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Approximation of Analytic Functions by Universal Vallee-Poussin Sums on the Chebyshev Polynomials

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Abstract. As it is known, Chebyshev polynomials provide the best uniform approach of a function. They are a special case of Faber polynomials. A. I. Shvay (1973) proved that the Vallee-Poussin sums are the best approach apparatus in comparison with the partial sums of the series in the Faber polynomials. Therefore, from the point of view of the best approach, it is natural to consider the approach of functions by means of the Vallee-Poussin sums over Chebyshev polynomials. The study of these sums from any point of view is of definite interest. As O.G. Rovenskaya and O. A. Novikov (2016) note, "during the last decades, the Vallee-Poussin sums and their special cases (Fourier sums and Fejer sums) have been extensively studied by many outstanding specialists in the theory of functions."

The authors (2017) of this article proved a theorem on the summability of a universal series in Chebyshev polynomials. In this paper we find a subsequence of transformed Vallee-Poussin sums satisfying the conditions of this theorem, that is, these sums are a special case of special sums constructed in the theorem of the authors mentioned above. Thus, the above subsequence of the Vallee-Poussin sums has the universality property. With the help of so-called matrix transformations, a generalization of this property is also obtained for these sums, which consists of the following: on the basis of the selected subsequence, sums are constructed, which uniformly approaches any function from a certain class on the specially defined compact sets. Thus, the sums constructed have the property of universality, which many authors have studied over the years for the functional series. In particular, it was studied for the Fourier series, Dirichlet, Faber, Hermite and other series. Then generalizations of this property were studied. For example, W. Luh (1976) generalized the universality property of a power series.

The existence of universal series and their generalizations was proved in various ways, depending on the specifics of the functions under consideration and the applicability of the methods. The first author (1990) developed a method of matrix transformations, which was subsequently used to solve similar problems (1997, 2012, 2013, 2017). The same method is used to prove the main result of this paper. W. Luh used another method.

Keywords: Vallee-Poussin sums, Chebyshev polynomials, universal series, uniform approach.

1. Introduction

The Chebyshev polynomials serve as the apparatus of the best approximation of the function [20, p.363]. As is well known, they are a particular case of the Faber polynomials [21, p. 56], for which the Vallee-Poussin sums in its turn are a better approximation than the partial sums of the series on the Faber polynomials [19].

In this paper, we consider the approximation of the function by special Valle-Poussin sums on Chebyshev polynomials, that is, we proved the universality property of these sums, which was studied in the papers [2; 3; 5-9; 11; 13; 14; 16; 17] for different functional series. Valle-Poussin sums were also studied by many authors, for example, in papers [1; 4; 10; 12; 15; 18; 19; 22].

We will introduce necessary definitions and notations.

The Chebyshev polynomials $T_k(z)$ are defined by the formula

$$T_k(z) = cos(karccosz),$$

where k = 0, 1, 2, ... [20, p. 75].

Let F be a compact set in the complex plane z, where the complement of this set is a domain containing an infinite point.

Denote by $C_A(F)$ the class of functions continuous on F and analytic at each inner point of this set.

Definition 1. *A series on Chebyshev polynomials*

$$\sum_{n=1}^{\infty} a_n T_n(z) \tag{1.1}$$

is called universal if for every set F specified above and any function $f(z) \in C_A(F)$ there is a subsequence $\{S_{n_k}\}_{k=1,2,...}$ of partial sums of the series (1.1), is converging to $f(z)$ uniformly on F.

The Vallee-Poussin polynomials are denoted as

$$v_{\lambda_n,m} = \frac{1}{m} \sum_{\nu=\lambda_n-m}^{\lambda_n-1} S_\nu(z), \ m = 1, 2, ..., \lambda_n,$$

where

$$S_\nu(z) = \sum_{k=0}^{\nu} a_k T_{\lambda_k}(z) \tag{1.2}$$

is the partial sum of a series on Chebyshev polynomials:

$$\sum_{n=0}^{\infty} a_n T_{\lambda_n}(z). \tag{1.3}$$

We define the domain in which the universality property of the studying sums will be executed. Let the function W=Φ(z) maps the exterior of the segment $[-1,+1]$ complex plane z onto the exterior of a certain circle Γ of the radius $\rho < 1$ with the center at the point $w = 0$ of the complex plane w.

Definition 2. *Under the curvilinear angle φ is understood as the domain described by the Jordan arc connecting the point of the circle Γ with the point $w = \infty$, when rotating the plane w by an angle φ around the point w=0 [17].*

Let D be the domain not containing the segment of the actual axis [-1,+1], this domain by mapping W=Φ(z) goes over in the curvilinear angle $2\pi\tau$, where $\tau \in (0,1]$.

Let $\{\lambda_n\}_{n=1,2,...}$ be a subsequence of the natural numbers which has the density

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \tau, \quad 0 < \tau < 1. \tag{1.4}$$

2. Universal Vallee-Poussin sums

We consider an arithmetic average of the Vallee-Poussin sums of the form

$$Q_{\lambda_n}(z) = \frac{1}{\lambda_n} \sum_{m=1}^{\lambda_n} \frac{1}{m} \sum_{\nu=\lambda_n-m}^{\lambda_n-1} S_\nu(z), \tag{2.1}$$

where S_ν(z) is the partial sum (1.2) of the series on Chebyshev polynomials (1.3), and we show that they satisfy the conditions of the theorem from [7]. The universality property of the sum (2.1) follows from this theorem.

Theorem 1. [7] *Let $\{\lambda_n\}_{n=0,1,...}$ be subsequence of the natural numbers which satisfies the condition* (1.4).

Let A=$\{\alpha_{n\nu}\}$ be a lower triangular infinite matrix whose elements satisfy the conditions:

$$\lim_{n\to\infty} \sum_{\nu=1}^{n} \alpha_{n\nu} = 1, \qquad \lim_{n\to\infty} \alpha_{n\nu} = 0 \quad \forall\nu. \tag{2.2}$$

Then there exists a series on Chebyshev polynomials

$$\sum_{n=0}^{\infty} a_n T_{\lambda_n}(z),$$

which has the following property: for each compact set F contained in an arbitrary domain D from the above class of domains, and any function $f \in C_A(F)$, there is a subsequence of the natural numbers $\{\lambda_{m_k}\}_{k=0,1,\dots}$, which depend from F and f, such that

$$S_{\lambda_{m_k}}(z) = \sum_{i=1}^{\lambda_{m_k}} \alpha_{\lambda_{m_k} i} Q_{\lambda_{n_i}}(z), \quad k = 1, 2, ..., \tag{2.3}$$

where

$$Q_{\lambda_{n_i}}(z) = \sum_{n=1}^{n_i} a_n T_{\lambda_n}(z),$$

converges to $f(z)$ uniformly on F.

Note that for sums (2.1) we will have a special case of the matrix whose elements satisfy the condition (2.2). From this, according to theorem 1, there exists a series on the Chebyshev polynomials having such a property that for any compact set F from the above domain D and for each function $f \in C_A(F)$ there is a subsequence of the natural numbers $\{\lambda_{N_k}\}_{k=0,1,\dots}$, which depends from F and f, such that

$$Q_{\lambda_{N_k}}(z) = \frac{1}{\lambda_{N_k}} \sum_{m=1}^{\lambda_{N_k}} \frac{1}{m} \sum_{\nu=\lambda_{N_k}-m}^{\lambda_{N_k}-1} S_\nu(z),$$

where $S_\nu(z)$ is a partial sum (1.2) of the series (1.3) uniformly converging to f(z) on F. Thus, from theorem 1 we obtain universal sums (2.3), which are substantially used in the following theorem 2, representing the main result of this paper.

3. Theorem on approximation of analytic functions by universal Valle-Poussin sums on Chebyshev polynomials

Using sums (2.3), we spread theorem 1 on special Vallee-Poussin sums, that is, for these sums we will obtain an analogue of the theorem 1, differing so that the columns of the considering matrix are taken with certain lacunas, and in theorem 1 they are given without lacunas.

Note that the law of the lacunas of columns is also different from the law of lacunas used in the theorem of the paper [8]. Thus, in this paper the investigating sums differ from the sums considered in the papers [7] and [8] in content and structure, although the consideration of Vallee-Poussin the sums with the same structures as in [7] and [8] also presents the certain interest. This is an analogy of how, for example, the power series were investigated with different lacunas structure (Hadamard, Ostrovski lacunas). This study made it possible to obtain overconverging series and universal series is a special case of overconverging series. In this work, we built the special sums generalizing the partial sums of the series on Chebyshev polynomials and having the property of universality. Namely, the generalization of the universality property of sums (2.3) represents

Theorem 2. *Let the conditions of theorem 1 be satisfied.*
Then there is a Vallee-Poussin sum of the form (2.1), having the following property: for each closed bounded set F contained in the domain D, and any function $f \in C_A(F)$, there exist subsequences of the natural numbers $\{\lambda_{m_k}\}_{k=0,1,...}$ and $\{\lambda_{n_k}\}_{k=0,1,...}$, these subsequences depend from F and f, such that

$$\sigma_{\lambda_{m_k}}(z) = \sum_{i=0}^{m_k} \alpha_{\lambda_{m_k}\lambda_i} S_{\lambda_{n_i}}(z), \ \ k = 0, 1, ..., \tag{3.1}$$

converges to $f(z)$ uniformly on F, where

$$S_{\lambda_{n_i}}(z) = \frac{1}{\lambda_{n_i}} \sum_{m=1}^{\lambda_{n_i}} \frac{1}{m} \sum_{\nu=\lambda_{n_i}-m}^{\lambda_{n_i}-1} R_\nu(z), \ \ R_\nu(z) = \sum_{n=0}^{\nu} a_n T_{\lambda_n}(z).$$

To prove the theorem 2, we need the following

Lemma. *Let the conditions accomplish:*

1) $B = \{\beta_{k\nu}\}$ — a lower triangular infinite matrix whose elements satisfy the conditions of (2.2)*;*

2) $\{\lambda_k\}_{k=1,2,...}$ — a sequence of positive integers which satisfies the condition (1.4)*, $\{\lambda_{k_m}\}_{m=1,2,...}$, $\{\lambda_{\widetilde{k}_m}\}_{m=1,2,...}$ — its subsequences;*

3) we have a universal series

$$\sum_{k=0}^{\infty} b_k T_{\lambda_k}(z). \tag{3.2}$$

Then on compact set F given the above what is in the theorem the function $f \in C_A(F)$ can be uniformly approximated by polynomials of the form

$$\sum_{i=m}^{k_{m+l}} \beta_{\lambda_{k_{m+l}}\lambda_i} \sigma_{\lambda_{\widetilde{k}_i}}(z), \tag{3.3}$$

where

$$\sigma_{\lambda_{\widetilde{k}_i}}(z) = \frac{1}{\lambda_{\widetilde{k}_i}} \sum_{m=1}^{\lambda_{\widetilde{k}_i}} \frac{1}{m} \sum_{\nu=\lambda_{\widetilde{k}_i}-m}^{\lambda_{\widetilde{k}_i}-1} R_\nu(z), \;\; \lambda_{k_{m+l}} \to \infty.$$

Proof. According to defining of the universal series of form (3.2) for each compact F of the above given, arbitrary $\varepsilon > 0$ and any function $f(z)/(\beta_{\lambda_{k_{m+l}}\lambda_i}(k_{m+l} - m + 1)) \in C_A(F)$ we have

$$|\sigma_{\lambda_{\widetilde{k}_i}}(z) - f(z)/(\beta_{\lambda_{k_{m+l}}\lambda_i}(k_{m+l} - m + 1))| < \varepsilon/(|\beta_{\lambda_{k_{m+l}}\lambda_i}|(k_{m+l} - m + 1)) \tag{3.4}$$

at $i = m, m+1, ..., k_{m+l};\; m > N$.

Using (3.4), we obtain

$$|\sum_{i=m}^{k_{m+l}} \beta_{\lambda_{k_{m+l}}\lambda_i}\sigma_{\lambda_{\widetilde{k}_i}}(z) - f(z)| \leq$$

$$\sum_{i=m}^{k_{m+l}} |\beta_{\lambda_{k_{m+l}}\lambda_i}||\sigma_{\lambda_{\widetilde{k}_i}}(z) - f(z)/(\beta_{\lambda_{k_{m+l}}\lambda_i}(k_{m+l} - m + 1)| < \varepsilon.$$

□

Proof of the theorem 2. For the proof we will apply the method of [5].
The set of polynomials of the form (3.3) we will place in the sequence

$$P_0(z), P_1(z), ..., P_m(z), ... \;. \tag{3.5}$$

Due to of the above proved Lemma, it can be uniformly approximated by polynomials (3.5) any function $f(z) \in C_A(F)$, where F is an arbitrary compact contained in the domain of D.

Inside the curvilinear angle $2\pi\tau$ we take a set of all possible closed bounded simply connected domains of the plane w, the boundary of each of these domains consists of the arcs Γ_1 and Γ_2 of the circles $|w| = n+2$ and $|w| = 1 + \frac{1}{n+2}$, $n = 0, 1, \ldots$, with ends at the points with rational polar coordinates and two non-intersecting with each other broken lines γ_1 and γ_2, having vertices at points with rational polar coordinates. We will place these domains in the sequence $D_0, D_1, ..., D_n, ...$. Let

$$F_n = \Phi^{-1}(D_n), \;\; n = 0, 1, 2, \ldots, \tag{3.6}$$

where $w = \Phi(z)$ is a function that maps the exterior of a segment $[-1, +1]$ onto the exterior of a circle $|w| = 1$.

Let $\{\varepsilon_k\}_{k=1,2,...}$ be a monotone decreasing sequence of positive numbers, $\varepsilon_k \to 0$ when $k \to \infty$.

To construct the sum of the form (3.1), uniformly converging on F from (3.6) to the function $f(z) \in C_A(F)$, it is enough to construct the sums

$$\sigma_{\lambda_{m_k^{(l)}}}(z) = \alpha_{\lambda_{m_k^{(l)}}\lambda_0} S_{\lambda_{n_0}}(z) + ... + \alpha_{\lambda_{m_k^{(l)}}\lambda_{m_k^{(l)}}} S_{\lambda_{n_{m_k^{(l)}}}}(z), \tag{3.7}$$

where $k = 0, 1, \ldots;\ l = 0, 1, \ldots, k$ and $\lambda_{m_k^{(l)}} < \lambda_{m_{k'}^{(l')}}$ at $k < k'$ or $k = k'$ and $l < l'$, which satisfy the conditions

$$|P_l(z) - \sigma_{\lambda_{m_k^{(l)}}}(z)| < \varepsilon_k \qquad (l = 0, 1, ..., k) \tag{3.8}$$

at $z \in F_k$ $(k = 0, 1, ...)$.

Indeed, assume that sums (3.7), satisfying the condition (3.8), are constructed. Using the Lemma proved above, for any $\varepsilon > 0$, compact set F from (3.6) and function $f(z) \in C_A(F)$ it is possible to choose in the sequence (3.5) the polynomial $P_l(z)$ such that

$$|f(z) - P_l(z)| < \frac{\varepsilon}{2} \quad \text{at} \quad z \in F.$$

After that, taking k so large in order to $F \subset F_k$ and $\varepsilon_k < \varepsilon/2$, we will have at any $z \in F$

$$|f(z) - \sigma_{\lambda_{m_k^{(l)}}}(z)| < |f(z) - P_l(z)| + |P_l(z) - \sigma_{\lambda_{m_k^{(l)}}}(z)| < \varepsilon_k.$$

The construction of the sum (3.7), satisfying (3.8), can be done in the following way.

Select from the sequence (3.5) the polynomial

$$\sigma_{\lambda_{m_0^{(0)}}}(z) = \alpha_{\lambda_{m_0^{(0)}}\lambda_0} S_{\lambda_{n_0}}(z) + \alpha_{\lambda_{m_0^{(0)}}\lambda_1} S_{\lambda_{n_1}}(z) + ... + \alpha_{\lambda_{m_0^{(0)}}\lambda_{m_0^{(0)}}} S_{\lambda_{n_{m_0^{(0)}}}}(z)$$

such that $|P_0(z) - \sigma_{\lambda_{n_0^{(0)}}}(z)| < \varepsilon_0$ at $z \in F_0$.

Due to above Lemma, take into consideration conditions (2.2), there are polynomials in the sequence (3.5)

$$\alpha_{\lambda_{m_1^{(0)}}\lambda_{\widetilde{m}_0^{(0)}+1}} S_{\lambda_{n_{\widetilde{m}_0^{(0)}+1}}} + ... + \alpha_{\lambda_{m_1^{(0)}}\lambda_{m_1^{(0)}}} S_{\lambda_{n_{m_1^{(0)}}}},$$

$$\alpha_{\lambda_{m_1^{(1)}}\lambda_{\widetilde{m}_1^{(0)}+1}} S_{\lambda_{n_{\widetilde{m}_1^{(0)}+1}}} + ... + \alpha_{\lambda_{m_1^{(1)}}\lambda_{m_1^{(1)}}} S_{\lambda_{n_{m_1^{(1)}}}},$$

$$m_0^{(0)} < \widetilde{m}_0^{(0)} < m_1^{(0)} < \widetilde{m}_1^{(0)} < m_1^{(1)},$$

such that at $z \in F_1$

$$|P_0(z) - \sigma_{\lambda_{m_1^{(0)}}}(z)| < \varepsilon_1,$$

$$|P_1(z) - \sigma_{\lambda_{m_1^{(1)}}}(z)| < \varepsilon_1,$$

where

$$\sigma_{\lambda_{m_1^{(0)}}}(z) = \alpha_{\lambda_{m_1^{(0)}}\lambda_0} S_{\lambda_{n_0}}(z) + ... + \alpha_{\lambda_{m_1^{(0)}}\lambda_{m_0^{(0)}}} S_{\lambda_{n_{m_0^{(0)}}}}(z) + ...$$

$$... + \alpha_{\lambda_{m_1^{(0)}}\lambda_{m_1^{(0)}}} S_{\lambda_{n_{m_1^{(0)}}}}(z),$$

$$\sigma_{\lambda_{m_1^{(1)}}}(z) = \alpha_{\lambda_{m_1^{(1)}}\lambda_0} S_{\lambda_{n_0}}(z) + ... + \alpha_{\lambda_{m_1^{(1)}}\lambda_{m_0^{(0)}}} S_{\lambda_{n_{m_0^{(0)}}}}(z) + ...$$

$$... + \alpha_{\lambda_{m_1^{(1)}}\lambda_{m_1^{(0)}}} S_{\lambda_{n_{m_1^{(0)}}}}(z) + ... + \alpha_{\lambda_{m_1^{(1)}}\lambda_{m_1^{(1)}}} S_{\lambda_{n_{m_1^{(1)}}}}(z).$$

Assuming that the next sums are constructed

$$\sigma_{\lambda_{m_0^{(0)}}}(z), \sigma_{\lambda_{m_1^{(0)}}}(z), \sigma_{\lambda_{m_1^{(1)}}}(z), ..., \sigma_{\lambda_{m_k^{(1)}}}(z), ..., \sigma_{\lambda_{m_k^{(k)}}}(z),$$

satisfying the condition (3.8), we choose from a sequence (3.5) polynomials

$$\alpha_{\lambda_{m_{k+1}^{(0)}}\lambda_{\widetilde{m}_k^{(k)}+1}} S_{\lambda_{n_{\widetilde{m}_k^{(k)}+1}}}(z) + ... + \alpha_{\lambda_{m_{k+1}^{(0)}}\lambda_{m_{k+1}^{(0)}}} S_{\lambda_{n_{m_{k+1}^{(0)}}}}(z),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_{\lambda_{m_{k+1}^{(k+1)}}\lambda_{\widetilde{m}_k^{(k)}+1}} S_{\lambda_{n_{\widetilde{m}_k^{(k)}+1}}}(z) + ... + \alpha_{\lambda_{m_{k+1}^{(k+1)}}\lambda_{m_{k+1}^{(k+1)}}} S_{\lambda_{n_{m_{k+1}^{(k+1)}}}}(z),$$

where $m_k^{(0)} < \widetilde{m}_k^{(0)} < m_{k+1}^{(0)} < \widetilde{m}_{k+1}^{(0)} < m_{k+1}^{(1)} < ... < m_{k+1}^{(k+1)}$,

such that, assuming in (3.7) $k+1$ instead of k, we obtain

$$|P_l(z) - \sigma_{\lambda_{m_{k+1}^{(l)}}}(z)| < \varepsilon_{k+1},$$

$(l = 0, 1, ..., k+1)$ at $z \in F_{k+1}$.

Hence, According to the mathematical induction, the existence of sums (3.7), satisfying (3.8), and hence the existence of sums (3.1) with the necessary approximation property, follows.

□

References

1. Bochkarev S.V. De la Vallee-Poussin Means of Fourier Series for Quadratic Spectrum and Power Density Spectra. *Dokl. Russia Akad. Nauk.*, 2011, vol. 439, no. 3, pp. 298-303. (in Russian) https://doi.org/10.4213/rm9566
2. Chashchina N.S. On the theory of a universal dirichlet series. *Izvestiya VUZ. Matematika*, 1963, no. 4, pp. 165-167. (in Russian)
3. Chui C.K., Parnes M.N. Approximation by overconvergence of a power series. *J. Math. Anal. Appl.*, 1971, vol. 36, no. 3, pp. 693-696. https://doi.org/10.1016/0022-247X(71)90049-7
4. Damen V. On best approximation and Vallee-Poussin sums. *Math. Zametki*, 1978, vol. 23, issue 5, pp. 671-683. (in Russian)

5. Dodunova L.K. A generalization of the universality property of Faber polynomial series. *Izvestiya VUZ. Matematika*, 1990, no. 12, pp. 31-34. (in Russian)
6. Dodunova L.K. Approximation of analytic functions by de la Vallee-Poussin sums. *Izvestiya VUZ. Matematika*, 1997, no. 3, pp. 34-37. (in Russian)
7. Dodunova L.K., Ageikin A.A. Summation of the Universal Series on the Chebyshev Polynomials. *Izvestiya Irkutskogo Gosudarstvennogo Universiteta. Seriya Matematika*, 2017, vol. 21, pp. 51-60. (in Russian) https://doi.org/10.26516/1997-7670.2017.21.51
8. Dodunova L.K., Okhatrina D.D. The generalization of the Universal Series in Chebyshev Polynomials. *Chebyshev sbornic*, 2017, vol. 18, issue 1, pp. 65-72. (in Russian) https://doi.org/10.22405/2226-8383-2017-18-1-65-72
9. Edge J.J. Universal trigonometric series. *J. Math. Anal. Appl.*, 1970, no. 29, pp. 507-511. https://doi.org/10.1016/0022-247X(70)90064-8
10. Efimov A.V. Approximation of periodic functions by de la Vallee-Poussin sums. *Izvestiya Akad. Nauk SSSR, ser. Mat.*, 1959, vol. 23, issue 5, pp. 737-770. (in Russian)
11. Katsoprinakis E., Nestoridis, V., Papadoperakis I. Universal Faber series. *Analysis (Munich)*, 2001, no. 21, pp. 339-363.
12. Korkmasov F.M. Approximation properties of the de la Vallee-Poussin averages for discrete Fourier – Jacobi sums. *Siberian Mathematical Journal*, 2004, vol. 45, no. 2, pp. 334-355. (in Russian)
13. Luh W. Uber den Sats von Mergelyan. *J. Approxim. Theory*, 1976, vol. 16, no. 2, pp. 194-198. https://doi.org/10.1016/0021-9045(76)90048-4
14. Men′shov D.E. On universal trigonometric series. *Dokl. Akad. Nauk SSSR*, 1945, vol. 49, no. 2, pp. 79-82. (in Russian)
15. Rovenskaya O.G., Novikov O.A. Approximation the analytical of periodic functions linear averages of the Fourier series. *Chebyshev sbornic*, 2016, vol. 17, issue 2, pp. 170-183. (in Russian) http://dx.doi.org/10.22405/2226-8383-2016-17-2-170-183
16. Seleznev A.I. On universal power series. *Mat. Sb.*, 1951, vol. 28, no. 2, pp. 453-460. (in Russian)
17. Seleznev A.I., Dodunova L.K. On two theorems of A.F. Leontev on the completeness of subsystems of Faber and Jacobi polynomials. *Izvestiya VUZ. Matematika*, 1982, no. 4, pp. 51-55. (in Russian)
18. Sharapudinov I.I. Approximation properties of the Vallee-Poussin averages of partial sums of a mixed series of Legendre polynomials. *Math. zametki*, 2008, vol. 84, issue 3, pp. 452–471. (in Russian)
19. Shvai A.I. Approximation of analytic functions by polynomials Vallee-Poussin. *Ukrainian Mathematical Journal*, 1973, vol. 25, no. 6, pp. 848–853. (in Russian)
20. Suetin P.K. *Classical orthogonal polynomials.* Moscow, Nauka Publ., 1979, 416 p. (in Russian)
21. Suetin P.K. *Series in Faber polynomials.* Moscow, Nauka Publ., 1984, 336 p. (in Russian)
22. Teljakovskii S.A. Approximation of differentiable functions by de la Vallee-Poussin sums. *Dokl. Akad. Nauk SSSR*, 1958, vol. 121, no. 3, pp. 426-429. (in Russian)

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Приближение аналитических функций универсальными суммами Валле- Пуссена по многочленам Чебышёва

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Аннотация. Как известно, многочлены Чебышёва обеспечивают наилучшее равномерное приближение функции. Они являются частным случаем многочленов Фабера. А. И. Швай (1973) доказал, что суммы Валле-Пуссена являются лучшим аппаратом приближения по сравнению с частичными суммами ряда по многочленам Фабера. Поэтому с точки зрения наилучших приближений естественно рассмотреть приближение функций с помощью сумм Валле-Пуссена по многочленам Чебышёва, хотя изучение этих сумм с любых точек зрения представляет определённый интерес. И как отмечают авторы О. Г. Ровенская и О. А. Новиков (2016), «в течение последних десятилетий суммы Валле-Пуссена и их особые случаи (суммы Фурье и суммы Фейера) интенсивно изучались многими выдающимися специалистами в теории функций».

Авторами (2017) данной статьи доказана теорема о суммируемости универсального ряда по многочленам Чебышёва. В настоящей работе найдена подпоследовательность преобразованных сумм Валле-Пуссена, удовлетворяющая условиям этой теоремы, то есть эти суммы являются частным случаем специальных сумм, построенных в упомянутой теореме авторов. Таким образом, указанная выше подпоследовательность сумм Валле-Пуссена обладает свойством универсальности. С помощью так называемых матричных преобразований получено также обобщение этого свойства для данных сумм, которое заключается в следующем: на основе выделенной подпоследовательности строятся суммы, равномерно приближающие любую функцию из определённого класса на компактных специальным образом определённых множествах. Таким образом, построенные суммы обладают свойством универсальности, которое в разное время изучали многие авторы для функциональных рядов. В частности, для рядов Фурье, Дирихле, Фабера, Эрмита и др. Затем изучались обобщения этого свойства. Например, W. Luh (1976) обобщил свойство универсальности степенного ряда.

Существование универсальных рядов и их обобщений доказывалось разными способами в зависимости от специфики рассматриваемых функций и применимости методов. Первым автором (1990) разработан метод матричных преобразований, который в дальнейшем применялся при решении подобных задач (1997, 2012, 2013, 2017). Этот же метод используется при доказательстве основного результата данной работы. W. Luh использовал другой метод.

Ключевые слова: суммы Валле-Пуссена, многочлены Чебышёва, универсальный ряд, равномерная сходимость.

Список литературы

1. Бочкарев С. В. Средние Валле-Пуссена рядов Фурье для квадратичного спектра и спектров степенной плотности // Докл. РАН. 2011. Т. 439, № 3. С. 298–303. https://doi.org/10.4213/rm9566
2. Чащина Н. С. К теории универсального ряда Дирихле // Матем. сб. 1963. № 4. С. 165–167.
3. Chui C. K., Parnes M. N. Approximation by overconvergence of a power series // J. Math. Anal. Appl. 1971. Vol. 36, N 3. P. 693–696. https://doi.org/10.1016/0022-247X(71)90049-7
4. Дамен В. О наилучшем приближении и суммах Валле-Пуссена // Мат. заметки. 1978. Т. 23, вып. 5. С. 671–683.
5. Додунова Л. К. Об одном обобщении свойства универсальности рядов по многочленам Фабера // Изв. вузов. Математика. 1990. № 12. С. 31–34.
6. Додунова Л. К. Приближение аналитических функций суммами Валле-Пуссена // Изв. вузов. Математика. 1997. № 3. С. 34–37.
7. Додунова Л. К., Агейкин А. А. Суммирование универсальных рядов по многочленам Чебышёва // Изв. Иркут. гос. ун-та. Сер. Математика. 2017. Т. 21. С. 51–60. https://doi.org/10.26516/1997-7670.2017.21.51
8. Додунова Л. К., Охатрина Д. Д. Обобщение универсального ряда по многочленам Чебышёва // Чебышев. сб. 2017. Т. 18, вып. 1. С. 65–72. https://doi.org/10.22405/2226-8383-2017-18-1-65-72
9. Edge J. J. Universal trigonometric series // J. Math. Anal. Appl. 1970. N 29. P. 507–511. https://doi.org/10.1016/0022-247X(70)90064-8
10. Ефимов А. В. О приближении периодических функций суммами Валле Пуссена // Изв. АН СССР. Сер. мат. 1959. Т. 23, вып. 5. С. 737–770.
11. Katsoprinakis E., Nestoridis V., Papadoperakis I. Universal Faber series // Analysis (Munich). 2001. N 21. P. 339–363.
12. Коркмасов Ф. М. Аппроксимативные свойства средних Валле-Пуссена для дискретных сумм Фурье – Якоби // Сиб. мат. журн. 2004. Т. 45, № 2. С. 334–355.
13. Luh W. Uber den Sats von Mergelyan // J. Approxim. Theory. 1976. Vol. l6, N 2. P. 194-–198. https://doi.org/10.1016/0021-9045(76)90048-4
14. Меньшов Д. Е. Об универсальных тригонометрических рядах // Доклады АН СССР. 1945. Т. 49, № 2. С. 79–82.
15. Ровенская О. Г., Новиков О. А. Приближение аналитических периодических функций линейными средними рядов Фурье // Чебышев. сб. 2016. Т. 17, вып. 2. С. 170–183. http://dx.doi.org/10.22405/2226-8383-2016-17-2-170-183
16. Селезнёв А.И. Об универсальных степенных рядах // Мат. сб. 1951. Т. 28, № 2. С. 453–460.
17. Селезнёв А. И., Додунова Л. К. К двум теоремам А. Ф. Леонтьева о полноте подсистем полиномов Фабера и Якоби // Изв. вузов. Математика. 1982. № 4. С. 51–55.
18. Шарапудинов И. И. Аппроксимативные свойства средних типа Валле-Пуссена частичных сумм смешанного ряда по полиномам Лежандра // Мат. заметки. 2008. Т. 84, вып. 3. С. 452–471.
19. Швай А. И. Приближение аналитических функций многочленами Валле-Пуссена // Украин. мат. журн. 1973. Т. 25, № 6. С. 848–853.
20. Суетин П. К. Классические ортогональные многочлены. Изд. 2-е, доп. М. : Наука, 1979. 416 с.
21. Суетин П. К. Ряды по многочленам Фабера. М. : Наука, 1984. 336 с.

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2018. Т. 24. Серия «Математика». С. 12–23

22. Теляковский С. А. Приближение дифференцируемых функций суммами Валле-Пуссена // Докл. АН СССР. 1958. Т. 121, № 3. С. 426–429.

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