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The Collection Formula and Regular p-Groups

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Abstract: For any elements x, y of a group G and for any positive integer n, we obtain the collection formula for the n-th power of the product xy when all commutators in G that have more than 3 occurrences of the element y are equal to 1. Suppose $n = p$ is a prime number, and any commutator weighting more than $p - 1$ has an order 1 or p. In terms of the segment of the formula including only commutators of weight p with the exponents modulo p, we provide a necessary condition for the regularity of a finite p-group. This result generalizes known necessary conditions that were used in proving the non-regularity of some matrix p-groups.

Keywords: P. Hall's collection formula, collection process, commutator, regular p-group

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Научная статья

Собирательная формула и регулярные p-группы

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Аннотация: Для любых элементов x, y группы G и для любого целого положительного числа n выводится собирательная формула для n-й степени произведения xy,

когда все коммутаторы в G, имеющие более 3 вхождений элемента y, равны 1. Пусть $n = p$ — простое число, и любой коммутатор веса больше $p-1$ имеет порядок 1 или p. В терминах отрезка данной формулы, включающего только коммутаторы веса p с показателями степеней по модулю p, получено необходимое условие регулярности конечной p-группы. Этот результат обобщает известные необходимые условия, использованные при доказательстве нерегулярности некоторых матричных p-групп.

Ключевые слова: собирательная формула Ф. Холла, собирательный процесс, коммутатор, регулярная p-группа

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1. Introduction

Suppose G is a group, $x, y \in G$. In [2, Theorem 3.1], P. Hall obtained the following collection formula:

$$(xy)^n \equiv x^n y^n q_3^{e_3} \dots q_{j(s)}^{e_{j(s)}} \pmod{\Gamma_s(G)}, \quad s \geqslant 2, \tag{1.1}$$

where $q_1 = x, q_2 = y, q_3, \dots, q_{j(s)}$ are commutators in x, y arranged in order of increasing weights; $\Gamma_s(G)$ is the s-th term of the lower central series of G, which is defined as follows: $\Gamma_1(G) = G$, $\Gamma_k(G) = [\Gamma_{k-1}(G), G]$, $k \geqslant 2$ (see [13, sections 10.2, 11.1]); for $H \lhd G$ and $a, b \in G$, we write $a \equiv b \pmod{H}$ iff $a = bc$ for some $c \in H$. It was proved [2, formula (3.21)] that the exponents e_i of the commutators can be expressed as integer-valued polynomials in n:

$$e_i(n) = \sum_{k=1}^{w(q_i)} a_k \binom{n}{k}, \tag{1.2}$$

where $w(q_i)$ is the weight of q_i, and non-negative integers a_k do not depend on n. In this paper we use the following notation for commutators: $[y, x] = y^{-1}x^{-1}yx$, $[y, {}_0x] = y$; $[y, {}_ix] = [[y, {}_{i-1}x], x]$, $i \geqslant 1$; $[y, x, z] = [[y, x], z]$.

Based on formula (1.1), P. Hall introduced a generalization of Abelian p-groups. A finite p-group G is called *regular* if, for any $x, y \in G$ and for any $\alpha \in \mathbb{N}_0$, there exist $R_3, \dots, R_t \in \Gamma_2(\langle x, y\rangle)$ such that

$$(xy)^{p^\alpha} = x^{p^\alpha} y^{p^\alpha} R_3^{p^\alpha} \dots R_t^{p^\alpha}. \tag{1.3}$$

From the collection formula (1.1) and expression (1.2) for $n = p^\alpha$ it follows that every finite p-group of nilpotency class less than p is regular since every binomial coefficient in (1.2) will be divisible by p^α.

In various works, there is a need for information (explicit form, divisibility properties, etc) about the exponents of the commutators in (1.1). For example, the exponent $\binom{n}{i+1}$ of the commutator $[y, {}_i x]$, $i \geqslant 1$, was used to prove the $(p-1)$-th Engel congruence for a group of prime exponent p, which was the key to investigation of the restricted Burnside problem for groups of prime exponent [13, p. 327]. With use of the exponents for more complex commutators, E. F. Krause proved the 14-th Engel congruence for groups of exponent 8 in [7; 8]. Also, explicit collection formulas (1.1), for metabelian and transmetabelian groups, are considered by A. I. Skopin and Yu. G. Teterin in the works [14; 16–18] and used, for example, to find the nilpotency class and estimate the order of groups of Burnside type [3; 15].

In [6, Corollary 1], the following necessary condition for the regularity of a finite p-group was obtained in terms of the segment of formula (1.1). Let G be a regular p-group, $p > 2$, and $x, y \in G$. Suppose that every commutator in x and y: 1) that has more than 2 occurrences of y, equals 1; 2) weighting more than $p-1$, has an order 1 or p. Then there exists an element $d \in \Gamma_2(\langle x, y\rangle)$ such that

$$d^p \equiv [y, {}_{p-1}x]\,[y, {}_{p-2}x, y]^{-1} \prod_{v=1}^{(p-3)/2} [[y, {}_{p-2-v}x], [y, {}_v x]]^{(-1)^{v+1}} \pmod{\Gamma_{p+1}(G)}.$$

This necessary condition was used to prove the non-regularity of the Sylow p-subgroup of the general linear group $GL_n(\mathbb{Z}_{p^m})$ for $n \geqslant (p+2)/3$, $m \geqslant 3$ when $(p+2)/3$ is an integer [6, Theorem 2]. This led to a partial solution to Wehrfritz's problem in the Kourovka Notebook [19, Question 8.3].

A condition, stronger than 1), assuming that every commutator with more than 1 occurrence of y equals 1, yields the congruence

$$d^p \equiv [y, {}_{p-1}x] \pmod{\Gamma_{p+1}(G)}.$$

This fact was used in solving Wehrfritz's problem for $m = 1, 2$ [4] and in solving its analogue for Chevalley groups over $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$ [1, Lemma 7].

The aim of this paper is to prove the following necessary condition for the regularity of a finite p-group, which generalizes the above.

Theorem 1. *Let G be a regular p-group, $p > 2$, and $x, y \in G$. Suppose that every commutator in x and y:*

1) that has more than 3 occurrences of y, equals 1;

2) weighting more than $p-1$, has an order 1 or p.

Then there exists an element $d \in \Gamma_2(\langle x, y\rangle)$ such that

$$d^p \equiv [y, {}_{p-1}x]\,[y, {}_{p-2}x, y]^{-1}[y, {}_{p-3}x, {}_2y] \prod_{v=1}^{(p-3)/2} [[y, {}_{p-2-v}x], [y, {}_vx]]^{(-1)^{v+1}}$$
$$\prod_{v=1}^{p-4}[[y, {}_{p-3-v}x, y], [y, {}_vx]]^{\alpha_v} \prod_{v=1}^{[(p-4)/3]} [[y, {}_{p-3-2v}x], {}_2[y, {}_vx]]^{\beta_v}$$
$$\prod_{v=1}^{[(p-5)/3]} \prod_{m=v+1}^{p-4-2v} [[y, {}_{p-3-v-m}x], [y, {}_vx], [y, {}_mx]]^{\gamma_{v,m}} \pmod{\Gamma_{p+1}(G)},$$

where $\alpha_v = (-1)^v(v+2)$, $\beta_v = \binom{2v+1}{v}$, $\gamma_{v,m} = (-1)^{v+m}\binom{v+m+2}{v+1}$, *and the symbol* $[\cdot]$ *denotes the integer part of a number.*

In Section 2, we derive the collection formula for $(xy)^n$, $n \geqslant 1$, when all commutators that have more than 3 occurrences of y are equal to 1 (Lemma 1). Note that we collect the commutators not in order of increasing weights to minimize the number of types of arisen commutators.

Section 3 is devoted to the universal existence conditions (Lemmas 2–4), i.e. logical formulas with use of which one can calculate the exponents of the commutators arising in collection formulas. Finding the exponents in an explicit form is a difficult problem, especially in the Hall's form (1.2), which is crucial when finding the exponents modulo prime n.

In Section 4, we note that the coefficient $a_{w(q_i)}$ in (1.2), which is congruent to the exponent $e_i(n)$ modulo n when n is prime and $w(q_i) = n$, can be calculated much more easily than the other ones (Remark 1). We find an explicit form of the coefficient $a_{w(q_i)}$ for the commutators we are interested in (Lemmas 5–7).

Using the obtained results, we prove Theorem 1 in Section 5.

2. The collection formula for $(xy)^n$

In [5, Theorem 1], the following parameterization of the uncollected part of formula (1.1) was introduced. Let G be a group, $x, y \in G$. Then, for any $n \in \mathbb{N}$, we have

$$(xy)^n = x^n y^n \prod_{k=1}^{n-1} \prod_{i_1=1}^{2^{n-k}-1} \prod_{i_2=0}^{2^{n-k}-1} [y, {}_{\omega(i_1)}x, {}_{\omega(i_2)}y], \tag{2.1}$$

where $\omega(i)$ is the binary weight function, which is equal to the number of units in the binary representation of the non-negative integer i. This formula is the result of two steps of the collection process when all x's and then all y's are collected. The right-hand side of (2.1) contains the following commutators:

$$[y, {}_ux, {}_vy], \quad n-1 \geqslant u \geqslant 1,\ n-1 \geqslant v \geqslant 0. \tag{2.2}$$

Suppose all commutators that have more than 3 occurrences of y are equal to 1. Then only three series of commutators remain:

$$[y, {}_u x],\ [y, {}_u x, y],\ [y, {}_u x, {}_2 y], \quad n-1 \geqslant u \geqslant 1.$$

Let us collect the commutators in the following order:

$$[y, x],\ [y, {}_2 x],\ \ldots,\ [y, {}_{n-1} x].$$

Then we get four new series:

$$[[y, {}_u x], [y, {}_v x]], \quad n-1 \geqslant u > v \geqslant 1; \tag{2.3}$$
$$[[y, {}_u x, y], [y, {}_v x]], \quad n-1 \geqslant u, v \geqslant 1; \tag{2.4}$$
$$[[y, {}_u x], {}_2[y, {}_v x]], \quad n-1 \geqslant u > v \geqslant 1; \tag{2.5}$$
$$[[y, {}_u x], [y, {}_v x], [y, {}_m x]], \quad n-1 \geqslant u, v, m \geqslant 1,\ u > v,\ m > v. \tag{2.6}$$

All the remaining commutators have at least 2 occurrences of y. Therefore, they can be collected in any order since they all commute with each other. The exponents $g_n(u, v)$ and $f_n(u, v)$ of commutators (2.2) and (2.3), respectively, are found in the Hall's form (1.2) in [10, Theorems 2 and 4]:

$$g_n(u, v) = \sum_{i=1}^{u+v+1} \binom{n}{i} \binom{i-1}{u} \binom{u}{i-v-1},$$

$$f_n(u, v) = \sum_{i=1}^{u+v+2} \binom{n}{i} \left(\binom{i-1}{v+1} \binom{v+1}{i-u-1} + \sum_{k=1}^{v} \binom{i-k-1}{i-v-1} \binom{v-k}{i-u-2} \right).$$

By [10, Corollary 2], if the weight of the commutators is a prime p, then

$$g_p(p-v-1, v) \equiv (-1)^{p-v-1} \equiv (-1)^v \pmod{p}; \tag{2.7}$$
$$f_p(p-v-2, v) \equiv (-1)^{p-v-2} \equiv (-1)^{v+1} \pmod{p}. \tag{2.8}$$

The exponents $a_n(u, v)$, $b_n(u, v)$, and $c_n(u, v, m)$ of commutators (2.5), (2.4), and (2.6), respectively, can be calculated by [10, Theorem 3], which gives an explicit expression for the exponents of complex commutators. However, that expression is quite cumbersome since it includes multiple combinatorial sums and complex binomial coefficients. So, analyzing this expression to find $a_n(u, v)$, $b_n(u, v)$, $c_n(u, v, m)$ modulo prime $n = p$ is very difficult. The Hall's form (1.2) for these series of commutators is not known.

Thus, we get the following

Lemma 1. *Let G be a group, $x, y \in G$. If all commutators in x and y that have more than 3 occurrences of y are equal to 1, then, for any $n \in \mathbb{N}$, we*

have

$$(xy)^n = x^n y^n \prod_{u=1}^{n-1} [y, {}_u x]^{g_n(u,0)} \prod_{u=1}^{n-1} [y, {}_u x, y]^{g_n(u,1)} \prod_{u=1}^{n-1} [y, {}_u x, {}_2 y]^{g_n(u,2)}$$
$$\cdot \prod_{n-1 \geqslant u > v \geqslant 1} [[y, {}_u x], [y, {}_v x]]^{f_n(u,v)} \prod_{n-1 \geqslant u,v \geqslant 1} [[y, {}_u x, y], [y, {}_v x]]^{a_n(u,v)}$$
$$\cdot \prod_{n-1 \geqslant u > v \geqslant 1} [[y, {}_u x], {}_2[y, {}_v x]]^{b_n(u,v)} \prod_{\substack{n-1 \geqslant u,v,m \geqslant 1; \\ u > v;\ m > v}} [[y, {}_u x], [y, {}_v x], [y, {}_m x]]^{c_n(u,v,m)},$$

where $a_n(u,v)$, $b_n(u,v)$, $c_n(u,v,m)$ are some integer-valued functions.

3. Universal existence conditions

In this paper we use the approach to studying the exponents of the commutators that arise in collection formulas obtained for positive words of a free group. This approach was introduced in [11;12] and it is based on P. Hall's idea that he used in the proof of formula (1.1). Here we will very briefly describe the basic concept.

The *collection process* is a construction of the sequence of words:

$$W_0 \equiv T_0,\ W_1 \equiv q_1^{e_1} T_1,\ W_2 \equiv q_1^{e_1} q_2^{e_2} T_2,\ \ldots,\ W_j \equiv q_1^{e_1} \ldots q_j^{e_j} T_j,\ \ldots \quad (3.1)$$

by the following rules. The initial word W_0 is a positive word of the free group $F = F(a_1, \ldots, a_n)$, $n \geqslant 2$. All occurrences of the letters $a_1, \ldots, a_n$ (commutators of weight 1) have *labels* (integer sequences) assigned to them, and different occurrences of the same letter have pairwise different labels of the same length. The word W_j, $j \geqslant 1$, is obtained from W_{j-1} by moving step by step all the occurrences of an arbitrary commutator q_j to the beginning of the *uncollected part* T_{j-1} with use of commutators:

$$Q(\Lambda_u) R(\Lambda_v) = R(\Lambda_v) Q(\Lambda_u) [Q, R](\Lambda_u \Lambda_v),$$

where $\Lambda_u \Lambda_v$ is the concatenation of the labels Λ_u and Λ_v.

The set $D(a_k)$, $k \in \overline{1,n}$ is an arbitrary fixed set of integer sequences of the same length that contains all labels of the occurrences of a_k in W_0. If a commutator R arose during the collection process (3.1) and the parenthesis-free notation of R is $(a_{i_1}, \ldots, a_{i_{w(R)}})$, then any occurrence of R has a label that belongs to the Cartesian product $D(R) = D(a_{i_1}) \times \cdots \times D(a_{i_{w(R)}})$.

Suppose that some uncollected part in (3.1) contains occurrences of the commutators R and Q. The *existence condition* of the commutator R is the predicate E_R^{Λ}, $\Lambda \in D(R)$, that is equal to 1 iff there exists a word in (3.1) such that its uncollected part contains the occurrence $R(\Lambda)$. The

precedence condition for the commutators R and Q is the predicate $P^{\Lambda_1\Lambda_2}_{Q,R}$, $\Lambda_1\Lambda_2 \in D(Q)\times D(R)$, that is equal to 1 iff there exists a word in (3.1) such that, in its uncollected part, $Q(\Lambda_1)$ precedes (is to the left of) $R(\Lambda_2)$.

For the exponent e_j, $j \geqslant 1$, in (3.1) we have

$$e_j = |\{\Lambda \in D(q_j) \mid E^{\Lambda}_{q_j} = 1\}|. \tag{3.2}$$

For commutators R_1, R_2, the predicate $R_1 \prec R_2$ is equal to 1 iff there exist commutators q_i, q_j in (3.1) such that $q_i = R_1$, $q_j = R_2$, $i < j$, i.e., the occurrences of R_1 were collected at an earlier stage than the occurrences of R_2 in the variant of the collection process (3.1).

The *universal existence condition* $\mathcal{E}_R$ of a commutator R is a logical formula with the following properties: 1) $\mathcal{E}_R$ contains at most the operations of conjunction, disjunction, and the following predicate symbols: E_{a_i}, P_{a_i,a_j}, $\prec$, $=$, where $i, j \in \overline{1,n}$; 2) if occurrences of R arose during some variant of the collection process, then $\mathcal{E}^{\Lambda}_R = E^{\Lambda}_R$, $\Lambda \in D(R)$, for the standard interpretation (according to the definitions above) of all predicate symbols in the formula $\mathcal{E}_R$.

In [12, Theorem 1], it was shown that $\mathcal{E}_R$ can be obtained with use of the following recurrence relations:

$$E^{\Lambda^1_0\dots\Lambda^1_u}_{[Q_1,\,{}_uR_1]} = P^{\Lambda^1_0\Lambda^1_1}_{Q_1,R_1} \bigwedge_{k=1}^{u-1} P^{\Lambda^1_k\Lambda^1_{k+1}}_{R_1,R_1}, \quad u \geqslant 1; \tag{3.3}$$

$P^{\Lambda^1_0\dots\Lambda^1_u\Lambda^2_0\dots\Lambda^2_v}_{[Q_1,\,{}_uR_1],[Q_2,\,{}_vR_2]}$ is equal to

$$E^{\Lambda^1_0\dots\Lambda^1_u}_{[Q_1,\,{}_uR_1]}E^{\Lambda^2_0\dots\Lambda^2_v}_{[Q_2,\,{}_vR_2]}F_1, \qquad \text{if } u+v \geqslant 1,\ R_1 = R_2,\ Q_1 = Q_2; \tag{3.4a}$$

$$E^{\Lambda^1_0\dots\Lambda^1_u}_{[Q_1,\,{}_uR_1]}E^{\Lambda^2_0\dots\Lambda^2_v}_{[Q_2,\,{}_vR_2]}P^{\Lambda^1_0\Lambda^2_0}_{Q_1,Q_2}, \quad \text{if } u+v \geqslant 1,\ R_1 = R_2,\ Q_1 \neq Q_2, \ u=0\Rightarrow w(Q_1)=1,\ v=0\Rightarrow w(Q_2)=1; \tag{3.4b}$$

$$E^{\Lambda^1_0\dots\Lambda^1_u}_{[Q_1,\,{}_uR_1]}E^{\Lambda^2_0\dots\Lambda^2_v}_{[Q_2,\,{}_vR_2]}F_2, \qquad \text{if } u, v \geqslant 1, \quad R_1 \neq R_2; \tag{3.4c}$$

where $[Q_1, {}_uR_1] \neq Q_2$ for $u \geqslant 1$ and $[Q_2, {}_vR_2] \neq Q_1$ for $v \geqslant 1$,

$$\Lambda^1_0 \in D(Q_1),\ \Lambda^2_0 \in D(Q_2),\ \Lambda^1_1, \dots, \Lambda^1_u \in D(R_1),\ \Lambda^2_1, \dots, \Lambda^2_v \in D(R_2),$$

$$F_1 = P^{\Lambda^1_0\Lambda^2_0}_{Q_1,Q_2} \vee (\Lambda^1_0 = \Lambda^2_0)\left((u<v)\bigwedge_{k=1}^{u}(\Lambda^2_k = \Lambda^1_k) \vee \bigvee_{k=1}^{\min\{u,v\}} P^{\Lambda^2_k\Lambda^1_k}_{R_1,R_2} \bigwedge_{h=1}^{k-1}(\Lambda^2_h = \Lambda^1_h)\right),$$

$$F_2 = (R_1 \prec R_2)P^{\Lambda^1_0\dots\Lambda^1_u\Lambda^1_0}_{[Q_1,\,{}_uR_1],Q_2} \vee (R_2 \prec R_1)P^{\Lambda^1_0\Lambda^2_0\dots\Lambda^2_v}_{Q_1,[Q_2,\,{}_vR_2]}.$$

Lemma 2. *The universal existence condition* $\mathcal{E}^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2\ldots\Lambda_v^2\Lambda_0^3\ldots\Lambda_v^3}_{[[y,\,_u x],\,_2[y,\,_v x]]}$, *for* $u, v \geqslant 1$, $u \neq v$, *is expressed as follows:*

$$P_{y,x}^{\Lambda_0^1\Lambda_1^1}\bigwedge_{k=1}^{u-1}P_{x,x}^{\Lambda_k^1\Lambda_{k+1}^1}\wedge P_{y,x}^{\Lambda_0^2\Lambda_1^2}\bigwedge_{k=1}^{v-1}P_{x,x}^{\Lambda_k^2\Lambda_{k+1}^2}\wedge P_{y,x}^{\Lambda_0^3\Lambda_1^3}\bigwedge_{k=1}^{v-1}P_{x,x}^{\Lambda_k^3\Lambda_{k+1}^3}$$

$$\wedge\Bigg(P_{y,y}^{\Lambda_0^1\Lambda_0^2}\vee(\Lambda_0^1=\Lambda_0^2)\Big((u<v)\bigwedge_{k=1}^{u}(\Lambda_k^2=\Lambda_k^1)\vee\bigvee_{k=1}^{\min\{u,v\}}P_{y,y}^{\Lambda_k^2\Lambda_k^1}\bigwedge_{h=1}^{k-1}(\Lambda_h^2=\Lambda_h^1)\Big)\Bigg)$$

$$\wedge\Bigg(P_{y,y}^{\Lambda_0^2\Lambda_0^3}\vee(\Lambda_0^2=\Lambda_0^3)\bigvee_{k=1}^{v}P_{y,y}^{\Lambda_k^3\Lambda_k^2}\bigwedge_{h=1}^{k-1}(\Lambda_h^3=\Lambda_h^2)\Bigg).$$

Proof. Applying relation (3.3) to $E^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2\ldots\Lambda_v^2\Lambda_0^3\ldots\Lambda_v^3}_{[[y,\,_u x],\,_2[y,\,_v x]]}$, we have

$$P^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^2\Lambda_1^2\ldots\Lambda_v^2}_{[y,\,_u x],[y,\,_v x]}P^{\Lambda_0^2\Lambda_1^2\ldots\Lambda_v^2\Lambda_0^3\Lambda_1^3\ldots\Lambda_v^3}_{[y,\,_v x],[y,\,_v x]}.$$

Applying (3.4a) to each predicate symbols P and then simplifying, we have

$$E^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1}_{[y,\,_u x]}E^{\Lambda_0^2\Lambda_1^2\ldots\Lambda_v^2}_{[y,\,_v x]}E^{\Lambda_0^3\Lambda_1^3\ldots\Lambda_v^3}_{[y,\,_v x]}$$

$$\wedge\Bigg(P_{y,y}^{\Lambda_0^1\Lambda_0^2}\vee(\Lambda_0^1=\Lambda_0^2)\Big((u<v)\bigwedge_{k=1}^{u}(\Lambda_k^2=\Lambda_k^1)\vee\bigvee_{k=1}^{\min\{u,v\}}P_{y,y}^{\Lambda_k^2\Lambda_k^1}\bigwedge_{h=1}^{k-1}(\Lambda_h^2=\Lambda_h^1)\Big)\Bigg)$$

$$\wedge\Bigg(P_{y,y}^{\Lambda_0^2\Lambda_0^3}\vee(\Lambda_0^2=\Lambda_0^3)\bigvee_{k=1}^{v}P_{y,y}^{\Lambda_k^3\Lambda_k^2}\bigwedge_{h=1}^{k-1}(\Lambda_h^3=\Lambda_h^2)\Bigg).$$

The application of (3.3) to the predicate symbols E completes the proof. □

Lemma 3. *The universal existence condition* $\mathcal{E}^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2\Lambda_0^3\ldots\Lambda_v^3}_{[[y,\,_u x,y],[y,\,_v x]]}$, *for* $u, v \geqslant 1$, *is expressed as follows:*

$$P_{y,x}^{\Lambda_0^1\Lambda_1^1}\bigwedge_{k=1}^{u-1}P_{x,x}^{\Lambda_k^1\Lambda_{k+1}^1}\wedge P_{y,x}^{\Lambda_0^3\Lambda_1^3}\bigwedge_{k=1}^{v-1}P_{x,x}^{\Lambda_k^3\Lambda_{k+1}^3}\wedge P_{y,y}^{\Lambda_0^1\Lambda_0^2}$$

$$\wedge\Bigg(P_{y,y}^{\Lambda_0^1\Lambda_0^3}\vee(\Lambda_0^1=\Lambda_0^3)\Big((u<v)\bigwedge_{k=1}^{u}(\Lambda_k^3=\Lambda_k^1)\vee\bigvee_{k=1}^{\min\{u,v\}}P_{y,y}^{\Lambda_k^3\Lambda_k^1}\bigwedge_{h=1}^{k-1}(\Lambda_h^3=\Lambda_h^1)\Big)\Bigg).$$

Proof. Applying relation (3.3) to $E^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2\Lambda_0^3\ldots\Lambda_v^3}_{[[y,\,_u x,y],[y,\,_v x]]}$, we have

$$P^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^2\Lambda_0^3\Lambda_1^3\ldots\Lambda_v^3}_{[y,\,_u x,y],[y,\,_v x]}.$$

If $u = v$, then we apply (3.4a) to the symbol P. If $u \neq v$, then we use (3.4c) and notice that $x \prec y$ is true for any variant of the collection process

where the commutator $[[y, {}_u x, y], [y, {}_v x]]$ has arisen. In any case, we get

$$E^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^2}_{[y,\,{}_u x,y]}E^{\Lambda_0^3\Lambda_1^3\ldots\Lambda_v^3}_{[y,\,{}_v x]}P^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^3\Lambda_1^3\ldots\Lambda_v^3}_{[y,\,{}_u x],[y,\,{}_v x]}.$$

Applying relation (3.3) to the first predicate symbol E on the left, we have

$$P^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^2}_{[y,\,{}_u x],y}E^{\Lambda_0^3\Lambda_1^3\ldots\Lambda_v^3}_{[y,\,{}_v x]}P^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^3\Lambda_1^3\ldots\Lambda_v^3}_{[y,\,{}_u x],[y,\,{}_v x]}.$$

Applying (3.4a) to all symbols P and simplifying, we have

$$E^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1}_{[y,\,{}_u x]}E^{\Lambda_0^3\Lambda_1^3\ldots\Lambda_v^3}_{[y,\,{}_v x]}P^{\Lambda_0^1\Lambda_0^2}_{y,y}$$
$$\wedge\Biggl(P^{\Lambda_0^1\Lambda_0^3}_{y,y}\vee(\Lambda_0^1=\Lambda_0^3)\Biggl((u<v)\bigwedge_{k=1}^{u}(\Lambda_k^3=\Lambda_k^1)\vee\bigvee_{k=1}^{\min\{u,v\}}P^{\Lambda_k^3\Lambda_k^1}_{y,y}\bigwedge_{h=1}^{k-1}(\Lambda_h^3=\Lambda_h^1)\Biggr)\Biggr).$$

The application of (3.3) to the predicate symbols E completes the proof. □

Lemma 4. *The universal existence condition* $\mathcal{E}^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2\ldots\Lambda_v^2\Lambda_0^3\ldots\Lambda_m^3}_{[[y,\,{}_u x],[y,\,{}_v x],[y,\,{}_m x]]}$, *for* $u, v, m \geqslant 1$, $u \neq v$, $v \neq m$, *is expressed as follows:*

$$P^{\Lambda_0^1\Lambda_1^1}_{y,x}\bigwedge_{k=1}^{u-1}P^{\Lambda_k^1\Lambda_{k+1}^1}_{x,x}\wedge P^{\Lambda_0^2\Lambda_1^2}_{y,x}\bigwedge_{k=1}^{v-1}P^{\Lambda_k^2\Lambda_{k+1}^2}_{x,x}\wedge P^{\Lambda_0^3\Lambda_1^3}_{y,x}\bigwedge_{k=1}^{m-1}P^{\Lambda_k^3\Lambda_{k+1}^3}_{x,x}$$
$$\wedge\Biggl(P^{\Lambda_0^1\Lambda_0^2}_{y,y}\vee(\Lambda_0^1=\Lambda_0^2)\Biggl((u<v)\bigwedge_{k=1}^{u}(\Lambda_k^2=\Lambda_k^1)\vee\bigvee_{k=1}^{\min\{u,v\}}P^{\Lambda_k^2\Lambda_k^1}_{y,y}\bigwedge_{h=1}^{k-1}(\Lambda_h^2=\Lambda_h^1)\Biggr)\Biggr)$$
$$\wedge\Biggl(P^{\Lambda_0^1\Lambda_0^3}_{y,y}\vee(\Lambda_0^1=\Lambda_0^3)\Biggl((u<m)\bigwedge_{k=1}^{u}(\Lambda_k^3=\Lambda_k^1)\vee\bigvee_{k=1}^{\min\{u,m\}}P^{\Lambda_k^3\Lambda_k^1}_{y,y}\bigwedge_{h=1}^{k-1}(\Lambda_h^3=\Lambda_h^1)\Biggr)\Biggr).$$

Proof. Applying relation (3.3) to $E^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2\ldots\Lambda_v^2\Lambda_0^3\ldots\Lambda_m^3}_{[[y,\,{}_u x],[y,\,{}_v x],[y,\,{}_m x]]}$, we have

$$P^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^2\Lambda_1^2\ldots\Lambda_v^2\Lambda_0^3\Lambda_1^3\ldots\Lambda_m^3}_{[[y,\,{}_u x],[y,\,{}_v x]],[y,\,{}_m x]}.$$

If $u = m$, then we apply (3.4a) to the symbol P. If $u \neq m$, then we use (3.4c) and notice that $x \prec [y, {}_v x]$ is true for any variant of the collection process where the commutator $[[y, {}_u x], [y, {}_v x], [y, {}_m x]]$ has arisen. In any case, we get

$$E^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^2\Lambda_1^2\ldots\Lambda_v^2}_{[[y,\,{}_u x],[y,\,{}_v x]]}E^{\Lambda_0^3\Lambda_1^3\ldots\Lambda_m^3}_{[y,\,{}_m x]}P^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^3\Lambda_1^3\ldots\Lambda_m^3}_{[y,\,{}_u x],[y,\,{}_m x]}.$$

Applying (3.3) to the first predicate symbol E on the left, we have

$$P^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^2\Lambda_1^2\ldots\Lambda_v^2}_{[y,\,{}_u x],[y,\,{}_v x]}E^{\Lambda_0^3\Lambda_1^3\ldots\Lambda_m^3}_{[y,\,{}_m x]}P^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1\Lambda_0^3\Lambda_1^3\ldots\Lambda_m^3}_{[y,\,{}_u x],[y,\,{}_m x]}.$$

Applying (3.4a) to all predicate symbols P and simplifying, we have

$$E^{\Lambda_0^1\Lambda_1^1\ldots\Lambda_u^1}_{[y,\,u\,x]}E^{\Lambda_0^2\Lambda_1^2\ldots\Lambda_v^2}_{[y,\,v\,x]}E^{\Lambda_0^3\Lambda_1^3\ldots\Lambda_m^3}_{[y,\,m\,x]}$$
$$\wedge\Biggl(P^{\Lambda_0^1,\Lambda_0^2}_{y,y}\vee(\Lambda_0^1=\Lambda_0^2)\Bigl((u<v)\bigwedge_{k=1}^{u}(\Lambda_k^2=\Lambda_k^1)\vee\bigvee_{k=1}^{\min\{u,v\}}P^{\Lambda_k^2\Lambda_k^1}_{y,y}\bigwedge_{h=1}^{k-1}(\Lambda_h^2=\Lambda_h^1)\Bigr)\Biggr)$$
$$\wedge\Biggl(P^{\Lambda_0^1\Lambda_0^3}_{y,y}\vee(\Lambda_0^1=\Lambda_0^3)\Bigl((u<m)\bigwedge_{k=1}^{u}(\Lambda_k^3=\Lambda_k^1)\vee\bigvee_{k=1}^{\min\{u,m\}}P^{\Lambda_k^3\Lambda_k^1}_{y,y}\bigwedge_{h=1}^{k-1}(\Lambda_h^3=\Lambda_h^1)\Bigr)\Biggr).$$

The application of (3.3) to the predicate symbols E completes the proof. □

4. The exponents of the commutators

We continue to use the approach described in the previous section. For the word $(xy)^n$, $n\geqslant 1$, we use the same labeling as P. Hall in [2]:

$$x(1)y(1)x(2)y(2)\ldots x(n)y(n).$$

Then $D(x)=D(y)=\{1,\ldots,n\}$ and $D(R)=\{1,\ldots,n\}^{w(R)}$ for any commutator R in x,y of weight $w(R)$ that has arisen during the collection process. As for the existence and the precedence conditions, we have

$$E_x^{(\lambda_1)}=E_y^{(\lambda_1)}=1,\ P_{x,y}^{(\lambda_1,\lambda_2)}=[\lambda_1<\lambda_2]\vee[\lambda_1=\lambda_2], \tag{4.1}$$
$$P_{y,x}^{(\lambda_1,\lambda_2)}=P_{x,x}^{(\lambda_1,\lambda_2)}=P_{y,y}^{(\lambda_1,\lambda_2)}=[\lambda_1<\lambda_2], \tag{4.2}$$

where $\lambda_1,\lambda_2\in\{1,\ldots,n\}$.

From the definition of the universal existence condition $\mathcal{E}_R^{\Lambda}$ and relations (3.3), (3.4) it follows that, for the commutator R of weight w, the existence condition $E_R^{(\lambda_1,\ldots,\lambda_w)}$ is expressed by an L-condition of rank at most w [9, Definition 2], i.e. a formula containing only the logical operations: disjunction, conjunction, negation and the predicate symbols: $[\lambda_i<\lambda_j]$, $[\lambda_i=\lambda_j]$, where $1\leqslant i,j\leqslant w$.

By [9, Theorem 1], the Cartesian product $\{1,\ldots,n\}^w$ is a union of pairwise disjoint classes:

$$\{1,\ldots,n\}^w=\bigcup_{t=1}^{w}\bigcup_{\substack{I_1\cup\ldots\cup I_t=\{\lambda_1,\ldots,\lambda_w\}\\ I_i\cap I_j=\varnothing\Leftrightarrow i\neq j}}\langle(I_1,\ldots,I_t)\rangle,$$

where the class

$$\langle(I_1,\ldots,I_t)\rangle=\Bigl\{(\lambda_1,\ldots,\lambda_w)\in\{1,\ldots,n\}^w\mid{\scriptstyle\begin{array}{l}\lambda_i=\lambda_j\Leftrightarrow\exists k\,(\lambda_i\in I_k)(\lambda_j\in I_k),\\ \lambda_i<\lambda_j\Leftrightarrow\exists p\,\exists q\,(\lambda_i\in I_p)(\lambda_j\in I_q)(p<q)\end{array}}\Bigr\}$$

has cardinality $\binom{n}{t}$, and, for any L-condition C of rank at most w, all elements of the class satisfy C or none of them [9, Lemma 1].

Thus, the exponent of the commutator R of weight w is equal to

$$|\{(\lambda_1,\dots,\lambda_w)\in\{1,\dots,n\}^w \mid E_R^{(\lambda_1,\dots,\lambda_w)}=1\}| = \sum_{k=1}^{w} a_k\binom{n}{k}, \qquad (4.3)$$

where a_k is the number of the classes of cardinality $\binom{n}{k}$ whose elements satisfy the L-condition $E_R^{(\lambda_1,\dots,\lambda_w)}$.

Remark 1. When finding the coefficient a_w, we must consider only the classes $\langle(I_1,\dots,I_w)\rangle$ whose elements $(\lambda_1,\dots,\lambda_w)$ have pairwise distinct components, i.e. $\lambda_i\neq\lambda_j$ for all $i\neq j$, since the number of the sets $I_1,\dots,I_w$ and the number of the variables $\lambda_1,\dots,\lambda_w$ coincide.

Lemma 5. *Suppose $u,v\geqslant 1$. If the commutator $[[y,\,{}_ux,y],[y,\,{}_vx]]$ has arisen during a collection process applied to the word $(xy)^n$, $n\geqslant 1$, then, in the Hall's form (1.2) of its exponent, we have*

$$a_{u+v+3}=\binom{u+v+2}{u}(v+2).$$

Proof. Taking into account Remark 1 and Lemma 3, we find a_{u+v+3} as the number of the classes $\langle(I_1,\dots,I_w)\rangle$ whose elements satisfy the condition

$$P_{y,y}^{\Lambda_0^1\Lambda_0^2}P_{y,y}^{\Lambda_0^1\Lambda_0^3}P_{y,x}^{\Lambda_0^1\Lambda_1^1}\bigwedge_{k=1}^{u-1}P_{x,x}^{\Lambda_k^1\Lambda_{k+1}^1}\wedge P_{y,x}^{\Lambda_0^3\Lambda_1^3}\bigwedge_{k=1}^{v-1}P_{x,x}^{\Lambda_k^3\Lambda_{k+1}^3},$$

where the sequence $\Lambda_i^j=(\lambda_i^j)$ for all possible i,j. By formulas (4.1), (4.2), we get the following system of relations:

$$\lambda_0^1<\lambda_0^2;\quad \lambda_0^1<\lambda_0^3;\quad \lambda_0^1<\ldots<\lambda_u^1;\quad \lambda_0^3<\ldots<\lambda_v^3.$$

Since λ_0^1 is the smallest variable, we have $I_1=\{\lambda_0^1\}$. Then we distribute the variables $\lambda_1^1,\dots,\lambda_u^1$ among $u+v+3-1$ sets in $\binom{u+v+2}{u}$ ways. Further we place λ_0^2 in any of $u+v+2-u=v+2$ sets. The remaining variables $\lambda_0^3,\dots,\lambda_v^3$ are distributed among $v+2-1=v+1$ sets in only one way. □

Lemma 6. *Suppose $u,v\geqslant 1$, $u\neq v$. If the commutator $[[y,\,{}_ux],{}_2[y,\,{}_vx]]$ has arisen during a collection process applied to the word $(xy)^n$, $n\geqslant 1$, then, in the Hall's form (1.2) of its exponent, we have*

$$a_{u+2v+3}=\binom{u+2v+2}{u}\binom{2v+1}{v}.$$

Proof. Taking into account Remark 1 and Lemma 2, we find a_{u+2v+3} as the number of the classes $\langle (I_1, \ldots, I_w) \rangle$ whose elements satisfy the condition

$$P_{y,y}^{\Lambda_0^1 \Lambda_0^2} P_{y,y}^{\Lambda_0^2 \Lambda_0^3} P_{y,x}^{\Lambda_0^1 \Lambda_1^1} \bigwedge_{k=1}^{u-1} P_{x,x}^{\Lambda_k^1 \Lambda_{k+1}^1} \wedge P_{y,x}^{\Lambda_0^2 \Lambda_1^2} \bigwedge_{k=1}^{v-1} P_{x,x}^{\Lambda_k^2 \Lambda_{k+1}^2} \wedge P_{y,x}^{\Lambda_0^3 \Lambda_1^3} \bigwedge_{k=1}^{v-1} P_{x,x}^{\Lambda_k^3 \Lambda_{k+1}^3},$$

where the sequence $\Lambda_i^j = (\lambda_i^j)$ for all possible i, j. By formulas (4.1), (4.2), we get the following system of relations:

$$\lambda_0^1 < \lambda_0^2 < \lambda_0^3; \quad \lambda_0^1 < \ldots < \lambda_u^1; \quad \lambda_0^2 < \ldots < \lambda_v^2; \quad \lambda_0^3 < \ldots < \lambda_v^3.$$

Since λ_0^1 is the smallest variable, we have $I_1 = \{\lambda_0^1\}$. Then we distribute the variables $\lambda_1^1, \ldots, \lambda_u^1$ among $u + 2v + 3 - 1$ sets in $\binom{u+2v+2}{u}$ ways. After that, λ_0^2 becomes the smallest variable. Therefore, $I_j = \{\lambda_0^2\}$ for the smallest j such that I_j is empty. Further we distribute the variables $\lambda_1^2, \ldots, \lambda_v^2$ among $u + 2v + 2 - u - 1$ sets in $\binom{2v+1}{v}$ ways. For the remaining variables $\lambda_0^3, \ldots, \lambda_v^3$, there is only one way to distribute them among $2v+1-v = v+1$ sets. □

Lemma 7. *Suppose $u, v, m \geqslant 1$, $u \neq v$, $v \neq m$. If the commutator $[[y, {}_u x], [y, {}_v x], [y, {}_m x]]$ has arisen during a collection process applied to the word $(xy)^n$, $n \geqslant 1$, then, in the Hall's form (1.2) of its exponent, we have*

$$a_{u+v+m+3} = \binom{u+v+m+2}{u} \binom{v+m+2}{v+1}.$$

Proof. Taking into account Remark 1 and Lemma 4, we find $a_{u+v+m+3}$ as the number of the classes $\langle (I_1, \ldots, I_w) \rangle$ whose elements satisfy the condition

$$P_{y,y}^{\Lambda_0^1 \Lambda_0^2} P_{y,y}^{\Lambda_0^1 \Lambda_0^3} P_{y,x}^{\Lambda_0^1 \Lambda_1^1} \bigwedge_{k=1}^{u-1} P_{x,x}^{\Lambda_k^1 \Lambda_{k+1}^1} \wedge P_{y,x}^{\Lambda_0^2 \Lambda_1^2} \bigwedge_{k=1}^{v-1} P_{x,x}^{\Lambda_k^2 \Lambda_{k+1}^2} \wedge P_{y,x}^{\Lambda_0^3 \Lambda_1^3} \bigwedge_{k=1}^{m-1} P_{x,x}^{\Lambda_k^3 \Lambda_{k+1}^3},$$

where the sequence $\Lambda_i^j = (\lambda_i^j)$ for all possible i, j. By formulas (4.1), (4.2), we get the following system of relations:

$$\lambda_0^1 < \lambda_0^2; \quad \lambda_0^1 < \lambda_0^3; \quad \lambda_0^1 < \ldots < \lambda_u^1; \quad \lambda_0^2 < \ldots < \lambda_v^2; \quad \lambda_0^3 < \ldots < \lambda_m^3.$$

Since λ_0^1 is the smallest variable, we have $I_1 = \{\lambda_0^1\}$. Then we distribute the variables $\lambda_1^1, \ldots, \lambda_u^1$ among $u + v + m + 3 - 1$ sets in $\binom{u+v+m+2}{u}$ ways. Further we distribute $\lambda_0^2, \ldots, \lambda_v^2$ among $u + v + m + 2 - u$ sets in $\binom{v+m+2}{v+1}$ ways. For the remaining variables $\lambda_0^3, \ldots, \lambda_m^3$, there is only one way to distribute them among $v + m + 2 - v - 1 = m + 1$ sets. □

5. Proof of Theorem 1

Consider the collection formula in Lemma 1 for $n = p$ modulo $\Gamma_{p+1}(G)$:

$$(xy)^p \equiv x^p y^p R_1^{e_1} \dots R_t^{e_t} \pmod{\Gamma_{p+1}(G)}, \tag{5.1}$$

here $R_1, \dots, R_t$ are the commutators of weight less than $p+1$. Note that, for any R and Q such that $w(R) = p$, $1 \leqslant w(Q) \leqslant p$, we have $RQ \equiv QR$ modulo $\Gamma_{p+1}(G)$. Therefore, by rearranging step by step consecutive occurrences of elements, we transform (5.1) into the following form:

$$(xy)^p \equiv x^p y^p R_{i_1}^{e_{i_1}} \dots R_{i_a}^{e_{i_a}} R_{j_1}^{e_{j_1}} \dots R_{j_b}^{e_{j_b}} \pmod{\Gamma_{p+1}(G)},$$

where $R_{i_1}, \dots, R_{i_a}$ are the commutators of weight less than p, $R_{j_1}, \dots, R_{j_b}$ are the commutators of weight p. Further we get

$$R_{i_a}^{-e_{i_a}} \dots R_{i_1}^{-e_{i_1}} y^{-p} x^{-p} (xy)^p \equiv R_{j_1}^{e_{j_1}} \dots R_{j_b}^{e_{j_b}} \pmod{\Gamma_{p+1}(G)}.$$

From (4.3) it follows that the exponents $e_{i_1}, \dots, e_{i_a}$ are divisible by p, i.e. $e_{i_1} = p\hat{e}_{i_1}, \dots, e_{i_a} = p\hat{e}_{i_a}$ Then, by [13, Corollary 12.4.1], there exist $c, d \in \Gamma_2(\langle x, y\rangle)$ such that

$$R_{i_a}^{-e_{i_a}} \dots R_{i_1}^{-e_{i_1}} y^{-p} x^{-p} (xy)^p = (R_{i_a}^{-\hat{e}_{i_a}} \dots R_{i_1}^{-\hat{e}_{i_1}} y^{-1} x^{-1} xyc)^p = d^p.$$

Thus, we have

$$d^p \equiv R_{j_1}^{e_{j_1}} \dots R_{j_b}^{e_{j_b}} \pmod{\Gamma_{p+1}(G)},$$

where $R_{j_1}, \dots, R_{j_b}$ are the following commutators of weight p:

$$[y, {}_{p-1}x]; [y, {}_{p-2}x, y]; [y, {}_{p-3}x, {}_2y]; [[y, {}_{p-2-v}x], [y, {}_v x]],\ \tfrac{p-3}{2} \geqslant v \geqslant 1; \tag{5.2}$$

$$[[y, {}_{p-3-v}x, y], [y, {}_v x]],\ p-4 \geqslant v \geqslant 1; \tag{5.3}$$

$$[[y, {}_{p-3-2v}x], {}_2[y, {}_v x]],\ [\tfrac{p-4}{3}] \geqslant v \geqslant 1; \tag{5.4}$$

$$[[y, {}_{p-3-v-m}x], [y, {}_v x], [y, {}_m x]],\ [\tfrac{p-5}{3}] \geqslant v \geqslant 1,\ p-4-2v \geqslant m \geqslant v+1. \tag{5.5}$$

By Lemma 1 and formulas (2.7), (2.8), the exponents of the commutators (5.2) modulo p are equal to: $1; -1; 1; (-1)^{v+1}$ respectively.

To find the exponents of the remaining commutators modulo p, we analyze the coefficient a_w in formula (4.3) with use of the congruence $\binom{p-1}{u} \equiv (-1)^u \pmod{p}$ [10, Proposition 1]. From Lemmas 5, 6, and 7 it follows that the exponents of (5.3), (5.4), and (5.5) modulo p are equal to: $\binom{p-1}{p-3-v}(v+2) \equiv (-1)^v (v+2) \pmod{p}$; $\binom{p-1}{p-3-2v}\binom{2v+1}{v} \equiv \binom{2v+1}{v} \pmod{p}$; $\binom{p-1}{p-3-v-m}\binom{v+m+2}{v+1} \equiv (-1)^{v+m}\binom{v+m+2}{v+1} \pmod{p}$ respectively. □

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