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Fixed Points of Multi-valued Almost Pseudo-contractions in Quasimetric Spaces

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Abstract: In this paper, we establish a new fixed point theorem for multi-valued almost pseudo-contractive mappings in quasimetric spaces, extending and improving several known results in the literature. Our approach generalizes earlier works by allowing the contractive constant to the whole interval $[0, 1)$, rather than being subject to more restrictive bounds. As an application, we derive new data dependence results for the fixed point sets of such mappings in quasimetric spaces.

Keywords: multi-valued mappings, almost pseudo-contractions, fixed point, data dependence, quasimetric spaces

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Научная статья

Неподвижные точки многозначных почти псевдосжимающих отображений в квазиметрических пространствах

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Аннотация: Установлена новая теорема о неподвижной точке для многозначных почти псевдосжимающих отображений в квазиметрических пространствах, которая расширяет и уточняет ряд известных в литературе результатов. Предлагаемый подход обобщает более ранние исследования, позволяя сжимающей константе принимать значения на всём интервале $[0, 1)$, а не подчиняться более строгим ограничениям. В качестве приложения получены новые результаты о зависимости от данных для множеств неподвижных точек таких отображений в квазиметрических пространствах.

Ключевые слова: многозначные отображения, почти псевдосжимающие отображения, неподвижные точки, зависимость от данных, квазиметрические пространства

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1. Introduction and preliminaries

Banach's fixed point theorem and its multi-valued version proved by Nadler [21] are fundamental results in the field of metric fixed point theory. Because of their applications to various areas of mathematics such as optimization theory, theory of fractals, solving equations and systems of equations, integral and differential equations, and differential inclusions, Banach's fixed point theorem and Nadler's fixed point theorem have been extended in several directions by weakening the contractive conditions or replacing metric spaces by generalized metric spaces. One of the important extensions is the result obtained by Dontchev and Hager in [14] where they proved a fixed point theorem for multi-valued pseudo-contractive mappings in metric spaces; see also [1;6;15;23] and references therein for some stability properties, applications and extensions. Some other interesting generalizations were given by S. Czerwik in [12; 13] where fixed point theorems for single-valued and multi-valued mappings were proved in the context of *b-metric spaces* (also known as *quasimetric spaces*).

Note that the concept of a quasimetric was first introduced in [11] where the authors used the name distance function. This concept is related to that of a quasi-norm, introduced by Hyers [17] under the name "pseudo-norm" and later by Bourgin [9] under the term "quasi-norm." Quasimetric spaces are linked to spaces of homogeneous type [10; 20] and play a crucial role in the study of optimal transport paths [24]. A quasimetric space is a special case of a (q_1, q_2)-quasimetric space introduced in [3] and, more generally, of an f-quasimetric space introduced in [4]. For terminologies related to quasimetric spaces, the early developments on fixed point theory in quasimetric spaces and some relevant bibliography related to this topic, we refer the reader to [8].

Definition 1 ([13; 20]). *Let X be a nonempty set and $s \geq 1$ be a real number. A function $d : X \times X \to \mathbb{R}$ is said to be a quasimetric with constant s on X if it satisfies the following conditions: for all $x, y, z \in X$,*

(1) $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.
The pair (X, d) *is called a quasimetric space with constant* s.

It is obvious that a metric space is a quasimetric space with constant $s = 1$. A quasimetric space with constant $s > 1$ is not a metric space. For example, the spaces ℓ^p and $L_p[0,1]$ with $0 < p < 1$ endowed with the quasimetrics respectively defined by

$$d(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{\frac{1}{p}}$$

and

$$d(x, y) = \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{\frac{1}{p}},$$

are quasimetric spaces with the same constant $s = 2^{\frac{1}{p}-1}$ (see [2; 8]).

Let (X, d) be a quasimetric space. The open ball and the closed ball centered at a point $x \in X$ with radius $r > 0$ are defined by $B(x, r) = \{y \in X : d(x, y) < r\}$, and $B[x, r] = \{y \in X : d(x, y) \leq r\}$, respectively. A subset E of X is said to be open in the quasimetric space (X, d) if, for each $x \in E$, there exists $r_x > 0$ such that $B(x, r_x) \subset E$. A subset C of X is said to be closed if $X \setminus C$ is an open set. The closure of a subset E of X, denoted by $\mathrm{cl}(E)$, is the smallest closed subset of X containing E. The set of all nonempty subsets of X and the set of all nonempty closed subsets of X will be denoted by $P(X)$ and $C(X)$, respectively. For any nonempty subset A of X and any point $x \in X$, the distance from x to A is defined by $D(x, A) = \inf\{d(x, a) : a \in A\}$, where $D(x, \emptyset) = +\infty$ by convention. For two nonempty subsets A and B of X, the excess of A over B is defined by $e(A, B) = \sup\{D(a, B) : a \in A\}$. Here, we adopt the convention $e(\emptyset, A) = 0$ and $e(A, \emptyset) = +\infty$ for a nonempty subset A of X. The Hausdorff-Pompeiu distance between two subsets A and B is defined by $H(A, B) = \max\{e(A, B), e(B, A)\}$. Let $F : X \to P(X)$ be a multi-valued mapping. The graph of F is defined by $\mathrm{gph}(F) = \{(x, y) \in X \times X : y \in F(x)\}$. A point $x \in X$ is called a fixed point of F if $x \in F(x)$. The set of all fixed points of F is denoted by $\mathrm{Fix}(F)$. A sequence $\{x_n\}$ in X is called a sequence of successive approximations for F starting from $\bar{x} \in X$ if $x_0 = \bar{x}$ and $x_n \in F(x_{n-1})$ for all $n \in \mathbb{N}$.

Definition 2 ([13; 20]). *A sequence* $\{x_n\}$ *in a quasimetric space* (X, d) *is said to be convergent to a point* $x \in X$, *written as* $\lim_{n\to\infty} x_n = x$, *if* $\lim_{n\to\infty} d(x_n, x) = 0$. *The sequence* $\{x_n\}$ *is called fundamental or Cauchy if* $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

It is known that any convergent sequence in a quasimetric space has a unique limit. Moreover, any convergent sequence is a Cauchy sequence. However, the converse is not true in general.

Definition 3 ([13; 20]). *A quasimetric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.*

Definition 4. *A function $f : X \to \mathbb{R}$ is said to be lower semicontinuous at $x \in X$ if for any sequence $\{x_n\}$ in X converging to x, we have $f(x) \leq \liminf_{n\to\infty} f(x_n)$. The function f is said to be lower semicontinuous if it is lower semicontinuous at every point x in X.*

A quasimetric space is a topological space with the topology induced by its convergence in the sense of Franklin [16]. As in the case of metric spaces, this topology coincides with the topology generated by the family of open subsets [2]. One of the main differences between a quasimetric and a metric is that a metric is always continuous, while a quasimetric is not a lower semicontinuous function of its variables. Moreover, an open ball in a quasimetric space is not necessarily an open set and a closed ball is not necessarily a closed set (see, e.g., [2, Example 3.9 and 3.10]).

In [22], the authors introduced the concept of multi-valued almost pseudo-contractions which generalizes the concept of pseudo-contractions and proved a fixed point theorem for such types of mappings in quasimetric spaces.

Definition 5 ([22]). *Let (X, d) be a quasimetric space and V, U be nonempty subsets of X. A mapping $F : X \to P(X)$ is said to be a multi-valued almost pseudo-contraction on (U, V) if there exist $\lambda \in [0, 1)$ and $\ell \geq 0$ such that*

$$e(F(x) \cap V, F(y)) \leq \lambda d(x, y) + \ell D(y, F(x) \cap V), \quad \text{for all } x, y \in U.$$

If $U = V$, then F is said to be a multi-valued almost pseudo-contraction on U. If $U = V = X$, then we say that F is a multi-valued almost pseudo-contraction.

In the above definition, if $\ell = 0$, then a multi-valued almost pseudo-contraction reduces to a *multi-valued pseudo-contraction*; see, e.g., [6; 14]. The concept of a pseudo-contractive mapping can be viewed as a particular case of the pseudo-Lipschitz continuity introduced by Aubin [5], a property that is equivalent to the well-known notion of metric regularity in variational analysis (see, e.g., [15; 23] and references therein). For several significant contributions to fixed point theory obtained via metric regularity, we refer the reader to, e.g., [3; 15; 18]. The following interesting fixed point result for almost pseudo-contractions in quasimetric spaces was proved in [22].

Theorem 1 ([22])**.** *Let (X, d) be a complete quasimetric space with constant $s \geq 1$, $\bar{x} \in X$ and $r > 0$. Let $F : X \to C(X)$ be a multi-valued almost pseudo-contraction on $B(\bar{x}, r)$ with constants $\lambda \in \left(0, \frac{1}{s}\right)$ and $\ell \geq 0$. Suppose that*

$$D(\bar{x}, F(\bar{x})) < \frac{(1 - s\lambda)r}{s^2}. \tag{1.1}$$

Then, there is a sequence $\{x_n\} \subset B(\bar{x}, r)$ of successive approximations for F starting from $\bar{x}$ converging to a point $x^* \in \mathrm{cl}(B(\bar{x}, r))$ such that $x^* \in \mathrm{Fix}(F) \cap B(\bar{x}, sr)$ and

$$D(\bar{x}, \mathrm{Fix}(F)) \leq \frac{s^2}{1 - s\lambda} D(\bar{x}, F(\bar{x})). \tag{1.2}$$

In Theorem 1, if $s = 1$, then one obtains an existence result for fixed points of a multi-valued almost pseudo-contraction in metric spaces. This result generalizes some results by Aze and Penot [6], Berinde and Berinde [7], and Dontchev and Hager in [14].

A limitation of Theorem 1 is that it applies only when the contractive constant λ of the mapping under consideration is less than $1/s$. Therefore, it is natural to ask whether this range can be extended to encompass the interval $[0, 1)$. In this paper, we provide an affirmative answer by proving a fixed point theorem for multi-valued almost pseudo-contractions in quasimetric spaces, thereby extending Theorem 1 to the case $\lambda \in [0, 1)$. Consequently, we derive several data dependence results for multi-valued almost pseudo-contractions in the setting of quasimetric spaces.

2. Main results

For a given nonnegative real number a and a given positive integer n, we set $\Phi(a, n) = 1 + a + \cdots + a^{n-1}$. We also assume that $\Phi(a, 0) = 0$ and $a^0 = 1$ for $a = 0$. It is evident that $\Phi(a, n) \leq \Phi(a, m)$ whenever $n \leq m$. The main result of this paper is stated as follows.

Theorem 2. *Let (X, d) be a complete quasimetric space with constant $s \geq 1$, $\bar{x} \in X$ and $r > 0$. Let $F : X \to C(X)$ be a multi-valued almost pseudo-contraction on $B(\bar{x}, r)$ with constants $\lambda \in [0, 1)$ and $\ell \geq 0$. Set $m_0 = \min\{m \in \mathbb{N} : s\lambda^m < 1\}$ and suppose that*

$$D(\bar{x}, F(\bar{x})) < \frac{(1 - s\lambda^{m_0})r}{s^2\Phi(s\lambda, m_0 - 1) + s(s\lambda)^{m_0 - 1}}. \tag{2.1}$$

Then, there exists a sequence $\{x_k\}$ in $B(\bar{x}, r)$ of successive approximations of F starting from $\bar{x}$ converging to some $x^* \in X$ such that $x^* \in \mathrm{Fix}(F) \cap$

cl$(B(\bar{x}, r))$ *and*

$$D(\bar{x}, \mathrm{Fix}(F)) \leq \frac{s^3\Phi(s\lambda, m_0 - 1) + s^2(s\lambda)^{m_0-1}}{1 - s\lambda^{m_0}} D(\bar{x}, F(\bar{x})). \tag{2.2}$$

If, in addition, d is a lower semicontinuous function, then $\{x_k\}$ converges to some $x^* \in \mathrm{Fix}(F) \cap B(\bar{x}, r)$ and it holds

$$D(\bar{x}, \mathrm{Fix}(F) \cap B(\bar{x}, r)) \leq \frac{s^2\Phi(s\lambda, m_0 - 1) + s(s\lambda)^{m_0-1}}{1 - s\lambda^{m_0}} D(\bar{x}, F(\bar{x})). \tag{2.3}$$

Proof. If $\lambda = 0$, then $m_0 = 1$ and (2.1) becomes $D(\bar{x}, F(\bar{x})) < r/s$. Let $\theta > 0$ be such that $D(\bar{x}, F(\bar{x})) < \theta < r/s$. Set $x_0 = \bar{x}$. There exists $x_1 \in F(x_0)$ such that $d(x_0, x_1) < \theta < r$. So $x_1 \in F(x_0) \cap B(\bar{x}, r)$. Since F is an almost pseudo-contraction with constant $\lambda = 0$, we have

$$D(x_1, F(x_1)) \leq \ell D(x_1, F(x_0) \cap B(\bar{x}, r)) = 0.$$

Together with the closedness of $F(x_1)$, this implies that $x_1 \in F(x_1)$, i.e., x_1 is a fixed point of F. Now, let $\{x_k\}$ be the sequence defined by: $x_0 = \bar{x}$ and $x_k = x_1$ for all $k \in \mathbb{N}$. Then, $x_k \in B(\bar{x}, r)$ for all $k \geq 0$ and $\{x_k\}$ converges to $x^* = x_1 \in B(\bar{x}, r)$. Moreover,

$$D(\bar{x}, \mathrm{Fix}(F) \cap B(\bar{x}, r)) \leq d(\bar{x}, x_1) < \theta.$$

Taking θ tend to $D(\bar{x}, F(\bar{x}))$, we get $D(\bar{x}, \mathrm{Fix}(F) \cap B(\bar{x}, r)) \leq D(\bar{x}, F(\bar{x}))$. Consequently, both (2.2) and (2.3) are fulfilled, as in this case they reduce to $D(\bar{x}, \mathrm{Fix}(F) \cap B(\bar{x}, r)) \leq s^2 D(\bar{x}, F(\bar{x}))$ and $D(\bar{x}, \mathrm{Fix}(F) \cap B(\bar{x}, r)) \leq sD(\bar{x}, F(\bar{x}))$, respectively.

Assume now that $\lambda \neq 0$. Let $\beta > 0$ be such that

$$D(\bar{x}, F(\bar{x})) < \beta < \frac{(1 - s\lambda^{m_0})r}{s^2\Phi(s\lambda, m_0 - 1) + s(s\lambda)^{m_0-1}}. \tag{2.4}$$

Then, one can easily see that $\beta < r$. We will construct, by induction, a sequence $\{x_k\}$ of elements in X with $x_0 = \bar{x}$ such that, for all $k \in \mathbb{N}$,

$$D(x_k, F(x_k)) < \beta\lambda^k, \tag{2.5}$$

$$x_{k+1} \in F(x_k) \cap B(\bar{x}, r) \tag{2.6}$$

and

$$d(x_k, x_{k+1}) < \beta\lambda^k. \tag{2.7}$$

Indeed, by (2.4), there exists $x_1 \in F(\bar{x})$ such that $d(\bar{x}, x_1) < \beta < r$. So, $x_1 \in F(x_0) \cap B(\bar{x}, r)$. Hence, (2.5), (2.6) and (2.7) hold for $k = 0$.

Assume now that, for some positive integer n, we have constructed $x_0, x_1, \cdots, x_n$ satisfying (2.5), (2.6) and (2.7) for $k = 0, 1, \cdots, n-1$. By the inductive hypothesis, one has $x_n \in F(x_{n-1}) \cap B(\bar{x}, r)$. Since F is almost pseudo-contractive,

$$\begin{aligned} D(x_n, F(x_n)) &\le e(F(x_{n-1}) \cap B(\bar{x}, r), F(x_n)) \\ &\le \lambda d(x_{n-1}, x_n) + \ell D(x_n, F(x_{n-1}) \cap B(\bar{x}, r)) = \lambda d(x_{n-1}, x_n). \end{aligned}$$

Then, by the inductive hypothesis, we get $D(x_n, F(x_n)) < \beta\lambda^n$. This leads to the existence of $x_{n+1} \in F(x_n)$ such that $d(x_n, x_{n+1}) < \beta\lambda^n$.

We will show that $x_{n+1} \in B(\bar{x}, r)$. Using the inductive hypothesis and the latter inequality, for all integers m and i with $0 \le m < m + i \le n + 1$, we have

$$\begin{aligned} d(x_m, x_{m+i}) &\le d(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+i}) \le sd(x_m, x_{m+1}) + \\ &+ s^2 d(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+i}) \le sd(x_m, x_{m+1}) + \\ &+ s^2 d(x_{m+1}, x_{m+2}) + \cdots + s^{i-1} d(x_{m+i-2}, x_{m+i-1}) + s^{i-1} d(x_{m+i-1}, x_{m+i}) \\ &< s\beta\lambda^m + s^2\beta\lambda^{m+1} + \cdots + s^{i-1}\beta\lambda^{m+i-2} + s^{i-1}\beta\lambda^{m+i-1} \\ &= s\beta\lambda^m \left[1 + s\lambda + \cdots + (s\lambda)^{i-2}\right] + \beta\lambda^m (s\lambda)^{i-1} \\ &= \beta\lambda^m \left[s\Phi(s\lambda, i-1) + (s\lambda)^{i-1}\right]. \end{aligned} \tag{2.8}$$

Set $q = \left[\frac{n+1}{m_0}\right]$ the integer part of $\frac{n+1}{m_0}$. Then, $0 \le n + 1 - qm_0 \le m_0 - 1$. If $n + 1$ is divisible by m_0, then $d(x_{qm_0}, x_{n+1}) = 0$. If $n + 1$ is not divisible by n, then $m_0 > 1$ and $n + 1 - qm_0 \ge 1$. In this case, $s\lambda \ge 1$ and $(s\lambda)^{n-qm_0} \le (s\lambda)^{m_0-1}$. Using (2.8), we have

$$\begin{aligned} d(x_{qm_0}, x_{n+1}) &< \beta\lambda^{qm_0} \left[s\Phi(s\lambda, n - qm_0) + (s\lambda)^{n-qm_0}\right] \\ &\le \beta\lambda^{qm_0} \left[s\Phi(s\lambda, m_0 - 1) + (s\lambda)^{m_0-1}\right]. \end{aligned}$$

Thus, in all cases, we have

$$d(x_{qm_0}, x_{n+1}) < \beta\lambda^{qm_0} \left[s\Phi(s\lambda, m_0 - 1) + (s\lambda)^{m_0-1}\right]. \tag{2.9}$$

Now, using (2.8), (2.9) and the fact that $s\lambda^{m_0} < 1$, one has

$$\begin{aligned} d(x_0, x_{n+1}) &\le sd(x_0, x_{m_0}) + s^2 d(x_{m_0}, x_{2m_0}) + \cdots \\ &+ s^q d(x_{(q-1)m_0}, x_{qm_0}) + s^q d(x_{qm_0}, x_{n+1}) < s\beta \left[s\Phi(s\lambda, m_0 - 1) + (s\lambda)^{m_0-1}\right] \\ &+ s^2\beta\lambda^{m_0} \left[s\Phi(s\lambda, m_0 - 1) + (s\lambda)^{m_0-1}\right] \\ &+ \cdots + s^q\beta\lambda^{(q-1)m_0} \left[s\Phi(s\lambda, m_0 - 1) + (s\lambda)^{m_0-1}\right] \\ &+ s^q\beta\lambda^{qm_0} \left[s\Phi(s\lambda, m_0 - 1) + (s\lambda)^{m_0-1}\right] \le s\beta \left[s\Phi(s\lambda, m_0 - 1) + (s\lambda)^{m_0-1}\right] \\ &\times \left[1 + s\lambda^{m_0} + \cdots + (s\lambda^{m_0})^{q-1} + (s\lambda^{m_0})^q\right] \\ &\le \frac{\beta \left[s^2\Phi(s\lambda, m_0 - 1) + s(s\lambda)^{m_0-1}\right]}{1 - s\lambda^{m_0}} < r. \end{aligned}$$

So, $x_{n+1} \in B(\bar{x}, r)$. Hence, the construction of the sequence $\{x_k\}$ satisfying (2.5), (2.6) and (2.7) is complete.

Arguing similarly to the proof of (2.8), we can show that for all integers n and i with $0 \leq n < n+i$ that

$$d(x_n, x_{n+i}) < \beta\lambda^n \left[s\Phi(s\lambda, i-1) + (s\lambda)^{i-1}\right]. \tag{2.10}$$

Let k, n be nonnegative integers with $k \geq 1$ and set $p = [k/m_0]$. Using (2.10) and proceeding similarly to the proof of (2.9), we obtain

$$d(x_{n+pm_0}, x_{n+k}) < \beta\lambda^{n+pm_0} \left[s\Phi(s\lambda, m_0 - 1) + (s\lambda)^{m_0-1}\right]. \tag{2.11}$$

By (2.10) and (2.11), we have

$$\begin{aligned} d(x_n, x_{n+k}) &\leq sd(x_n, x_{n+m_0}) + s^2 d(x_{n+m_0}, x_{n+2m_0}) + \cdots \\ &\quad + s^p d(x_{n+(p-1)m_0}, x_{n+pm_0}) + s^p d(x_{n+pm_0}, x_{n+k}) \\ &< s\beta\lambda^n \left[s\Phi(s\lambda, m_0-1) + (s\lambda)^{m_0-1}\right] \\ &\quad + s^2\beta\lambda^{n+m_0}\left[s\Phi(s\lambda, m_0-1) + (s\lambda)^{m_0-1}\right] \\ &\quad + \cdots + s^p\beta\lambda^{n+(p-1)m_0}\left[s\Phi(s\lambda, m_0-1) + (s\lambda)^{m_0-1}\right] \\ &\quad + s^p\beta\lambda^{n+pm_0}\left[s\Phi(s\lambda, m_0-1) + (s\lambda)^{m_0-1}\right] \\ &\leq s\beta\lambda^n\left[s\Phi(s\lambda, m_0-1) + (s\lambda)^{m_0-1}\right] \\ &\quad \times \left[1 + s\lambda^{m_0} + \cdots + (s\lambda^{m_0})^{p-1} + (s\lambda^{m_0})^p\right] \\ &\leq \frac{\beta\lambda^n[s^2\Phi(s\lambda, m_0-1) + s(s\lambda)^{m_0-1}]}{1 - s\lambda^{m_0}}. \end{aligned} \tag{2.12}$$

Let $n = 0$ in (2.12), one has for all $k \geq 1$ that

$$d(x_0, x_k) < \frac{\beta[s^2\Phi(s\lambda, m_0-1) + s(s\lambda)^{m_0-1}]}{1 - s\lambda^{m_0}}. \tag{2.13}$$

It follows from (2.12) that $\{x_k\}$ is a Cauchy sequence in X. By the completeness of X, the Cauchy sequence $\{x_k\}$ converges to some $x^* \in X$. Moreover,

$$\begin{aligned} D(x^*, F(x^*)) &\leq s[d(x^*, x_k) + D(x_k, F(x^*))] \\ &\leq s[d(x_k, x^*) + e(F(x_{k-1}) \cap B(\bar{x}, r), F(x^*))] \\ &\leq sd(x_k, x^*) + s\lambda d(x_{k-1}, x^*) + s\ell D(x^*, F(x_{k-1}) \cap B(\bar{x}, r)) \\ &\leq s(1+\ell)d(x_k, x^*) + s\lambda d(x_{k-1}, x^*). \end{aligned}$$

By passing $k \to \infty$, we get $D(x^*, F(x^*)) \leq 0$ and thus $D(x^*, F(x^*)) = 0$. The closedness of $F(x^*)$ implies $x^* \in F(x^*)$, i.e., $x^* \in \mathrm{Fix}(F)$. Since $\{x_n\}$ is in $B(\bar{x}, r)$, one has $x^* \in \mathrm{cl}(B(\bar{x}, r))$. Moreover, using (2.13), we have

$$\begin{aligned} d(\bar{x}, x^*) &\leq sd(\bar{x}, x_k) + sd(x_k, x^*) \\ &< \frac{\beta[s^3\Phi(s\lambda, m_0-1) + s^2(s\lambda)^{m_0-1}]}{1 - s\lambda^{m_0}} + sd(x_k, x^*). \end{aligned}$$

Letting $k \to \infty$, one gets

$$d(\bar{x}, x^*) \leq \frac{\beta[s^3\Phi(s\lambda, m_0 - 1) + s^2(s\lambda)^{m_0-1}]}{1 - s\lambda^{m_0}}. \tag{2.14}$$

This, together with (2.4), implies that $d(\bar{x}, x^*) < sr$, i.e., $x^* \in B(\bar{x}, sr)$. Thus, $x^* \in \mathrm{Fix}(F) \cap B(\bar{x}, sr)$. Thus, by (2.14), one has

$$D(\bar{x}, \mathrm{Fix}(F)) \leq d(\bar{x}, x^*) \leq \frac{\beta[s^3\Phi(s\lambda, m_0 - 1) + s^2(s\lambda)^{m_0-1}]}{1 - s\lambda^{m_0}}.$$

Letting β tend to $D(\bar{x}, F(\bar{x}))$, we obtain (2.2).

If d is lower semicontinuous, then it follows from (2.13) that

$$d(\bar{x}, x^*) \leq \liminf_{k\to\infty} d(\bar{x}, x_k) \leq \frac{\beta[s^2\Phi(s\lambda, m_0 - 1) + s(s\lambda)^{m_0-1}]}{1 - s\lambda^{m_0}}.$$

Combining with (2.4), we get $x^* \in B(\bar{x}, r)$ and (2.3). This completes the proof. □

Remark 1. (i) In Theorem 2, if $\lambda \in (0, 1/s)$, i.e., $m_0 = 1$, then condition (2.1) reduces to

$$D(\bar{x}, F(\bar{x})) < \frac{(1 - s\lambda)r}{s},$$

which is more flexible than condition (1.1) in Theorem 1.

(ii) In the proof of Theorem 2 for the case $\lambda = 0$, we obtain the estimate

$$D(\bar{x}, \mathrm{Fix}(F) \cap B(\bar{x}, r)) \leq D(\bar{x}, F(\bar{x})),$$

which is sharper than both (2.2) and (2.3), although it does not require the lower semicontinuity of d.

(iii) Theorem 2 improves and extends Theorem 1 to the case $\lambda \in [0, 1)$, and thus represents a generalization of several results obtained in [6; 7; 13; 14; 21].

We now present an example to illustrate Theorem 2.

Example 1. Let $X = \left\{\frac{1}{2^n} : n = 0, 1, 2 \cdots\right\} \cup \{0\}$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x, y) = (x - y)^2$ for all $x, y \in X$. Then, (X, d) is a complete quasimetric space with constant $s = 2$ and d is continuous. We define $F : X \to P(X)$ by

$$F(x) = \begin{cases} \{0\} & \text{if} \quad x = 0, \\ \left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\right\} & \text{if} \quad x = \frac{1}{2^n}, \quad n = 1, 2, \cdots \\ \{1\} & \text{if} \quad x = 1. \end{cases}$$

Let $\bar{x} = 1/4$ and $r = 1$. We show that F is an almost pseudo-contraction on $B(\bar{x}, r)$ with constants $\lambda = \dfrac{1}{4}$ and $\ell = 4$, i.e.,

$$e(F(x) \cap B(\bar{x}, r), F(y)) \leq \frac{1}{4} d(x, y) + 4D(y, F(x) \cap B(\bar{x}, r)), \tag{2.15}$$

for all $x, y \in B(\bar{x}, r)$. We have

$$B(\bar{x}, r) = \{x \in X : d(\bar{x}, x) < r\} = \{x \in X : (x - 1/4)^2 < 1\} = X.$$

Hence, we have the following cases.

Case 1. $x = 0$ and $y = 1$. Then, $e(F(0), F(1)) = e(\{0\}, \{1\}) = 1$ and $D(1, F(0)) = D(1, \{0\}) = 1$. Thus, (2.15) holds.

Case 2. $x = 1$ and $y = 0$. Then, $e(F(1), F(0)) = e(\{1\}, \{0\}) = 1$ and $D(0, F(1)) = D(0, \{1\}) = 1$. Thus, (2.15) holds.

Case 3. $x = 0$ and $y = \dfrac{1}{2^n}$ with $n \geq 1$. Then, (2.15) holds since

$$e\left(F(0), F\left(\frac{1}{2^n}\right)\right) = e\left(\{0\}, \left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\right\}\right) = \frac{1}{2^{2n+4}}$$

and

$$D\left(\frac{1}{2^n}, F(0)\right) = \frac{1}{2^{2n}}.$$

Case 4. $x = \dfrac{1}{2^n}$ and $y = 0$ with $n \geq 1$. Then, (2.15) holds since

$$e\left(F\left(\frac{1}{2^n}\right), F(0)\right) = e\left(\left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\right\}, \{0\}\right) = \frac{1}{2^{2n+2}}$$

and

$$D\left(0, F\left(\frac{1}{2^n}\right)\right) = \frac{1}{2^{2n+4}}.$$

Case 5. $x = 1$ and $y = \dfrac{1}{2^n}$ with $n \geq 1$. Then, (2.15) holds since

$$e\left(F(1), F\left(\frac{1}{2^n}\right)\right) = e\left(\{1\}, \left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\right\}\right) = \left(1 - \frac{1}{2^{n+1}}\right)^2$$

and

$$D\left(\frac{1}{2^n}, F(1)\right) = \left(1 - \frac{1}{2^n}\right)^2.$$

Case 6. $x = \dfrac{1}{2^n}$ and $y = 1$ with $n \geq 1$. Then, (2.15) holds since

$$e\left(F\left(\frac{1}{2^n}\right), F(1)\right) = e\left(\left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\right\}, \{1\}\right) = \left(1 - \frac{1}{2^{n+2}}\right)^2$$

and

$$D\left(1, F\left(\frac{1}{2^n}\right)\right) = \left(1 - \frac{1}{2^{n+1}}\right)^2.$$

Case 7. $x = \dfrac{1}{2^n}$ and $y = \dfrac{1}{2^m}$ with $n \geq m \geq 1$. Then, (2.15) holds since

$$\begin{aligned} e\left(F\left(\frac{1}{2^n}\right), F\left(\frac{1}{2^m}\right)\right) &= e\left(\left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\right\}, \left\{\frac{1}{2^{m+1}}, \frac{1}{2^{m+2}}\right\}\right) \\ &= \left(\frac{1}{2^{n+2}} - \frac{1}{2^{m+2}}\right)^2 = \frac{1}{2^{2n+4}}\left(1 - \frac{1}{2^{n-m}}\right)^2 \end{aligned}$$

and

$$D\left(\frac{1}{2^m}, F\left(\frac{1}{2^n}\right)\right) = \left(\frac{1}{2^m} - \frac{1}{2^{n+1}}\right)^2 = \frac{1}{2^{2m}}\left(1 - \frac{1}{2^{n+1-m}}\right)^2.$$

Case 8. $x = \dfrac{1}{2^n}$ and $y = \dfrac{1}{2^m}$ with $m > n \geq 1$. Then, (2.15) holds since

$$\begin{aligned} e\left(F\left(\frac{1}{2^n}\right), F\left(\frac{1}{2^m}\right)\right) &= e\left(\left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\right\}, \left\{\frac{1}{2^{m+1}}, \frac{1}{2^{m+2}}\right\}\right) \\ &= \left(\frac{1}{2^{n+1}} - \frac{1}{2^{m+1}}\right)^2 = \frac{1}{2^{2n+2}}\left(1 - \frac{1}{2^{m-n}}\right)^2 \end{aligned}$$

and

$$d\left(\frac{1}{2^n}, \frac{1}{2^m}\right) = \left(\frac{1}{2^n} - \frac{1}{2^m}\right) = \frac{4}{2^{2n+2}}\left(1 - \frac{1}{2^{m-n}}\right)^2.$$

Therefore, F is a multi-valued almost pseudo-contraction on $B(\bar{x}, r)$ with constant $\lambda = \dfrac{1}{4}$ and $\ell = 4$. We have

$$D(\bar{x}, F(\bar{x})) = D\left(\frac{1}{4}, F\left(\frac{1}{4}\right)\right) = \frac{1}{8},$$

and

$$\frac{(1 - s\lambda^{m_0})r}{s^2\Phi(s\lambda, m_0 - 1) + s(s\lambda)^{m_0-1}} = \frac{(1 - s\lambda)r}{s} = \frac{1}{4}.$$

Thus, (2.1) is satisfied, and all the conditions of Theorem 2 are fulfilled. We can see that $\mathrm{Fix}(F) = \{0, 1\}$ and (2.3) holds since

$$D(\bar{x}, \mathrm{Fix}(F)) = \frac{1}{4} < \frac{1}{2} = \frac{s}{1 - s\lambda} D(\bar{x}, F(\bar{x})).$$

Note that F is not an almost pseudo-contraction on $B(\bar{x}, r)$ with $\lambda \in [0, 1)$ and $\ell = 0$. Assume to the contrary that F is pseudo-contractive on $B(\bar{x}, r)$ with constant $\lambda \in [0, 1)$. Then, for every $n \geq 1$,

$$e\left(F(1) \cap B(\bar{x}, r), F\left(\frac{1}{2^n}\right)\right) \leq \lambda d\left(1, \frac{1}{2^n}\right),$$

equivalently,

$$\left(1-\frac{1}{2^{n+1}}\right)^2 \le \alpha\left(1-\frac{1}{2^n}\right)^2.$$

Letting $n \to \infty$, we get $1 \le \lambda$, which contradicts $\lambda < 1$. Thus, F is not a pseudo-contraction on $B(\bar{x}, r)$.

In Theorem 2, if we take $r = +\infty$, then we get the following fixed point result for a multi-valued almost pseudo-contraction.

Corollary 1. *Let (X, d) be a complete quasimetric space with constant $s \ge 1$ and $F : X \to C(X)$ be a multi-valued almost pseudo-contraction with constants $\lambda \in [0, 1)$ and $\ell \ge 0$. Set $m_0 = \min\{m \in \mathbb{N} : s\lambda^m < 1\}$. Then,* $\mathrm{Fix}(F) \neq \emptyset$ *and*

$$D(x, \mathrm{Fix}(F)) \le \frac{s^3\Phi(s\lambda, m_0-1) + s^2(s\lambda)^{m_0-1}}{1-s\lambda^{m_0}} D(x, F(x)), \quad \forall x \in X.$$

If, in addition, d is lower semicontinuous, then

$$D(x, \mathrm{Fix}(F)) \le \frac{s^2\Phi(s\lambda, m_0-1) + s(s\lambda)^{m_0-1}}{1-s\lambda^{m_0}} D(x, F(x)), \quad \forall x \in X.$$

The following example illustrates Corollary 1.

Example 2. Let $X = [0, \infty)$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x, y) = (x-y)^2$ for all $x, y \in X$. Then, (X, d) is a complete quasimetric space with constant $s = 2$, and d is continuous. Define $F : X \to P(X)$ by

$$F(x) = \left[\frac{x}{2}, \frac{9x}{10}\right] \quad \text{for all } x \in X.$$

For $x, y \in X$, we have

$$e(F(x), F(y)) = e\left(\left[\frac{x}{2}, \frac{9x}{10}\right], \left[\frac{y}{2}, \frac{9y}{10}\right]\right) = \begin{cases} \dfrac{(x-y)^2}{4} & \text{if} \quad x \le y, \\ \dfrac{81(x-y)^2}{100} & \text{if} \quad x > y. \end{cases}$$

Thus,

$$e(F(x), F(y)) \le \frac{81}{100} d(x, y), \ \forall x, y \in X.$$

That is, F is a pseudo-contraction with constant $\lambda = 0.81$. Moreover, we obtain $m_0 = \min\{m \in \mathbb{N} : s\lambda^m < 1\} = 4$ and the fixed point of F is $\mathrm{Fix}(F) = \{0\}$. For each $x \in X$, we have $D(x, \mathrm{Fix}(F)) = x^2$ and $D(x, F(x)) = x^2/100$ and, by direct computation, one can verify that

$$D(x, \mathrm{Fix}(F)) \le \frac{s^2\Phi(s\lambda, m_0-1) + s(s\lambda)^{m_0-1}}{1-s\lambda^{m_0}} D(x, F(x)).$$

From the proof of Theorem 2, we can derive the following data dependence result for the fixed point set of a multi-valued almost pseudo-contraction that extends [22, Theorem 3.2] to the case $\lambda \in [0, 1)$ and thus generalizes the data dependence results presented in [6; 19].

Theorem 3. *Let (X, d) be a complete quasimetric space with constant $s \geq 1$, $\bar{x} \in X$ and $r > 0$. Let $F : X \to C(X)$ be a multi-valued almost pseudo-contraction on $B(\bar{x}, sr)$ with constants $\lambda \in [0, 1)$ and $\ell \geq 0$. Set $m_0 = \min\{m \in \mathbb{N} : s\lambda^m < 1\}$ and assume that $G : X \to P(X)$ is a multi-valued mapping such that for any $t \in (0, r)$ we have $\mathrm{Fix}(G) \cap B(\bar{x}, t) \neq \emptyset$ and*

$$e(G(x), F(x)) < \frac{(1 - s\lambda^{m_0})(r - t)}{s^2\Phi(s\lambda, m_0 - 1) + s(s\lambda)^{m_0 - 1}}, \quad \forall x \in B(\bar{x}, t).$$

Then $\mathrm{Fix}(F) \neq \emptyset$ and for each $t \in (0, r)$

$$e(\mathrm{Fix}(G) \cap B(\bar{x}, t), \mathrm{Fix}(F)) \leq s(r - t).$$

Proof. Let $t \in (0, r)$ and let $y \in \mathrm{Fix}(G) \cap B(\bar{x}, t)$. For any $z \in B(y, r - t)$, we have

$$d(\bar{x}, z) \leq s[d(\bar{x}, y) + d(y, z)] < s[t + r - t] = sr.$$

This implies that $z \in B(\bar{x}, sr)$ and thus $B(y, r - t) \subset B(\bar{x}, sr)$. It follows that F is a multi-valued almost pseudo-contraction on $B(y, r - t)$ with constants λ and ℓ. Moreover,

$$\begin{aligned} D(y, F(y)) &\leq e(G(y) \cap B(y, t), F(y)) \leq e(G(y), F(y)) \\ &< \frac{(1 - s\lambda^{m_0})(r - t)}{s^2\Phi(s\lambda, m_0 - 1) + s(s\lambda)^{m_0 - 1}}. \end{aligned}$$

Then, by Theorem 2, $\mathrm{Fix}(F)$ is nonempty and

$$D(y, \mathrm{Fix}(F)) \leq \frac{s^3\Phi(s\lambda, m_0 - 1) + s^2(s\lambda)^{m_0 - 1}}{1 - s\lambda^{m_0}} D(y, F(y)) < s(r - t).$$

Taking the supremum over $y \in \mathrm{Fix}(G) \cap B(\bar{x}, t)$, we get

$$e(\mathrm{Fix}(G) \cap B(\bar{x}, t), \mathrm{Fix}(F)) \leq s(r - t).$$

This ends the proof. □

The following result is an extension of Lim's lemma [19, Lemma 1] to multi-valued almost pseudo-contractive mappings in the setting of quasimetric spaces.

Theorem 4. *Let (X, d) be a complete quasimetric space with constant $s \geq 1$ and $F_1, F_2 : X \to C(X)$ be multi-valued almost pseudo-contractions with* *the same constants* $\lambda \in [0,1)$ *and* $\ell \geq 0$*. Set* $m_0 = \min\{m \in \mathbb{N} : s\lambda^m < 1\}$*. Then,* $\mathrm{Fix}(F_1) \neq \emptyset$*,* $\mathrm{Fix}(F_2) \neq \emptyset$ *and*

$$e(\mathrm{Fix}(F_1), \mathrm{Fix}(F_2)) \leq \frac{s^4\Phi(s\lambda, m_0-1) + s^3(s\lambda)^{m_0-1}}{1-s\lambda^{m_0}} \sup_{x\in X} e(F_1(x), F_2(x)). \tag{2.16}$$

and

$$H(\mathrm{Fix}(F_1), \mathrm{Fix}(F_2)) \leq \frac{s^4\Phi(s\lambda, m_0-1) + s^3(s\lambda)^{m_0-1}}{1-s\lambda^{m_0}} \sup_{x\in X} H(F_1(x), F_2(x)). \tag{2.17}$$

Proof. By Corollary 1, we have that $\mathrm{Fix}(F_1)$ and $\mathrm{Fix}(F_2)$ are nonempty. Moreover, for any $x \in X$,

$$D(x, \mathrm{Fix}(F_2)) \leq \frac{s^3\Phi(s\lambda, m_0-1) + s^2(s\lambda)^{m_0-1}}{1-s\lambda^{m_0}} D(x, F_2(x)) =: \theta D(x, F_2(x)).$$

For any $\eta > 0$, there exists $y \in F_1(x)$ such that $d(x,y) < D(x, F_1(x)) + \eta$. Thus,

$$\begin{aligned} D(x, \mathrm{Fix}(F_2)) &\leq \theta D(x, F_2(x)) \leq \theta s[d(x,y) + D(y, F_2(x))] \\ &< s\theta[D(x, F_1(x)) + \eta + e(F_1(x), F_2(x))]. \end{aligned}$$

Letting $\eta \to 0^+$, one gets $D(x, \mathrm{Fix}(F_2)) \leq s\theta[D(x, F_1(x)) + e(F_1(x), F_2(x))]$. Taking the supremum on both sides of the latter inequalities over $x \in \mathrm{Fix}(F_1)$, we have

$$\begin{aligned} e(\mathrm{Fix}(F_1), \mathrm{Fix}(F_2)) &\leq s\theta \sup_{x\in \mathrm{Fix}(F_1)} [D(x, F_1(x)) + e(F_1(x), F_2(x))] \\ &\leq s\theta \sup_{x\in X} e(F_1(x), F_2(x)). \end{aligned}$$

This proves (2.16). Estimation (2.17) follows (2.16) and the definition of the Hausdorff-Pompeiu distance. □

3. Conclusion

In this paper, we studied the theory of fixed points for multi-valued almost pseudo-contractive mappings within the framework of quasimetric spaces. Building upon and generalizing previous results, we established a new fixed point theorem that applies to a broader class of contractive constants $\lambda \in [0,1)$, thereby relaxing the more restrictive condition $\lambda < 1/s$ assumed in earlier works. Our results also include quantitative estimates for the distance to the set of fixed points. Furthermore, we derived data dependence results that demonstrate the stability of fixed points under perturbations of the mapping, thereby extending existing results. These findings offer a deeper understanding of the structure and behavior of multi-valued almost pseudo-contractive mappings in generalized distance settings. Overall, our work provides a significant generalization of fixed point theory in quasimetric spaces and may be useful for future studies involving variational analysis, optimization, or numerical methods.

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