Серия «Математика»
2018. Т. 24. С. 24—36

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского
государственного
университета

УДК 517.547
MSC 30C45, 30C80
DOI https://doi.org/10.26516/1997-7670.2018.24.24

Certain subclasses of analytic functions defined by a new general linear operator

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Abstract. Hypergeometric functions are of special interests among the complex analysts especially in looking at the properties and criteria of univalent. Hypergeometric functions have been around since 1900's and have special applications according to their own needs. Recently, we had an opportunity to study on q-hypergeometric functions and quite interesting to see the behavior of the functions in the complex plane. There are many different versions by addition of parameters and choosing suitable variables in order to impose new set of q-hypergeometric functions. The aim of this paper is to study and introduce a new convolution operator of q-hypergeometric typed. Further, we consider certain subclasses of starlike functions of complex order. We derive some geometric properties like, coefficient bounds, distortion results, extreme points and the Fekete-Szego inequality for these subclasses.

Keywords: analytic functions, univalent functions, starlike functions, linear operator, Fekete-Szego problem

1. Introduction

Recently, Darus and others in [1] and [2] have used the q-hypergeometric functions in studying certain families of analytic functions in the open unit disk. The q-hypergeometric functions are the generalized form of the classical hypergeometric function. Then by letting the limit $q \to 1$, it will return to the classical hypergeometric function. The notion of hypergeometric functions have been used and introduced by many great mathematicians started by Euler in (1748), Gauss (1813) and Cauchy (1852) and after that, Heine(1846) converted a simple notation $\lim_{q\to 1}\frac{1-q^a}{1-q} = a$ into a systematic theory of hypergeometric functions parallel to the theory of Gauss hypergeometric function.

Here, we can say that many of the results for classical hypergeometric functions can be generalized to q-hypergeometric functions.

In this work, we shall introduce a new subclasses of univalent functions involving new operator $\mathcal{L}_q^{s,a}(a_i, b_j)$ which generalizes many well-known operators and derived some geometric properties for this new subclasses.

Denote $\mathcal{A}$ the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open unit disk $\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be starlike of complex order if the following condition (see[3]) is satisfied:

$$\Re\{1 + \frac{1}{d}(\frac{zf'(z)}{f(z)} - 1)\} > \beta, (0 \leq \beta < 1 \text{ and } d \in \mathbb{C}\backslash\{0\}) \tag{1.2}$$

For complex parameters $a_1, ..., a_t$ and $b_1, ..., b_r (b_j \in \mathbb{C}\backslash\{0, -1, -2, -3, ...\}$, $j = 1, ...r, |q| < 1)$, the q-hypergeometric

$${}_t\Psi_r = \sum_{n=0}^{\infty} \frac{(a_1, q)_n....(a_t, q)_n}{(q, q)_n(b_1, q)_n....(b_r, q)_n} z^n \tag{1.3}$$

$$(t = r + 1; t, r \in \mathbb{N}_0 = \{0, 1, 2, ...\}; z \in \mathcal{U}).$$

The q-shifted factorial is defined by

$$(a, q)_0 = 1 \quad and \quad (a, q)_n = (1 - a)(1 - aq)(1 - aq^2)....(1 - aq^{n-1}), n \in \mathbb{N},$$

where a any complex number and in terms of the Gamma function

$$(q^\alpha, q)_n = \frac{\Gamma_q(\alpha + n)(1 - q)^n}{\Gamma_q(\alpha)},$$

such that $\Gamma_q(y) = \frac{(q,q)_\infty (1-q)1-y}{(q^y,q)_\infty}, 0 < q < 1$. We note that and by using ratio test, the series (1.3) converges absolutely in open unit disk $\mathcal{U}, |q| < 1$ and $t = r + 1$. Now,if $t = 2$ and $r = 1$, then we have the following

$${}_2\Psi_1 = \sum_{n=0}^{\infty} \frac{(a_1, q)_n(a_2, q)_n}{(q, q)_n(b_1, q)_n} z^n (|q| < 1, z \in \mathcal{U})$$

is the q-Gauss hypergeometric function [4].

Recently, Mohammed and Darus [1] defined the the following: $\mathcal{I}(a_i;b_j;q)f:\mathcal{A}\to\mathcal{A}:$

$$\mathcal{I}(a_i;b_j;q)f(z)=z+\sum_{n=2}^{\infty}\frac{(a_1,q)_{n-1}....(a_t,q)_{n-1}}{(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}}c_nz^n. \tag{1.4}$$

The Srivastava-Attiya operator $\mathcal{J}_{s,a}:\mathcal{A}\to\mathcal{A}$ is defined in [10] as:

$$\mathcal{J}_{s,a}f(z)=z+\sum_{n=2}^{\infty}\left(\frac{1+a}{n+a}\right)^s c_nz^n, \tag{1.5}$$

where $z\in\mathcal{U},a\in\mathbb{C}\backslash\{0,-1,-2,-3,...\},s\in\mathbb{C}\quad and\quad f\in\mathcal{A})$. This linear operator $\mathcal{J}_{s,a}$ can be written as

$$\mathcal{J}_{s,a}f(z)=G_{s,a}(z)*f(z)=(1+a)^s(\phi(z,s,a)-a^{-s})*f(z),$$

by using the Hadamard product(convolution). Here,

$$\phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s},$$

is the well-known Hurwitz -Lerch zeta function(see [5], [6]). It is also an important function of Analytic Number Theory such the De Jonquiere function:

$$Li_s(z)=\sum_{n=0}^{\infty}\frac{z^n}{(n)^s}=z\Phi(z,s,1),\quad (Re(s)>1\quad if\quad |z|=1).$$

We define the linear operator $\mathcal{L}_q^{s,a}(a_i,b_j)(f):\mathcal{A}\to\mathcal{A}$ as follows:

$$\mathcal{L}_q^{s,a}(a_i,b_j)f(z)=z+\sum_{n=2}^{\infty}\frac{(a_1,q)_{n-1}....(a_t,q)_{n-1}}{(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}}\left(\frac{1+a}{n+a}\right)^s c_nz^n. \tag{1.6}$$

$(z\in\mathcal{U},a\in\mathbb{C}\backslash\{0,-1,-2,-3,...\},s\in\mathbb{C},a_i,b_j\in\mathbb{C},b_j\in\mathbb{C}\setminus\{0,-1,-2,...\},$ $|q|<1\ and\ t=r+1)$.

It should be noted that the linear operator (1.6) generalised many operators studied by several earlier authors as follows: 1- If s=0, then

$$\mathcal{L}_q^{0,a}(a_i,b_j)f(z)=\mathcal{M}_q(a_i,b_j),$$

where $\mathcal{M}_q(a_i,b_j)$ is the linear operator introduced by Mohammed and Darus [1].

For $q\to 1,a_i=q^{\alpha_i},b_j=q^{\beta_j}$, where $\alpha_i,\beta_j\in\mathbb{C}$ and $\beta_j\neq 0$ $(i=1,2,...,t$ and $j=1,2,..,r)$, we have the following operators:
2- If $t=2,r=1,\alpha_1=\lambda+1,\alpha_2=\lambda+1,\beta_1=\nu+1$, then we obtain the

operator considered by Prajapat and Bulboaca [7].
3- If $t = 2, r = 1, \alpha_1 = \lambda, \alpha_2 = 1, \beta_1 = \nu + 1$, then we have the operator considered by Noor and Bukhari [8].
4- If $s = 0, a = 0, t = 2, r = 1, \alpha_1 = \lambda, \alpha_2 = 1, \beta_1 = \nu + 1$, then we obtain the Choi-Saigo-Srivastava operator [9].
5- If $t = 2, r = 1, \alpha_1 = \beta_1, \alpha_2 = 1$, then we obtain the Srivastava-Attiya operator [10].
6- If $s = -x, t = 2, r = 1, \alpha_1 = \beta_1, \alpha_2 = 1$, then we obtain the Cho and Srivastava operator [11].
7- If $s = -k(k \in \mathbb{N}), a = 0, t = 2, r = 1, \alpha_1 = \lambda = \beta_1, \alpha_2 = 1$, then we obtain the Salagean operator [12].
8- If $s = 1, t = 2, r = 1, \alpha_1 = \beta_1, \alpha_2 = 1, a \geq -1$, then we obtain the Bernardi operator [13].
9- If $s = 0, a = 0, t = 2, r = 1, \alpha_1 = \lambda, \alpha_2 = 1, \beta_1 = \nu$, then we obtain the Carlson-Shaffer operator [14].
10- If $s = 0, t = r + 1$, then we obtain the Dziok-Srivastava Operator [15].
Some of these operators contained some other operators (see for example [16; 17])

Definition 1. *A function $f \in \mathcal{U}$ is said to be in the class$S_q^{s,a}(d, \beta)$ if the following condition holds:*

$$Re\left\{1 + \frac{1}{d}\left(\frac{z\mathcal{L}_q^{s,a}(a_i, b_j)f(z)}{\mathcal{L}_q^{s,a}(a_i, b_j)f(z)} - 1\right)\right\} > \beta \tag{1.7}$$

$(d \in \mathbb{C}\backslash\{0\}$ *and* $0 \leq \beta < 1)$

Motivated by the work given by Srivastava and Gaboury [18], we investigate some geometric properties like coefficients estimate, distortion bounds, extreme points and the Fekete-Szegö problem for the current function class.

2. Coefficient Estimate

Theorem 1. *For $0 \leq \beta < 1, d \in \mathbb{C} \setminus \{0\}$ and if $f(z) \in \mathcal{A}$ satisfies the following*

$$\sum_{n=2}^{\infty}(n-\beta|d|)\frac{(a_1, q)_{n-1}....(a_t, q)_{n-1}}{(q, q)_{n-1}(b_1, q)_{n-1}....(b_r, q)_{n-1}}\left(\frac{1+a}{n+a}\right)^s |c_n| \leq 1-\beta|d|, \tag{2.1}$$

then $f \in S_q^{s,a}(d, \beta)$

Proof. Suppose that (2.1) holds. Then if

$$H(z) = \frac{\frac{z(\mathcal{L}_q^{s,a}(a_i,b_j)f(z))'}{d\mathcal{L}_q^{s,a}(a_i,b_j)f(z)} - (\frac{1}{d} + \beta - 1) - 1}{\frac{z(\mathcal{L}_q^{s,a}(a_i,b_j)f(z))'}{d\mathcal{L}_q^{s,a}(a_i,b_j)f(z)} - (\frac{1}{d} + \beta - 1) + 1}$$

Therefore its enough to show that $|H(z)| < 1$,that is

$$\left| \frac{\frac{z(\mathcal{L}_q^{s,a}(a_i,b_j)f(z))'}{d\mathcal{L}_q^{s,a}(a_i,b_j)f(z)} - (\frac{1}{d} + \beta - 1) - 1}{\frac{z(\mathcal{L}_q^{s,a}(a_i,b_j)f(z))'}{d\mathcal{L}_q^{s,a}(a_i,b_j)f(z)} - (\frac{1}{d} + \beta - 1) + 1} \right|$$

$$= \left| \frac{\beta dz + \sum_{n=2}^{\infty} \frac{(a_1,q)_{n-1}....(a_t,q)_{n-1}}{(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}} \left(\frac{1+a}{n+a}\right)^s (n-1-\beta d) c_n z^n}{(2-\beta)dz - \sum_{n=2}^{\infty} \frac{(a_1,q)_{n-1}....(a_t,q)_{n-1}}{(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}} \left(\frac{1+a}{n+a}\right)^s (n+1-\beta d) c_n z^n} \right|,$$

thus by using (2.1) we have

$$|H(z)| \leq$$

$$\leq \frac{\beta|d||z| + \sum\limits_{n=2}^{\infty} \frac{(a_1,q)_{n-1}....(a_t,q)_{n-1}}{(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}} \left(\frac{1+a}{n+a}\right)^s (n-1-\beta|d|) c_n |z|^n}{(2-\beta)|d||z| - \sum\limits_{n=2}^{\infty} \frac{(a_1,q)_{n-1}....(a_t,q)_{n-1}}{(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}} \left(\frac{1+a}{n+a}\right)^s (n+1-\beta|d|) c_n |z|^n} <$$

$$< \frac{\beta|d| + \sum\limits_{n=2}^{\infty} \frac{(a_1,q)_{n-1}....(a_t,q)_{n-1}}{(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}} \left(\frac{1+a}{n+a}\right)^s (n-1-\beta|d|) c_n}{(2-\beta)|d| - \sum\limits_{n=2}^{\infty} \frac{(a_1,q)_{n-1}....(a_t,q)_{n-1}}{(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}} \left(\frac{1+a}{n+a}\right)^s (n+1-\beta|d|) c_n} \leq 1,$$

and the proof is complete. □

3. Distortion Bounds

Theorem 2. *For* $0 \leq \beta < 1, d \in \mathbb{C} \setminus \{0\}$ *and let* $f(z)$ *of the form (1.1) be in the class* $S_q^{s,a}(d,\beta)$. *Then*

$$r - \frac{(1-\beta)|d|^2}{Re(d)} r^2 \sum_{n=2}^{\infty} \frac{(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}}{(a_1,q)_{n-1}....(a_t,q)_{n-1}(n-1+\frac{(1-\beta)|d|^2}{Re(d)})} \left(\frac{n+a}{1+a}\right)^s \leq$$

$$\leq |f(z)| \leq$$

$$\leq r + \frac{(1-\beta)|d|^2}{Re(d)} r^2 \sum_{n=2}^{\infty} \frac{(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}}{(a_1,q)_{n-1}....(a_t,q)_{n-1}(n-1+\frac{(1-\beta)|d|^2}{Re(d)})} \left(\frac{n+a}{1+a}\right)^s,$$

$$(|z| = r < 1) \quad (3.1)$$

$$1-\frac{(1-\beta)|d|^2}{Re(d)}r\sum_{n=2}^{\infty}\frac{n(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}}{(a_1,q)_{n-1}....(a_t,q)_{n-1}(n-1+\frac{(1-\beta)|d|^2}{Re(d)})}\left(\frac{n+a}{1+a}\right)^s \leq$$
$$\leq |f'(z)| \leq$$
$$\leq 1+\frac{(1-\beta)|d|^2}{Re(d)}r\sum_{n=2}^{\infty}\frac{n(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}}{(a_1,q)_{n-1}....(a_t,q)_{n-1}(n-1+\frac{(1-\beta)|b|^2}{Re(b)})}\left(\frac{n+a}{1+a}\right)^s,$$
$$(|z|=r<1) \quad (3.2)$$

Proof. Let $f(z)$ be of the form (1.1). Then by using Theorem 2.1, we have

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty}|c_n||z^n| \geq$$
$$\geq r-\frac{(1-\beta)|d|^2}{Re(d)}r^2\sum_{n=2}^{\infty}\frac{(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}}{(a_1,q)_{n-1}....(a_t,q)_{n-1}(n-1+\frac{(1-\beta)|d|^2}{Re(d)})}|\left(\frac{n+a}{1+a}\right)^s|$$

and

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty}|c_n||z^n| \leq$$
$$\leq r+\frac{(1-\beta)|d|^2}{Re(d)}r^2\sum_{n=2}^{\infty}\frac{(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}}{(a_1,q)_{n-1}....(a_t,q)_{n-1}(n-1+\frac{(1-\beta)|d|^2}{Re(d)})}|\left(\frac{n+a}{1+a}\right)^s|.$$
$$(|z|=r<1)$$

Also from(1.1), we have

$$|f'(z)| \geq 1 - \sum_{n=2}^{\infty}n|c_n||z^{n-1}| \geq$$
$$\geq 1-\frac{(1-\beta)|d|^2}{Re(d)}r\sum_{n=2}^{\infty}\frac{n(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}}{(a_1,q)_{n-1}....(a_t,q)_{n-1}(n-1+\frac{(1-\beta)|d|^2}{Re(d)})}\left(\frac{n+a}{1+a}\right)^s$$

and

$$|f'(z)| \leq 1 + \sum_{n=2}^{\infty}n|c_n||z^{n-1}| \leq$$
$$\leq 1+\frac{(1-\beta)|d|^2}{Re(d)}r\sum_{n=2}^{\infty}\frac{n(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}}{(a_1,q)_{n-1}....(a_t,q)_{n-1}(n-1+\frac{(1-\beta)|d|^2}{Re(d)})}\left(\frac{n+a}{1+a}\right)^s,$$
$$(|z|=r<1).$$

Therefore, the results (3.1) and (3.2) are obtained. The proof is complete.

□

4. Extreme Points

In this section we take the subclass $\widehat{S}_q^{s,a}(d,\beta)$ of the class $S_q^{s,a}(d,\beta)$ consisting of all the functions $f(z) \in \mathcal{A}$ of the form (1.1) and satisfy (2.1).The following theorem determine the extreme points of the subclass $\widehat{S}_q^{s,a}(d,\beta)$

Theorem 3. *Let*

$$f_1(z) = z \tag{4.1}$$

and

$$f_n(z) = \\ = z + \frac{|(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}|\,(1-\beta|d|)}{(n-\beta|d|)\,|(a_1,q)_{n-1}....(a_t,q)_{n-1}|}\left|\left(\frac{n+a}{1+a}\right)^s\right| z^n \\ (n \neq 1). \tag{4.2}$$

Then $f \in \widehat{S}_q^{s,a}(d,\beta)$ *if and only if*

$$f(z) = \sum_{n=1}^{\infty} \eta_n f_n(z) \left(\eta_n > 0; \sum_{n=1}^{\infty} \eta_n = 1\right) \tag{4.3}$$

Proof. Let $f \in \widehat{S}_q^{s,a}(d,\beta)$. Then in virtue of (2.1), we can set

$$\eta_n = \\ = (n-\beta|d|)\frac{(a_1,q)_{n-1}....(a_t,q)_{n-1}}{(1-\beta|d|)(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}}\left(\frac{1+a}{n+a}\right)^s |c_n|, \\ (n \neq 1). \tag{4.4}$$

which give our result (4.3).

Conversely,let

$$f(z) = \sum_{n=1}^{\infty} \eta_n f_n(z)$$

$$= z + \sum_{n=1}^{\infty} \eta_n \frac{|(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}|\,(1-\beta|d|)}{(n-\beta|d|)\,|(a_1,q)_{n-1}....(a_t,q)_{n-1}|}\left|\left(\frac{n+a}{1+a}\right)^s\right| z^n.$$

Then

$$\sum_{n=1}^{\infty}(n-\beta|d|)\frac{(a_1,q)_{n-1}....(a_t,q)_{n-1}}{(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}}\left(\frac{1+a}{n+a}\right)^s$$

$$.\eta_n \frac{|(q,q)_{n-1}(b_1,q)_{n-1}....(b_r,q)_{n-1}|\,(1-\beta|d|)}{(n-\beta|d|)\,|(a_1,q)_{n-1}....(a_t,q)_{n-1}|}\left|\left(\frac{n+a}{1+a}\right)^s\right|$$

$$= 1-\beta|d|\sum_{n=2}^{\infty}\eta_n = (1-\beta|d|)(1-\eta_1).$$

Therefore, we have $f \in \widehat{S}_q^{s,a}(d,\beta)$.

□

5. The Fekete-Szego Inequality

Let $S_q^{s,a}(\phi)$ be a a class consisting of all the functions $f(z) \in \mathcal{A}$ of the form (1.1) and satisfies the following

$$1+\frac{1}{d}\left(\frac{z\mathcal{L}_q^{s,a}(a_i,b_j)f(z)}{\mathcal{L}_q^{s,a}(a_i,b_j)f(z)}-1\right) \prec \frac{1+Az}{1+Bz} = \phi(z), \tag{5.1}$$

$(z \in \mathcal{U}, a \in \mathbb{C}\backslash\{0,-1,-2,-3,...\}, s \in \mathbb{C}, a_i, b_j \in \mathbb{C}, b_j \in \mathbb{C}\setminus\{0,-1,-2,...\},$ $|q| < 1$ *and* $t = r+1, d \in \mathbb{C}\backslash\{0\})$, where $\prec$ denotes subordination $, -1 \leq B < A \leq 1$.

In this section, we shall find the upper bounds of the Fekete-Szegö functional for the class$S_q^{s,a}(\phi)$. We need the following lemma due to Ma and Minda [19] to prove our theorem involving the Fekete-Szegö inequality.

Lemma 1. *If* $p(z) = 1+d_1z+d_2z^2+....$ *is an analytic function in* $\mathcal{U}$ *with positive real part, the for any complex number* μ,

$$|d_2-\mu d_1^2| \leq 2max\{1, |2\mu-1|\}. \tag{5.2}$$

The result is sharp for functions given by

$$p(z) = \frac{1+z}{1-z} \quad and \quad p(z) = \frac{1+z^2}{1-z^2}$$

Theorem 4. *Let* $\phi(z) = 1+B_1z+B_2z^2+...$*,and* $f(z)$ *given by (1.1) belongs to* $\mathfrak{S}_q^{s,a}(\phi)$*. Then*

$$|c_3-\mu c_2^2| \leq 2max\{1, |\frac{2\mu}{\sigma_2}-1|\}. \tag{5.3}$$

Proof. Let $f(z) \in S_q^{s,a}(\phi)$. Then by definition of subordination, there exists a Schwarz function$w(z)$ with$w(0) = 0$ *and* $|w(z)| < 1$, analytic in the open unit disk such that

$$1+\frac{1}{d}\left(\frac{z\mathcal{L}_q^{s,a}(a_i,b_j)f(z)}{\mathcal{L}_q^{s,a}(a_i,b_j)f(z)}-1\right) = \phi(w(z)). \tag{5.4}$$

Define

$$p_1(z) := \frac{1+w(z)}{1-w(z)} = 1 + d_1 z + d_2 z^2 + \dots, \tag{5.5}$$

it is clear that $Re(p_1(z)) > 0$ and $p_1(0) = 1$. Let

$$p(z) := 1 + \frac{1}{d}\left(\frac{z\mathcal{L}_q^{s,a}(a_i, b_j)f(z)}{\mathcal{L}_q^{s,a}(a_i, b_j)f(z)} - 1\right) = 1 + h_1 z + h_2 z^2 + \dots \tag{5.6}$$

Therefore, in view of the above equations, we have

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)$$

and since

$$\frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[d_1 z + (d_2 - \frac{d_1^2}{2})z^2 + (d_3 + \frac{d_1^3}{4} - d_1 d_2)z^3 + \dots\right]$$

thus

$$\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1 d_1 z + \left[\frac{1}{2}B_1(d_2 - \frac{1}{2}d_1^2) + \frac{1}{4}B_2 c_1^2\right] z^2 + \dots,$$

By comparing the coefficients for z we obtain

$$h_1 = \frac{1}{2}B_1 d_1$$

and

$$h_2 = \frac{1}{2}B_1(d_2 - \frac{1}{2}d_1^2) + \frac{1}{4}B_2 d_1^2.$$

Then, with the help of (1.6), we get

$$\begin{aligned}
\frac{z\mathcal{L}_q^{s,a}(a_i, b_j)f(z)}{\mathcal{L}_q^{s,a}(a_i, b_j)f(z)} &= 1 + c_2\left(\frac{(a_1,q)_1 \dots (a_t,q)_1}{(q,q)_1(b_1,q)_1\dots(b_r,q)_1}\left(\frac{1+a}{2+a}\right)^s\right) z + \\
&+ \left(2c_3\frac{(a_1,q)_2 \dots (a_t,q)_2}{(q,q)_2(b_1,q)_2\dots(b_r,q)_2}\left(\frac{1+a}{3+a}\right)^s - \right.\\
&\left. - c_2\frac{(a_1,q)_1 \dots (a_t,q)_1}{(q,q)_1(b_1,q)_1\dots(b_r,q)_1}\left(\frac{1+a}{2+a}\right)^s\right)^2 z^2 + \\
&+ \left(3c_4\frac{(a_1,q)_3 \dots (a_t,q)_3}{(q,q)_3(b_1,q)_3\dots(b_r,q)_3}\left(\frac{1+a}{4+a}\right)^s + \right.\\
&\left. + c_2\frac{(a_1,q)_1 \dots (a_t,q)_1}{(q,q)_1(b_1,q)_1\dots(b_r,q)_1}\left(\frac{1+a}{2+a}\right)^s\right)^3 - \\
&- 3\left(c_3\frac{(a_1,q)_2 \dots (a_t,q)_2}{(q,q)_2(b_1,q)_2\dots(b_r,q)_2}\left(\frac{1+a}{3+a}\right)^s\right) \cdot \\
&\quad \cdot \left(c_2\frac{(a_1,q)_1 \dots (a_t,q)_1}{(q,q)_1(b_1,q)_1\dots(b_r,q)_1}\left(\frac{1+a}{2+a}\right)^s\right) z^3 + \dots
\end{aligned}$$

Therefore, from (5.6) we obtain

$$bh_1 = c_2 \frac{(a_1,q)_1 ....(a_t,q)_1}{(q,q)_1(b_1,q)_1....(b_r,q)_1} \left(\frac{1+a}{2+a}\right)^s$$

$$bh_2 = 2c_3 \frac{(a_1,q)_2....(a_t,q)_2}{(q,q)_2(b_1,q)_2....(b_r,q)_2} \left(\frac{1+a}{3+a}\right)^s - \\ - \left(c_2 \frac{(a_1,q)_1....(a_t,q)_1}{(q,q)_1(b_1,q)_1...(b_r,q)_1} \left(\frac{1+a}{2+a}\right)^s\right)^2,$$

then

$$c_2 = \frac{dB_1 d_1}{2} \frac{(q,q)_1(b_1,q)_1....(b_r,q)_1}{(a_1,q)_1....(a_t,q)_1} \left(\frac{2+a}{1+a}\right)^s$$

$$c_3 = \\ = \left(\frac{d}{4}B_1 d_2 + \frac{d_1^2}{8}(b^2 B_1^2 \frac{(q,q)_1(b_1,q)_1....(b_r,q)_1}{(a_1,q)_1....(a_t,q)_1} \left(\frac{2+a}{1+a}\right)^s - d(B_1 - B_2))\right) \cdot \\ \cdot \frac{(q,q)_2(b_1,q)_2....(b_r,q)_2}{(a_1,q)_2....(a_t,q)_2} \left(\frac{3+a}{1+a}\right)^s.$$

Therefore, we obtain

$$c_3 - \mu c_2^2 = \\ = \left(\frac{d}{4}B_1 d_2 + \frac{d_1^2}{8}(b^2 B_1^2 \frac{(q,q)_1(b_1,q)_1....(b_r,q)_1}{(a_1,q)_1....(a_t,q)_1} \left(\frac{2+a}{1+a}\right)^s - d(B_1 - B_2))\right) \cdot \\ \cdot \frac{(q,q)_2(b_1,q)_2....(b_r,q)_2}{(a_1,q)_2....(a_t,q)_2} \left(\frac{3+a}{1+a}\right)^s - \\ - \mu \left(\frac{dB_1 d_2}{2} \frac{(q,q)_1(b_1,q)_1....(b_r,q)_1}{(a_1(,q)_1....(a_t,q)_1} \left(\frac{2+a}{1+a}\right)^s\right)^2.$$

Then

$$c_3 - \mu c_2^2 = \frac{dB_1\sigma_2}{4}\left(d_2 - d_1^2\tau\right),$$

where

$$\tau = \frac{1}{2}(1 - \frac{B_2}{B_1} + (2\mu\frac{1}{\sigma_2} - 1)dB_1\sigma_1)$$

and

$$\sigma_1 = \frac{(q,q)_1(b_1,q)_1....(b_r,q)_1}{(a_1,q)_1....(a_t,q)_1} \left(\frac{2+a}{1+a}\right)^s$$

$$\sigma_2 = \frac{(q,q)_2(b_1,q)_2....(b_r,q)_2}{(a_1,q)_2....(a_t,q)_2} \left(\frac{3+a}{1+a}\right)^s.$$

Now, using Lemma 5.1 we get

$$|c_3 - \mu c_2^2| \leq 2max\{1, |\frac{2\mu}{\sigma_2} - 1|\}.$$

The result is sharp for the functions f given by

$$p(z) = \frac{1+z^2}{1-z^2}, p(z) = \frac{1+z}{1-z}.$$

□

Here, we can mention that this result generalizes many recently results which investigated in several earlier works. In fact, some other work related to q-analoque can also be seen in [20–22].

Conflict of interest: The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgement: The work here is supported by UKM grant: GUP-2017-064.

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Received 27.11.2017

Подклассы аналитических функций, определяемые общим линейным оператором

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Аннотация. Гипергеометрические функции вызывают особый интерес в теории функций комплексных переменных, особенно при рассмотрении свойств и критериев однолистных функций. Гипергеометрические функции существуют с 1900-х годов и имеют специальные приложения в соответствии с их собственными потребностями. Недавно у нас была возможность изучить q-гипергеометрические функции и увидеть довольно интересное поведение функций в комплексной плоскости. Существует множество различных версий путем добавления параметров и выбора подходящих переменных, чтобы получить новый набор q-гипергеометрических функций. Целью настоящей работы является изучение и введение нового оператора свертки с q-гипергеометрической функцией. Рассмотрены некоторые подклассы звездообразных функций сложного порядка. Получены некоторые геометрические свойства, такие как оценки коэффициентов, теоремы искажений, экстремальные точки и неравенство Фекете – Сегё для этих подклассов.

Ключевые слова: аналитические функции, однолистные функции, звездообразные функции, линейный оператор, задача Фекете – Сегё.

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Поступила в редакцию 27.11.2017