Серия «Математика»
2016. Т. 15. С. 38—49

Онлайн-доступ к журналу:
http://isu.ru/izvestia

И З В Е С Т И Я
Иркутского
государственного
университета

УДК 519.853
MSC 90C26

Global search method for solving Malfatti's four-circle problem

R. Enkhbat
Institute of Mathematics, National University of Mongolia

M. Barkova
Matrosov Institute for System Dynamics and Control Theory, SB of RAS

Abstract. We consider Malfatti's problem formulated 200 years ago. In the beginning, Malfatti's problem was supposed to be solved in a geometric construction way. In 1994, it was done by Zalgaller and Los for the original Malfatti's problem using so-called greedy algorithm. There is still a conjecture about solving Malfatti's problem for more than four circles by the greedy algorithm. We generalize Malfatti's problem formulated for the case of three circles inscribed in a triangle for four circles. We examine six cases for inscribed circles in a triangle. The problem has been formulated as the convex maximization problem over a nonconvex set. Global optimality conditions by Strekalovsky have been applied to this problem. For solving numerically Malfatti's problem, we propose an algorithm which converges globally. Subproblems of the proposed algorithm were quadratic programming problems with quadratic constraints. These problem can be solved by Lagrangian methods. For a computational purpose, we consider a triangle with given vertices. Some computational results are provided.

Keywords: Malfatti's problem, triangle set, circle, global optimization, algorithm, optimality conditions.

1. Introduction

In 1803 Italian mathematician Malfatti posed the following problem: how to pack three non-overlapping circles of maximum total area in a given triangle? We call this problem as three-circle problem.

Malfatti originally assumed that the solution to this problem are three circles inscribed in a triangle such that each circle tangent to other two and touches two sides of the triangle.

Now it is well known that Malfatti's solution is not optimal. There are works devoted to solving Malfatti's problem[2, 5-12]. The most common methods used for finding the best solutions to Malfatti's problem were algebraic and geometric approaches. In 1994 Zalgaller and Los [6] showed that the greedy arrangement is the best one. Based on trigonometric equations and inequalities, using so called rigit systems they did attempt to find the best solution to Malfatti's problem. In this paper, we generalize Malfatti's problem for the case of four circles inscribed in a triangle. The paper is organized as follows. In section 2, we formulate Malfatti's problem as the convex maximization problem. Global optimality conditions for Malfatti's problem are given in Section 3. In section 4 computational results are provided.

2. Malfatti's Problem and Convex Maximization

In order to generalize Malfatti's problem as an optimization problem, we need to do following steps.

First, we equivalently formulate the problem in terms of convex sets such as a circle and a triangle set. Secondly, we characterize inscribed conditions of circles into a triangle set. For this purpose, we introduce the following sets. Denote by $B(x,z)$ a circle with a center $x \in \mathbb{R}^2$ and a radius $z \in R$:

$$B(x,z) = \{y \in \mathbb{R}^2 | \|y - x\| \leq z\}. \tag{2.1}$$

A bounded and closed triangle set $D \subset \mathbb{R}^2$ is given by

$$D = \{x \in \mathbb{R}^2 | \langle a^i, x\rangle \leq b_i, a^i \in \mathbb{R}^2, b_i \in \mathbb{R}, i = 1,2,3\}, \tag{2.2}$$

here $\langle , \rangle$ denotes the scalar product of two vectors in $\mathbb{R}^2$, and $\|\cdot\|$ is Euclidean norm, $a^i \nparallel a^j,\ i \neq j;\ i,j = 1,2,3.$

Theorem 1. *$B(x,z) \subset D$ if and only if*

$$\langle a^i, x\rangle + z\|a^i\| \leq b_i, i = 1,2,3. \tag{2.3}$$

Proof.
Necessity. Let $y \in B(x,z)$ and $y \in D$. A point $y \in B(x,z)$ can be easily presented as $y = x + zh, h \in \mathbb{R}^2, \|h\| \leq 1$. Condition $y \in D$ follows that $\langle a^i, y\rangle \leq b_i, i = 1,2,3$ or equivalently, $\langle a^i, x\rangle + z\langle a^i, h\rangle \leq b_i, i = 1,2,3, \quad \forall h \in \mathbb{R}^2$. Hence, we have
$\langle a^i, x\rangle + z \max_{\|h\| \leq 1} \langle a^i, h\rangle \leq b_i, i = 1,2,3$
or
$\langle a^i, x\rangle + z\langle a^i, \frac{a^i}{\|a^i\|}\rangle \leq b_i, i = 1,2,3,$
which yield

$\langle a^i, x\rangle + z\|a^i\| \le b_i, i = 1, 2, 3.$

Sufficiency. Let condition (2.3) be held and on the contrary, assume that there exists $\tilde{y} \in B(x, z)$ such that $\tilde{y} \notin D$. Clearly, there exists $\tilde{h} \in \mathbb{R}^2$ so that $\tilde{y} = x + z\tilde{h}, \|\tilde{h}\| \le 1$. Since $\tilde{y} \notin D$, there exists $j \in \{1, 2, 3\}$ for which $\langle a^j, \tilde{y}\rangle > b_j$ or $\langle a^j, x + z\tilde{h}\rangle = \langle a^j, x\rangle + z\langle a^j, \tilde{h}\rangle > b_j$.
On the other hand, we have $\langle a^j, x\rangle + z\|a^j\| > b_j$ which contradicts (2.3). □

Now we formulate inscribed conditions of four circles into a triangle set. There are 6 main cases:
Case 1. Four circles are mutually tangent to each other.
Case 2. Three circles are mutually tangent to each other and fourth circle is tangent to two of them.
Case 3. Three circles are mutually tangent to each other and fourth circle is tangent to one of them.
Case 4. One of the circles is tangent to two of them while fourth circle is tangent to one of them.
Case 5. In addition to case 4, centers of 4 circles lie on the same line.
Case 6. In addition to case 4, centers of only 3 circles lie on the same line.

Denote by $c^1(x_1, x_2)$, $c^2(x_3, x_4)$, $c^3(x_5, x_6)$ and $c^4(x_7, x_8)$ centers of four circles inscribed in a triangle set D given by (2.2). Let r_1, r_2, r_3 and r_4 be their corresponding radii.
Now we are ready to formulate Malfatti's generalized problem for Case 1.

$$\max f = \pi(r_1^2 + r_2^2 + r_3^2 + r_4^2), \tag{2.4}$$
$$\langle a^i, c^j\rangle + r_j\|a^i\| \le b_i,\ i = 1, 2, 3;\ j = 1, 2, 3, 4; \tag{2.5}$$
$$\|c^i - c^j\|^2 = (r_i + r_j)^2,\ i \ne j; i, j = 1, 2, 3, 4; \tag{2.6}$$
$$r_1 \ge 0, r_2 \ge 0, r_3 \ge 0, r_4 \ge 0. \tag{2.7}$$

The function f in (2.4) denotes a total area of the four circles. Conditions (2.5) – (2.7) characterize inscribed conditions of four circles into a triangle set while conditions (2.6) correspond to Case 1. We can easily see that conditions (2.5),(2.7) and (2.6)for $i = 2; j = 4$ with

$$\|c^4 - c^3\|^2 \ge (r_3 + r_4)^2, \tag{2.8}$$

describe Case 2. Case 3 is defined by conditions (2.5),(2.7) and (2.6) for $i = 2; j = 4$ with

$$\|c^4 - c^3\|^2 \ge (r_3 + r_4)^2,$$

$$\|c^2 - c^4\|^2 \ge (r_2 + r_4)^2.$$

Case 4 is defined by conditions (2.5),(2.7) and (2.6) for $i = 1; j = 3$ with

$$\|c^3 - c^1\|^2 \geq (r_1 + r_3)^2,$$
$$\|c^4 - c^2\|^2 \geq (r_4 + r_2)^2,$$
$$\|c^4 - c^1\|^2 \geq (r_4 + r_1)^2.$$

Case 5 is defined by conditions (2.5),(2.7) and (2.6) for $i = 2; j = 3, 4$ with

$$\|c^3 - c^1\|^2 = (r_1 + 2r_2 + r_3)^2,$$
$$\|c^4 - c^2\|^2 = (r_2 + 2r_3 + r_4)^2,$$
$$\|c^4 - c^1\|^2 = (r_1 + 2r_2 + 2r_3 + r_4)^2.$$

Case 6 is defined by conditions (2.5),(2.7) and (2.6) for $i = 2; j = 4$ with

$$\|c^4 - c^1\|^2 \geq (r_4 + r_1)^2,$$
$$\|c^4 - c^2\|^2 \geq (r_4 + r_2)^2,$$
$$\|c^4 - c^3\|^2 \geq (r_4 + r_3)^2.$$

Denote by $S_i, \ \ i = \overline{1, 6}$ sets defined by conditions corresponding to cases 1-6, respectively. The sets $S_i, \ \ i = \overline{1, 6}$ are nonconvex compact sets. Thus, problem (2.4) – (2.7) becomes the convex maximization problem over a nonconvex set. A stationary point of this problem satisfies a system of 33 equations and inequalities with 24 variables including Lagrange multipliers.

3. Global Optimality Conditions and Algorithm

In previous section, we note that the solution to Malfatti's problem is: $f^* = \max\{\max\limits_{x\in S_1} f, \max\limits_{x\in S_2} f, \max\limits_{x\in S_3} f, \max\limits_{x\in S_4} f, \max\limits_{x\in S_5} f, \max\limits_{x\in S_6} f\}$.

Let us consider again these problems

$$\max_{x\in S_i} f, S_i \subset \mathbb{R}^{12}, \tag{3.1}$$

where $x = (x_1, x_2, \ldots, x_8, r_1, r_2, r_3, r_4) \in S_i, \ i = \overline{1, 6}$.

Problems (3.1) belong to a class of concave programming or equivalently, convex maximization problem.

Global Optimality conditions for the convex maximization problem first formulated by Strekalovsky, A.S. in 1987 [4]. Now we apply this result to problem (3.1) which is the following:

Theorem 2. *[4] Let $z \in S_i$ satisfy $f'(z) \neq 0$, then z is a solution to problem* (3.1) *if and only if*
$\langle f'(y), x - y\rangle \leq 0$ *for all* $y \in E_{f(z)}(f)$ *and* $x \in S_i$,

where $E_c(f) = \{y \in \mathbb{R}^{12} | f(y) = c\}$ is the level set of f at c and $f'(y)$ is the gradient of f at y.

Before presenting an algorithm for solving problem (3.1) it is useful to restate Theorem 2 in a convenient way via the function $\Theta(z)$ defined for $z \in S_i$:

$$\Theta(z) = \max_{y \in E_{f(z)}(f)} \Pi(y),$$

where $\Pi(y) = \max_{x \in S_i} \langle f'(y), x - y \rangle$.
It has been shown in [1] that the function $\Pi(y)$ is continuous and differentiable in directions. Since f is strongly convex, the set $E_{f(z)}(f)$ is compact. Thus, $\Theta(z) < +\infty$. We note that $\Pi(y) \leq \Theta(z)$ for all $y \in E_{f(z)}(f)$.

Theorem 3. *Let $z \in S_i$ satisfy $f'(z) \neq 0$, if $\Theta(z) = 0$ then z is a global solution to problem* (3.1).

Proof. follows from the following inequalities:
$\langle f'(y), x - y \rangle \leq \Pi(y) = \max_{x \in S_i} \langle f'(y), x - y \rangle \leq \Theta(z) = 0$
which hold for all $x \in S_i$ and $y \in E_{f(z)}(f)$. □

Now we apply the Algorithm MAX in [1] to solve problem (3.1) numerically.

Algorithm MAX

Step 1. Choose a point $x^0 \in S_i$ such that $f'(x^0) \neq 0$. Set $k := 0$.
Step 2. Solve the problem $\max_{y \in E_{f(x^k)}(f)} \Pi(y)$.
Let y^k be a solution to this problem, i.e.

$$\Pi(y^k) = \max_{y \in E_{f(x^k)}(f)} \max_{x \in D} \langle f'(y), x - y \rangle.$$

Let $\Theta(x^k) := \Pi(y^k)$, and let x^{k+1} be a solution satisfying

$$\Pi(y^k) = \langle f'(y^k), x^{k+1} - y^k \rangle.$$

Step 3. If $\Theta(x^k) = 0$ then stop, and x^k is a global solution.
Otherwise, set $k = k + 1$ and return to **Step 2**.

The convergence of the Algorithm is given by the following theorem.

Theorem 4. *[1] The sequence$\{x^k, k = 1, 2, \dots\}$ generated by Algrorithm MAX is a maximizing sequence for problem* (3.1)*,that is,*

$$\lim_{k \to \infty} f(x^k) = \max_{x \in S_i} f(x),$$

and every accumulation point of the sequence $\{x^k, k = 1, 2, \dots\}$ *is a global maximizer of the problem.*

4. Computational Results

Algorithm MAX starts with an arbitrary local maximizer x^k found by *fmincon* in Matlab. Note that in numerical experiments we solved subproblem $\max\limits_{y \in E_{f(x^k)}(f)} \Pi(y)$ at each iterations $k = 1, 2, \dots,$ as problems with the single equality constraint by the set covering method [3] while problems $\max\limits_{x \in S_i} \langle f'(y^k), x - y^k \rangle$ have been solved by Lagrangian method. For a test purpose, the triangle with vertices $A(0,0)$,$B(3,4)$ and $C(8,6)$ has been considered. As we can see in Section 2 that solving Malfatti's problem consisted of six main cases. Then this problem is the following for Case 1:

$$\max f = \pi(r_1^2 + r_2^2 + r_3^2 + r_4^2) \tag{4.1}$$

$$\begin{gathered}
-4x_1 + 3x_2 + 5r_1 \le 0, \\
3x_1 - 4x_2 + 5r_1 \le 0, \\
-2x_1 + 5x_2 + \sqrt{29} r_1 \le 14, \\
-4x_3 + 3x_4 + 5r_2 \le 0, \\
3x_3 - 4x_4 + 5r_2 \le 0, \\
-2x_3 + 5x_4 + \sqrt{29} r_2 \le 14, \\
-4x_5 + 3x_6 + 5r_3 \le 0, \\
3x_5 - 4x_6 + 5r_3 \le 0, \\
-2x_5 + 5x_6 + \sqrt{29} r_3 \le 14, \\
-4x_7 + 3x_8 + 5r_4 \le 0, \\
3x_7 - 4x_8 + 5r_4 \le 0, \\
-2x_7 + 5x_8 + \sqrt{29} r_4 \le 14, \\
(x_3 - x_1)^2 + (x_4 - x_2)^2 - (r_1 + r_2)^2 = 0, \\
(x_5 - x_1)^2 + (x_6 - x_2)^2 - (r_1 + r_3)^2 = 0, \\
(x_5 - x_3)^2 + (x_6 - x_4)^2 - (r_2 + r_3)^2 = 0, \\
(x_7 - x_1)^2 + (x_8 - x_2)^2 - (r_1 + r_4)^2 = 0, \\
(x_7 - x_3)^2 + (x_8 - x_4)^2 - (r_2 + r_4)^2 = 0, \\
(x_7 - x_5)^2 + (x_8 - x_6)^2 - (r_3 + r_4)^2 = 0, \\
r_1 \ge 0, r_2 \ge 0, r_3 \ge 0, r_4 \ge 0.
\end{gathered} \tag{4.2}$$

Similarly, we solve the problems:

$$\max f = \pi(r_1^2 + r_2^2 + r_3^2 + r_4^2),$$

subject to $x \in S_i \subset \mathbb{R}^{12}, \quad i = \overline{2,6}$.

The performance of the proposed algorithm was tested on six cases of Malfatti's problem. The programming code for the algorithm was written in Matlab and run on a computer Pentium Core 2. The results are given for each case in Table 1.

Table 1

Numerical results

Case	lnitial local value	Global value	Computational time(min:sec)
Case 1	1.7727	2.4584	2:28
Case 2	1.8837	2.8062	2:06
Case 3	1.7538	3.2650	1:49
Case 4	2.1941	3.7104	2:19
Case 5	2.1083	3.0629	2:58
Case 6	2.9544	3.6689	1:55

The global solution to Malfatti's problem (4.1) – (4.2) corresponding to Case 4 was $f^* = 3.7104$ and the centers of the circles were:
$(x_1^*, x_2^*) = (5.3057, 4.4858)$,
$(x_3^*, x_4^*) = (4.4925, 4.0288)$,
$(x_5^*, x_6^*) = (3.4339, 3.4339)$,
$(x_7^*, x_8^*) = (2.5830, 2.5830)$.

During computational process, local and stationary points were examined by Algorithm MAX. Geometric pictures showing global solutions of six cases are given in Figures 1–6.

Conclusion

200 years old Malfatti's problem has been generalized for four-circle case. The problem was reduced to the convex maximization problem with 12 variables. The global optimality conditions by Strekalovsky as well as a global search algorithm in [1] have been applied to the problem. Computational experiment was done for a given triangle.

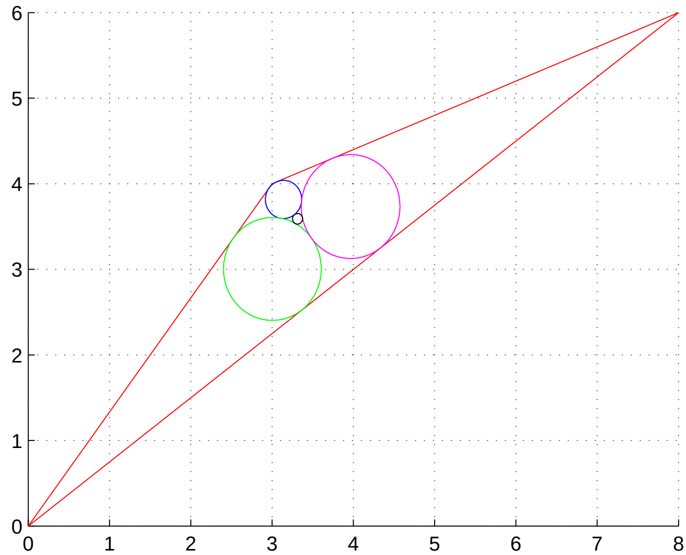

Figure 1. Case 1

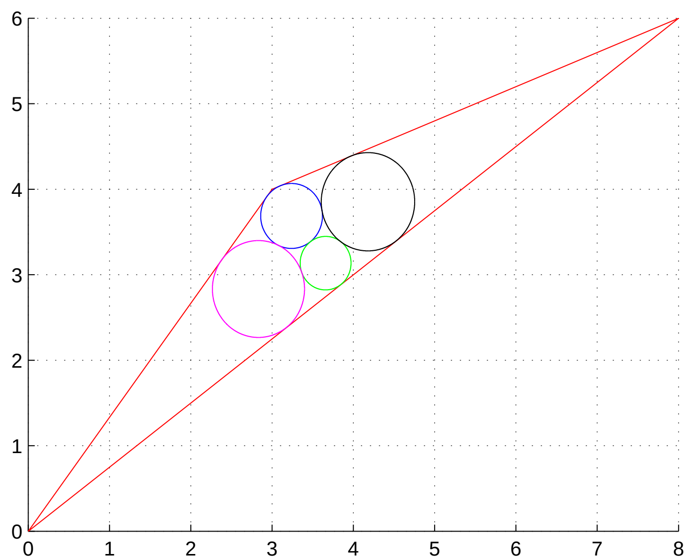

Figure 2. Case 2

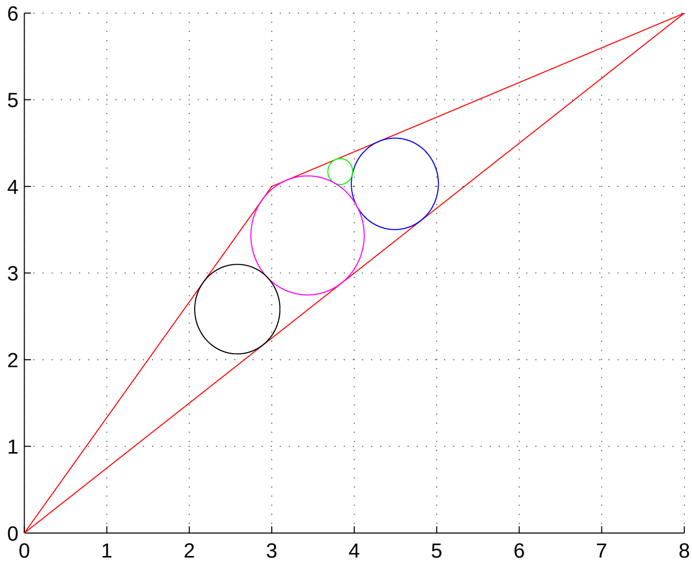

Figure 3. Case 3

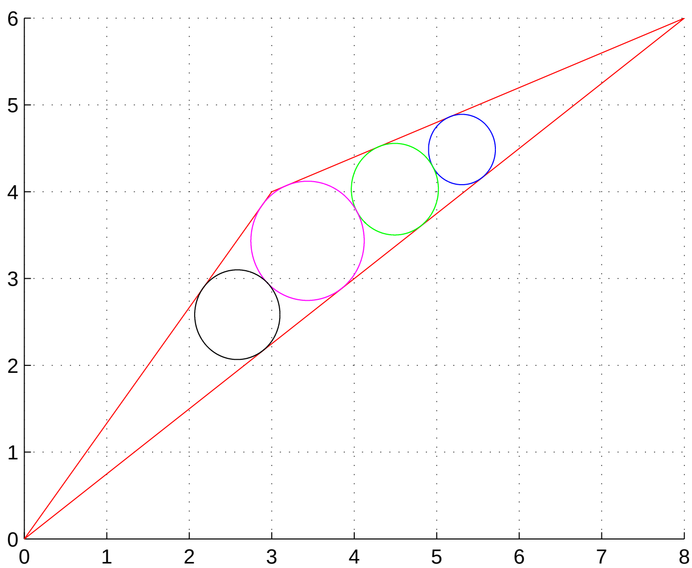

Figure 4. Case 4

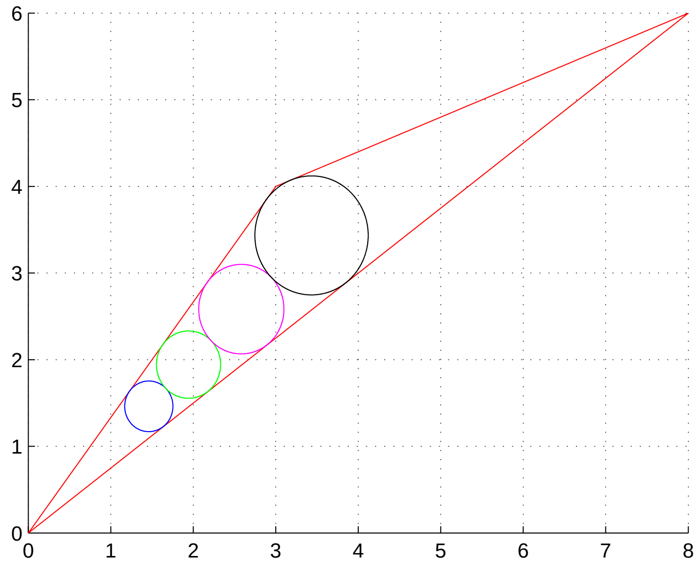

Figure 5. Case 5

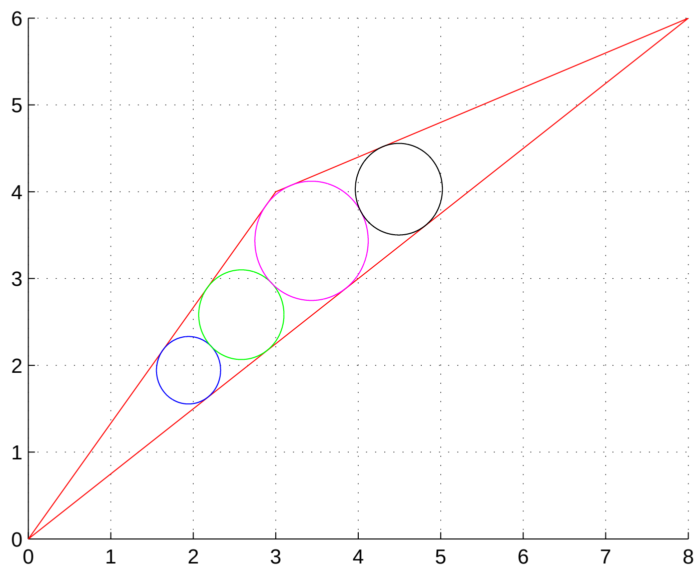

Figure 6. Case 6

References

1. Enkhbat R. An algorithm for maximizing a convex function over a simple set. *Journal of Global Optimization*, 1996, vol.8, pp. 379-391.
2. Marco Andreatta, Andrs Bezdek and Jan P. Boroski, The Problem of Malfatti: Two Centuries of Debate. *The Mathematical Intelligencer*, 2011, vol. 33, issue 1, pp. 72-76.
3. V.N. Nefedov, Finding the Global Maximum of a Function of Several Variables on a Set Given by Inequality Constraints. *Journal of Numerical Mathematics and Mathematical Physics*, 1987, vol. 27(1), pp. 35-51.
4. Strekalovsky A.S. On the global extrema problem. *Soviet Math. Doklad*, 1987, vol. 292(5), pp. 1062-1066.
5. V.A. Zalgaller, An inequality for acute triangles. *Ukr. Geom. Sb.*, 1991, vol. 34, pp. 10-25.
6. V.A. Zalgaller, The solution of Malfatti's problem. *Journal of Mathematical Sciences*, 1994, vol.72, no 4, pp. 3163-3177.
7. G.A. Los, Malfatti's Optimization Problem. [in Russian]. Dep. Ukr. NIINTI, July 5, 1988.
8. Saaty T. Integer Optimization Methods and Related Extremal Problems [Russian translation]. Moscow, Nauka, 1973.
9. Gabai H., Liban E. On Goldberg's inequality associated with the Malfatti problem. *Math. Mag.*, 1967, vol. 41, no 5, pp. 251-252.
10. Goldberg M. On the original Malfatti problem. *Math. Mag.*, 1967, vol. 40, no 5, pp. 241-247.
11. H. Lob and H. W. Richmond, On the solutions of the Malfatti problem for a triangle. *Proc. London Math. Soc.*, 1930, vol. 2, no 30, pp. 287-301.
12. C. Malfatti, Memoria sopra una problema stereotomico. *Memoria di Matematica e di Fisica della Societa italiana della Scienze*, 1803, vol. 10, no 1, pp. 235-244.

Rentsen Enkhbat, Dr. Sc., Professor, Director of Institute of Mathematics, National University of Mongolia, 4, Baga toiruu, Sukhbaatar district, Ulaanbaatar, Mongolia, tel.: 976-99278403
(e-mail: renkhbat46@yahoo.com)

Barkova Maria, Matrosov Institute for System Dynamics and Control Theory, SB of RAS, 134, Lermontov st., Irkutsk, 664033, tel.: (3952)453082
(e-mail: mariabarkovamail@gmail.com)

Р. Энхбат, М. Баркова

Метод глобального поиска для задачи Мальфатти: случай четырёх кругов

Аннотация. Рассматривается задача Мальфатти, сформулированная 200 лет назад. Изначально, решение этой задачи предполагалось геометрическим методом. В 1994 году Залгаллер и Лос предложили для решения так называемый жадный алгоритм. Однако до сих пор не известна оптимальность этого алгоритма для задачи с $n \geq 4$ кругами. В статье обобщается задача Мальфатти, сформулированная для трёх кругов, на случай четырёх кругов. Были исследованы шесть возможных вариантов вписания кругов в треугольник. Исследуемая задача формулируется как задача максимизации выпуклой функции на невыпуклом множестве, для решения которой используются условия глобальной оптимальности А. С. Стрекаловского. Предложен алгоритм для численного решения задачи Мальфатти, который сходится к глобальному решению. Вспомогательными задачами предложенного алгоритма являются задачи квадратичного программирования с квадратичными ограничениями. Эти задачи могут быть решены методом Лагранжа. Для проведения вычислительного эксперимента был рассмотрен треугольник с заданными вершинами. В работе приводятся численные результаты для заданного треугольника.

Ключевые слова: Задача Мальфатти, треугольник, круг, глобальная оптимизация, алгоритм, условия оптимальности.

Рэнцэн Энхбат, доктор физико-математических наук, профессор, директор Института математики, Национальный университет Монголии, Монголия, Округ Сухэ-Батора, Улан-Батор, ул. Бага Тойру, 4, тел.: 976-99278403 (e-mail: renkhbat46@yahoo.com)

Баркова Мария, Институт динамики систем и теории управления имени В. М. Матросова СО РАН, 664033, Иркутск, ул. Лермонтова, 134, тел.: (3952)453082 (e-mail: mariabarkovamail@gmail.com)