E.

The structure $\mathcal{F}$ is called *discrete* if F does not contain accumulation points.

By Corollary 2, if $\mathcal{F}$ is discrete then for any $J \in F$, $T_J \notin \mathrm{Cl}_E(\mathcal{T}_{F \setminus \{J\}})$. Thus we get

Proposition 7. *For any discrete $\mathcal{F}$, $\mathcal{T}_F$ is the least generating set for $\mathrm{Cl}_E(\mathcal{T}_F)$.*

By Proposition 7, for any discrete $\mathcal{F}$, $\mathcal{T}_F$ can be reconstructed from $\mathrm{Cl}_E(\mathcal{T}_F)$ removing accumulation points, which always exist. For instance, if $\langle F; \subseteq \rangle$ is isomorphic to $\langle \omega; \leq \rangle$ or $\langle \omega^*; \leq \rangle$ (respectively, isomorphic to $\langle \mathbb{Z}; \leq \rangle$)

then $\mathrm{Cl}_E(\mathcal{T}_F)$ has exactly one (two) new element(s) $\overline{\lim} F$ or $\underline{\lim} F$ (both $\overline{\lim} F$ and $\underline{\lim} F$).

Consider an opposite case: with dense $\mathcal{F}$. Here, if $\mathcal{F}$ is countable then, similarly to $\langle \mathbb{Q}; \leq \rangle$, taking cuts for $\mathcal{F}$, i. e., partitions (F^-, F^+) of F with $F^- < F^+$, we get the closure $\overline{F}$ with continuum many elements. Thus, the following proposition holds.

Proposition 8. *For any dense $\mathcal{F}$, $|\overline{F}| \geq 2^\omega$.*

Clearly, there are dense $\mathcal{F}$ with dense and non-dense $\overline{\mathcal{F}}$. If $\overline{\mathcal{F}}$ is dense then, since $\overline{\overline{F}} = \overline{F}$, there are dense $\mathcal{F}_1$ with $|F_1| = |\overline{F_1}|$. In particular, it is followed by Dedekind theorem on completeness of $\mathbb{R}$.

Answering the question in Section 1 we have

Proposition 9. *If $\overline{\mathcal{F}}$ is dense then $\mathrm{Cl}_E(\mathcal{T}_F)$ does not contain the least generating set.*

Proof. Assume on contrary that $\mathrm{Cl}_E(\mathcal{T}_F)$ contains the least generating set with a set $F_0 \subseteq F$ of indexes. By the minimality F_0 does not contain both the least element and the greatest element. Thus taking an arbitrary $J \in F_0$ we have that for the cut $(F^-_{0,J}, F^+_{0,J})$, where $F^-_{0,J} = \{J^- \in F_0 \mid J^- \subset J\}$ and $F^+_{0,J} = \{J^+ \in F_0 \mid J^+ \supset J\}$, $J = \overline{\lim} F^-_{0,J}$ and $J = \underline{\lim} F^+_{0,J}$. Thus, $F_0 \setminus \{J\}$ is again a set of indexes for a generating set for $\mathrm{Cl}_E(\mathcal{T}_F)$. Having a contradiction we obtain the required assertion. □

Combining Proposition 2 and Proposition 9 we obtain

Corollary 3. *If $\overline{\mathcal{F}}$ is dense then $\mathrm{Th}(\mathcal{A}_E)$ does not have e-least models and, in particular, it is not small.*

Remark 2. The condition of the density of $\overline{\mathcal{F}}$ for Proposition 9 is essential. Indeed, we can construct step-by step a countable dense structure $\mathcal{F}$ without endpoints such that for each $J \in F$ and for its cut (F^-_J, F^+_J), where $F^-_J = \{J^- \in F \mid J^- \subset J\}$ and $F^+_J = \{J^+ \in F \mid J^+ \supset J\}$, $J \supset \overline{\lim} F^-_J$ and $J \subset \underline{\lim} F^+_J$. In this case $\mathrm{Cl}_E(\mathcal{T}_F)$ contains the least generating set $\{T_J \mid J \in F\}$.

In general case, if an element J of F has a successor J' or a predecessor J^{-1} then J defines a connected component with respect to the operations $\cdot'$ and $\cdot^{-1}$. Indeed, taking closures of elements in F with respect to $\cdot'$ and $\cdot^{-1}$ we get a partition of F defining an equivalence relation such that two elements J_1 and J_2 are equivalent if and only if J_2 is obtained from J_1 applying $\cdot'$ or $\cdot^{-1}$ several (maybe zero) times.

Now for any connected component C we have one of the following possibilities:

(i) C is a singleton consisting of an element J such that $J \neq \overline{\lim} F^-_J$ and $J \neq \underline{\lim} F^+_J$; in this case J is not an accumulation point for $F \setminus \{J\}$ and T_J belongs to any generating set for $\mathrm{Cl}_E(\mathcal{T}_F)$;

(ii) C is a singleton consisting of an element J such that $J = \overline{\lim} F^-_J$ or $J = \underline{\lim} F^+_J$, and $\overline{\lim} F^-_J \neq \underline{\lim} F^+_J$; in this case J is an accumulation point for exactly one of F^-_J and F^+_J, J separates F^-_J and F^+_J, and T_J can be removed from any generating set for $\mathrm{Cl}_E(\mathcal{T}_F)$ preserving the generation of $\mathrm{Cl}_E(\mathcal{T}_F)$; thus T_J does not belong to minimal generating sets;

(iii) C is a singleton consisting of an element J such that $J = \overline{\lim} F^-_J = \underline{\lim} F^+_J$; in this case J is a (unique) accumulation point for both F^-_J and F^+_J, moreover, again T_J can be removed from any generating set for $\mathrm{Cl}_E(\mathcal{T}_F)$ preserving the generation of $\mathrm{Cl}_E(\mathcal{T}_F)$, and T_J does not belong to minimal generating sets;

(iv) $|C| > 1$ (in this case, for any intermediate element J of C, T_J belongs to any generating set for $\mathrm{Cl}_E(\mathcal{T}_F)$), $\underline{\lim}\, C \supset \overline{\lim} F^-_{\underline{\lim}\, C}$ and $\overline{\lim}\, C \subset \underline{\lim} F^+_{\overline{\lim}\, C}$; in this case, for the endpoint(s) J^* of C, if it (they) exists, T_{J^*} belongs to any generating set for $\mathrm{Cl}_E(\mathcal{T}_F)$;

(v) $|C| > 1$, and $\underline{\lim}\, C = \overline{\lim} F^-_{\underline{\lim}\, C}$ or $\overline{\lim}\, C = \underline{\lim} F^+_{\overline{\lim}\, C}$; in this case, for the endpoint $J^* = \underline{\lim}\, C$ of C, if it exists, T_{J^*} does not belong to minimal generating sets of $\mathrm{Cl}_E(\mathcal{T}_F)$, and for the endpoint $J^{**} = \overline{\lim}\, C$ of C, if it exists, $T_{J^{**}}$ does not belong to minimal generating sets of $\mathrm{Cl}_E(\mathcal{T}_F)$.

Summarizing (i)–(v) we obtain the following assertions.

Proposition 10. *A partition of F by the connected components forms* discrete intervals *or, in particular, singletons of $\mathcal{F}$, where only endpoints J of these intervals can be among elements J^{**} such that $T_{J^{**}}$ does not belong to minimal generating sets of* $\mathrm{Cl}_E(\mathcal{T}_F)$.

Proposition 11. *If (F^-, F^+) is a cut of F with $\overline{\lim} F^- = \underline{\lim} F^+$ (respectively $\overline{\lim} F^- \subset \underline{\lim} F^+$) then any generating set $\mathcal{T}^0$ for $\mathrm{Cl}_E(\mathcal{T}_F)$ is represented as a (disjoint) union of generating set $\mathcal{T}^0_{F^-}$ for $\mathrm{Cl}_E(\mathcal{T}_{F^-})$ and of generating set $\mathcal{T}^0_{F^+}$ for $\mathrm{Cl}_E(\mathcal{T}_{F^+})$, moreover, any (disjoint) union of a generating set for $\mathrm{Cl}_E(\mathcal{T}_{F^-})$ and of a generating set for $\mathrm{Cl}_E(\mathcal{T}_{F^+})$ is a generating set $\mathcal{T}^0$ for $\mathrm{Cl}_E(\mathcal{T}_F)$.*

Proposition 11 implies

Corollary 4. *If (F^-, F^+) is a cut of F then $\mathrm{Cl}_E(\mathcal{T}_F)$ has the least generating set if and only if $\mathrm{Cl}_E(\mathcal{T}_{F^-})$ and $\mathrm{Cl}_E(\mathcal{T}_{F^+})$ have the least generating sets.*

Considering $\subset$-ordered connected components we have that *discretely ordered* intervals in $\overline{\mathcal{F}}$, consisting of discrete connected components and their limits $\underline{\lim}$ and $\overline{\lim}$, are alternated with densely ordered intervals including their limits. If $\overline{\mathcal{F}}$ contains an (infinite) dense interval, then by Proposition

9, $\mathrm{Cl}_E(\mathcal{T}_F)$ does not have the least generating set. Conversely, if $\overline{\mathcal{F}}$ does not contain dense intervals then $\mathrm{Cl}_E(\mathcal{T}_F)$ contains the least generating set. Thus, answering Questions 1 and 2 [6] for $\mathrm{Cl}_E(\mathcal{T}_F)$ including the question in Section 1, we have

Theorem 2. *For any linearly ordered set $\mathcal{F}$, the following conditions are equivalent:*
(1) $\mathrm{Cl}_E(\mathcal{T}_F)$ *has the least generating set;*
(2) $\overline{\mathcal{F}}$ *does not have dense intervals.*

Remark 3. Theorem 2 does not hold for some non-linearly ordered $\mathcal{F}$. Indeed, taking countably many disjoint, incomparable with respect to nonempty predicates modulo their intersections, copies $\mathcal{F}_q$, $q \in \mathbb{Q}$, of linearly ordered sets isomorphic to $\langle \omega, \leq \rangle$ and ordering limits $J_q = \underline{\lim} F_q$ by the ordinary dense order on $\mathbb{Q}$ such that $\{J_q \mid q \in \mathbb{Q}\}$ is densely ordered, we obtain a dense interval $\{J_q \mid q \in \mathbb{Q}\}$ whereas the set $\cup\{F_q \mid q \in \mathbb{Q}\}$ forms the least generating set $\mathcal{T}_0$ of theories for $\mathrm{Cl}_E(\mathcal{T}_0)$.

The above operation of extensions of theories for $\{J_q \mid q \in \mathbb{Q}\}$ by theories for $\mathcal{F}_q$ as well as expansions of theories of the empty language to theories for $\{J_q \mid q \in \mathbb{Q}\}$ confirm that the (non)existence of a least/minimal generating set for $\mathrm{Cl}_E(\mathcal{T}_0)$ is not preserved under restrictions and expansions of theories.

Remark 4. Taking an arbitrary theory T with a non-empty predicate R of an arity n, we can modify Theorem 2 in the following way. Extending the language $\Sigma(T)$ by infinitely many n-ary predicates interpreted exactly as R and by infinitely many empty n-ary predicates we get a class $\mathcal{T}_{T,R}$ of theories *R-generated* by T. The class $\mathcal{T}_{T,R}$ satisfies the following: any linearly ordered $\mathcal{F}$ as above is isomorphic to some family $\mathcal{F}'$, under inclusion, sets of indexes of non-empty predicates for theories in $\mathcal{T}_{T,R}$ such that strict inclusions $J_1 \subset J_2$ for elements in $\mathcal{F}'$ imply that cardinalities $J_2 \setminus J_1$ are infinite and do not depend on choice of J_1 and J_2. Theorem 2 holds for linearly ordered $\mathcal{F}'$ involving the given theory T.

3. On e-spectra for families of language uniform theories

Remark 5. Remind [5, Proposition 4.1, (7)] that if $T = \mathrm{Th}(\mathcal{A}_E)$ has an e-least model $\mathcal{M}$ then $e\text{-}\mathrm{Sp}(T) = e\text{-}\mathrm{Sp}(\mathcal{M})$. Then, following [5, Proposition 4.1, (5)], $e\text{-}\mathrm{Sp}(T) = |\mathcal{T}_0 \setminus \mathcal{T}_0'|$, where $\mathcal{T}_0'$ is the (least) generating set of theories for E-classes of $\mathcal{M}$, and $\mathcal{T}_0$ is the closed set of theories for E-classes of an e-largest model of T. Note also that $e\text{-}\mathrm{Sp}(T)$ is infinite if $\mathcal{T}_0$ does not have the least generating set.

Remind that, as shown in [5, Propositions 4.3], for any cardinality λ there is a theory $T = \mathrm{Th}(\mathcal{A}_E)$ of a language Σ such that $|\Sigma| = |\lambda + 1|$ and e-$\mathrm{Sp}(T) = \lambda$. Modifying this proposition for the class of LU-theories we obtain

Proposition 12. (1) *For any $\mu \leq \omega$ there is an E-combination $T = \mathrm{Th}(\mathcal{A}_E)$ of* IILU-*theories in a language Σ of the cardinality ω such that T has an e-least model and e-$\mathrm{Sp}(T) = \mu$.*

(2) *For any uncountable cardinality λ there is an E-combination $T = \mathrm{Th}(\mathcal{A}_E)$ of* IILU-*theories in a language Σ of the cardinality λ such that T has an e-least model and e-$\mathrm{Sp}(T) = \lambda$.*

Proof. In view of Propositions 2, 7 and Remark 5, it suffices to take an E-combination of IILU-theories of a language Σ of the cardinality λ and with a discrete linearly ordered set $\mathcal{F}$ having:

1) $\mu \leq \omega$ accumulation points if $\lambda = \omega$;

2) λ accumulation points if $\lambda > \omega$.

We get the required $\mathcal{F}$ for (1) taking:

(i) finite F for $\mu = 0$;

(ii) $\mu/2$ discrete connected components, forming $\mathcal{F}$, with the ordering type $\langle \mathbb{Z}; \leq \rangle$ and having pairwise distinct accumulation points, if $\mu > 0$ is even natural;

(iii) $(\mu - 1)/2$ discrete connected components, forming $\mathcal{F}$, with the ordering type $\langle \mathbb{Z}; \leq \rangle$ and one connected components with the ordering type $\langle \omega; \leq \rangle$ such that all accumulation points are distinct, if $\mu > 0$ is odd natural;

(iv) ω discrete connected components, forming $\mathcal{F}$, with the ordering type $\langle \mathbb{Z}; \leq \rangle$, if $\mu = \omega$.

The required $\mathcal{F}$ for (2) is formed by (uncountably many) λ discrete connected components, forming $\mathcal{F}$, with the ordering type $\langle \mathbb{Z}; \leq \rangle$. $\square$

Combining Propositions 2, 9, Theorem 2, and Remark 5 with $\overline{\mathcal{F}}$ having dense intervals, we get

Proposition 13. *For any infinite cardinality λ there is an E-combination $T = \mathrm{Th}(\mathcal{A}_E)$ of* IILU-*theories in a language Σ of cardinality λ such that T does not have e-least models and e-$\mathrm{Sp}(T) \geq \max\{2^\omega, \lambda\}$.*

Assertion of Proposition 13 can be improved as follows.

Proposition 14. *For any infinite cardinality λ there is an E-combination $T = \mathrm{Th}(\mathcal{A}_E)$ of* LU-*theories in a language Σ of cardinality λ such that T does not have e-least models and e-$\mathrm{Sp}(T) = 2^\lambda$.*

Proof. Let Σ be a language consisting, for some natural n, of n-ary predicate symbols R_i, $i < \lambda$. Choose a cardinality $\mu \in (\omega \setminus \{0\}) \cup \{\omega\}$. For any $\Sigma' \subseteq \Sigma$ we take a structure $\mathcal{A}_{\Sigma'}$ of the cardinality μ such that $R_i = (A_{\Sigma'})^n$

for $R_i \in \Sigma'$, and $R_i = \varnothing$ for $R_i \in \Sigma \setminus \Sigma'$. Clearly, each structure $\mathcal{A}_{\Sigma'}$ has a LU-theory and $\mathcal{A}_{\Sigma'} \not\equiv \mathcal{A}_{\Sigma''}$ for $\Sigma' \neq \Sigma''$. For the E-combination $\mathcal{A}_E$ of the structures $\mathcal{A}_{\Sigma'}$ we obtain the theory $T = \mathrm{Th}(\mathcal{A}_E)$ having a model of the cardinality λ. At the same time $\mathcal{A}_E$ has 2^λ distinct theories of the E-classes $\mathcal{A}_{\Sigma'}$. Thus, e-Sp$(T) = 2^\lambda$. Finally we note that T does not have e-least models by Theorem 1 and arguments for Proposition 6. □

Remark 6. LU-theories in the proof of Proposition 14 can be easily transformed to IILU-theories with the same effect for the e-spectrum.

4. Cantor-Bendixson ranks for language uniform theories

Recall the definition of the Cantor-Bendixson rank. It is defined on the elements of a topological space X by induction: $\mathrm{CB}_X(p) \geq 0$ for all $p \in X$; $\mathrm{CB}_X(p) \geq \alpha$ if and only if for any $\beta < \alpha$, p is an accumulation point of the points of CB_X-rank at least β. $\mathrm{CB}_X(p) = \alpha$ if and only if both $\mathrm{CB}_X(p) \geq \alpha$ and $\mathrm{CB}_X(p) \not\geq \alpha + 1$ hold; if such an ordinal α does not exist then $\mathrm{CB}_X(p) = \infty$. Isolated points of X are precisely those having rank 0, points of rank 1 are those which are isolated in the subspace of all non-isolated points, and so on. For a non-empty $C \subseteq X$ we define $\mathrm{CB}_X(C) = \sup\{\mathrm{CB}_X(p) \mid p \in C\}$; in this way $\mathrm{CB}_X(X)$ is defined and $\mathrm{CB}_X(\{p\}) = \mathrm{CB}_X(p)$ holds. If X is compact and C is closed in X then the sup is achieved: $\mathrm{CB}_X(C)$ is the maximum value of $\mathrm{CB}_X(p)$ for $p \in C$; there are finitely many points of maximum rank in C and the number of such points is the CB_X*-degree* of C. If X is countable and compact then $\mathrm{CB}_X(X)$ is a countable ordinal and every closed subset has ordinal-valued rank and finite CB_X-degree.

Clearly, for any set $\mathcal{F}$, where $\mathrm{Cl}_E(\mathcal{T}_F)$ does not have the least generating set, $\mathrm{CB}_{\mathcal{T}_F}(\mathcal{T}_F) = \infty$.

Theorem 3. *For any countable ordinal α and a natural number $n > 0$, there is an E-closed family $\mathcal{T}_{F_\alpha}$ of* LU*-theories such that $\mathrm{CB}_{\mathcal{T}_{F_\alpha}}(\mathcal{T}_{F_\alpha}) = \alpha$ and its $\mathrm{CB}_{\mathcal{T}_{F_\alpha}}$-degree is equal to n.*

Proof. If $\alpha = 0$ it suffices to take n singletons $\mathcal{F}_{0,1}, \ldots, \mathcal{F}_{0,n}$. If $\alpha = 1$ we take n disjoint copies $\mathcal{F}_{1,j}$, $j = 1, \ldots, n$, of $\mathcal{F}_q$ in Remark 3, each of which is ordered as $\langle \omega, \leq \rangle$ and $\cup \mathcal{F}_{0,j} = \underline{\lim}\, F_{1,j}$, $j = 1, \ldots, n$. We set $F_0 = F_{0,1} \cup \ldots \cup F_{0,n}$, $F_1 = F_0 \cup \bigcup\limits_{j=1}^{n} F_{1,j}$. If $\alpha > 1$ is finite and $\mathcal{F}_\alpha$ is already defined then we add ω new disjoint copies $\mathcal{F}_{\alpha+1,m}$ of $\mathcal{F}_q$ related to each element in $F_\alpha \setminus F_{\alpha-1}$, each of which is ordered as $\langle \omega, \leq \rangle$ and $f_m = \underline{\lim}\, F_{\alpha+1,m}$, $f_m \in F_\alpha \setminus F_{\alpha-1}$. In such a case, $\mathrm{CB}(\mathcal{F}_{0,j}) = \alpha + 1$ and CB-degree is equal to n.

In general case, if α is limit we take $\mathcal{T}_{F_\alpha}$ as the union of $\mathcal{T}_{F_\beta}$ for $\beta < \alpha$ with ω disjoint copies of $\mathcal{F}_q$ such that each element in $\mathcal{T}_{F_\beta}$ is the limit $\underline{\lim}$ of unique new copy of $\mathcal{F}_q$ and vice versa. Otherwise, if $\alpha = \beta + 1$, we add ω disjoint copies of $\mathcal{F}_q$ such that the set of these new copies $\mathcal{F}$ are in the bijective correspondence with the set of elements f, added in the step β, and $f = \underline{\lim}\, F$.

The inductive process guarantees that $\mathrm{CB}_{\mathcal{T}_{F_\alpha}}(\mathcal{T}_{F_\alpha}) = \alpha$ and $\mathrm{CB}_{\mathcal{T}_{F_\alpha}}$-degree is equal to n. $\square$

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С. В. Судоплатов

Семейства сигнатурно однородных теорий и их порождающие множества

Аннотация. Вводится понятие сигнатурно однородной теории и изучаются топологические свойства, относящиеся к семействам сигнатурно однородных теорий и их E-совмещениям. Показано, что класс сигнатурно однородных теорий достаточно широк. Найдены достаточные условия сигнатурного подобия сигнатурно однородных теорий. Изучены свойства сигнатурного доминирования и бесконечного сигнатурного доминирования. Найдена характеризация для E-замыкания семейства сигнатурно однородных теорий в терминах индексных множеств. Рассмотрен класс линейно упорядоченных семейств сигнатурно однородных теорий и к этому классу применена указанная характеризация. Исследованы свойства дискретных и плотных индексных множеств: показано, что любое дискретное индексное множество задает наименьшее порождающее множество, в то время как плотные индексные множества определяют по меньшей мере континуальное число точек накопления и замыкания без наименьших порождающих множеств. В частности, при наличии плотного индексного множества теория соответствующего E-совмещения не имеет e-наименьшей модели и не является малой. Используя дихотомию для дискретных и плотных индексных множеств, решается проблема существования наименьшего порождающего множества относительно E-совмещений и характеризуется это существование в терминах порядков.

Получены значения e-спектров для семейств сигнатурно однородных теорий. Показано, что любой e-спектр может быть реализован некоторым E-совмещением сигнатурно однородных теорий. Найдены нижние оценки для e-спектров относительно мощностей сигнатур.

Показано, что семейства сигнатурно однородных теорий задают произвольный ранг Кантора – Бендиксона и произвольную степень относительно этого ранга.

Ключевые слова: E-совмещение, P-совмещение, оператор замыкания, порождающее множество, сигнатурно однородная теория.

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