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Noncommutative Products on Categories and Chu Construction

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Abstract: The category of Chu spaces is constructed from a given symmetric monoidal closed category K and a fixed object in it. If the object is not fixed, we obtain the category $Chu(K)$, whose objects are Chu spaces and whose morphisms are defined in a more general way. E.E. Skurikhin defined and studied the category of $\mathcal{T}$-Chu spaces associated with an arbitrary functor $\mathcal{T}$ from the product of categories N^{op} and M to the category of sets. In this paper, we prove that if the functor $\mathcal{T}$ is closed, then the category M is isomorphic to a reflective subcategory of the category of $\mathcal{T}$-Chu spaces. In the case where the categories N and M are complete, we present constructions of limits and colimits of arbitrary functors with values in the category of $\mathcal{T}$-Chu spaces. We also prove that if K is a symmetric monoidal closed category, then $Chu(K)$ admits the structure of a closed right-monoidal category. As a consequence of the results on $\mathcal{T}$-Chu spaces, it follows that the category $Chu(K)$, as well as the categories of Chu spaces over it, are complete and cocomplete whenever K is.

Keywords: Chu spaces, Chu construction, monoidal category, completeness

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Научная статья

Некоммутативные произведения на категориях и конструкция Чу

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Аннотация: Категория пространств Чу строится по данной замкнутой моноидальной категории K и фиксированному объекту в ней. Если объект не фиксировать, появляется категория $Chu(K)$, объектами которой являются пространства Чу, а морфизмы определяются более общим образом. Е. Е. Скурихин определил и изучил категорию $\mathcal{T}$-пространств Чу, сопоставляющуюся произвольному функтору $\mathcal{T}$ на произведении категорий N^{op} и M со значением в категории множеств. Доказывается, что в случае, когда функтор $\mathcal{T}$ замкнут, категория M изоморфна рефлективной подкатегории категории $\mathcal{T}$-пространств Чу. Для случая, когда категории N и M полны, приведены конструкции пределов и копределов произвольных функторов со значениями в категории $\mathcal{T}$-пространств Чу. Доказано, что если K — замкнутая моноидальная симметрическая категория, то $Chu(K)$ снабжается структурой замкнутой правомоноидальной категории. Как следствие результатов о $\mathcal{T}$-пространствах Чу получаем, что категория $Chu(K)$, а также категории пространств Чу над ней полны и кополны, если таковой является категория K.

Ключевые слова: пространства Чу, конструкция Чу, моноидальная категория, полнота

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1. Introduction

In [2], Po-Hsiang Chu proposed a construction that allows building a new category (later called the category of Chu spaces) from a given symmetric monoidal closed category K and a fixed object in it. If the object is not fixed, we obtain the category $Chu(K)$ of Chu spaces defined in the work of A.A. Stepanova, E.E. Skurikhin, and A.G. Sukhonos [14], where objects are Chu spaces and morphisms are defined in a more general way. I. Petrakis [6] constructed a category of generalized Chu spaces from a category with products and an endofunctor acting on it. And if the endofunctor is identity, the resulting category coincides with $Chu(K)$. Chu spaces have been studied both for category-theoretic needs and due to their emerging interpretations of various mathematical and natural science concepts [1,3,5,7-13]. E.E. Skurikhin defined and studied the category $\mathcal{T}Chu$ of generalized Chu spaces associated with an arbitrary functor $\mathcal{T} : N^{op} \times M \to Set$. This work presents the corresponding definitions and results. Special cases with different functors $\mathcal{T}$ and $M = K^{op} \times K$ or $M = K^{op}$ yield the aforementioned generalizations of both Chu spaces themselves and their morphisms. The main results concern the case when the functor $\mathcal{T}$ is $\mathcal{H}$-closed. We prove that in this case, the category M is isomorphic to a reflective subcategory of $\mathcal{T}Chu$. For complete categories N and M, we provide constructions of limits and colimits for arbitrary functors with values in $\mathcal{T}Chu$, thereby proving its completeness. An important property of the category of Chu spaces [2] is the presence on it of the structure of a closed symmetric monoidal category. In this work, we define a tensor product on $Chu(K)$ that is non-commutative but associative and has a right unit, making $Chu(K)$ a right-monoidal category. We prove that this category is closed and that the category of Chu spaces over it, $Chu(Chu(K))$, is complete and cocomplete whenever K is.

2. Category $\mathcal{T}Chu$

Let K be a category. The class of all objects of the category K is denoted by $Ob(K)$, and the set of all morphisms from object c to object c' is denoted by $Hom_K(c, c')$. The dual category is denoted by K^{op}. The identity of object c is denoted by 1_c, so that $h \circ 1_c = h = 1_{c'} \circ h$.

Let N, M be categories,

$$\mathcal{T} : N^{op} \times M \to Set \text{ be a functor, } h \in Hom_N(n', n), u \in Hom_M(m, m').$$

We will omit $\mathcal{T}$ in the notation associated with the mapping $\mathcal{T}(h, u) : \mathcal{T}(n, m) \to \mathcal{T}(n', m')$ and use the following abbreviations:

$$uah = \mathcal{T}(h, u)(a);\ ua = \mathcal{T}(1_n, u)(a);\ ah = \mathcal{T}(h, 1_m)(a), \tag{1}$$

where $a \in \mathcal{T}(n, m)$. Note that $uah \in \mathcal{T}(n', m')$, $ua \in \mathcal{T}(n, m')$, $ah \in \mathcal{T}(n', m)$.

Since $\mathcal{T}$ is a functor, then for any $h' \in Hom_N(n'', n')$, $u' \in Hom_M(m', m'')$ we have

$$u'(uah)h' = (u' \circ u)a(h \circ h');\ 1_m a 1_n = 1_m a = a 1_n = a; \tag{2}$$

$$u(ah) = uah = (ua)h; \quad u'(ua) = (u' \circ u)a; \quad (ah)h' = a(h \circ h'). \tag{3}$$

Let $\mathcal{H} : M \to N$ be a functor. By $\mathcal{T}_\mathcal{H} : N^{op} \times M \to Set$ we denote the functor $Hom_N(-, \mathcal{H}(-))$, so $\mathcal{T}_\mathcal{H}(n, m) = Hom_N(n, \mathcal{H}(m))$ and for $w \in Hom_N(n, \mathcal{H}(m)) = \mathcal{T}_\mathcal{H}(n, m)$ we have

$$\mathcal{T}_\mathcal{H}(h, u)(w) = uwh = \mathcal{H}(u) \circ w \circ h. \tag{4}$$

The functor $\mathcal{T}$ is called $\mathcal{H}$-*closed* if $\mathcal{T}$ is natural isomorphic to $\mathcal{T}_\mathcal{H}$.

For families of mappings $p = \{p_{n,m} : \mathcal{T}(n, m) \to Hom_N(n, \mathcal{H}(m)) \mid n \in Ob(N), m \in Ob(M)\}$, $q = \{q_{n,m} : Hom_N(n, \mathcal{H}(m)) \to \mathcal{T}(n, m) \mid n \in Ob(N), m \in Ob(M)\}$ we introduce notations

$$r_m = q_{\mathcal{H}(m),m}(1_\mathcal{H}(m)) \in \mathcal{T}(\mathcal{H}(m), m);\ \hat{r} = p_{n,m}(r) \in Hom_N(n, \mathcal{H}(m))$$

The following lemma formulates general properties of the functors $\mathcal{T}$ and $\mathcal{T}_\mathcal{H}$, and presents relations describing their natural transformations. The results follow directly from the definition of the natural transformation of functors and relations (1)–(4). They are used further in defining and studying $\mathcal{T}$Chu spaces and related products on categories.

Lemma 1. *Let N, M be categories, $\mathcal{T} : N^{op} \times M \to Set$, $\mathcal{T}_1 : N^{op} \times M \to Set$, $\mathcal{T}_2 : N^{op} \times M \to Set$ be functors.*

Let $x = \{x_{n,m} : \mathcal{T}_1(n, m) \to \mathcal{T}_2(n, m) \mid n \in Ob(N), m \in Ob(M)\}$ be a family of mappings of sets. Then x is a natural transformation of functors $\mathcal{T}_1 \to \mathcal{T}_2$ if and only if

$$x_{n',m'}(uah) = ux_{n,m}(a)h \tag{5}$$

for all $h \in Hom_N(n', n)$, $u \in Hom_M(m, m')$, $a \in \mathcal{T}_1(n, m)$. In particular, if $\mathcal{H} : M \to N$ is a functor, then the family of mappings $p = \{p_{n,m} : \mathcal{T}(n, m) \to Hom_N(n, \mathcal{H}(m)) \mid n \in Ob(N), m \in Ob(M)\}$ is a natural transformation $p : \mathcal{T} \to \mathcal{T}_\mathcal{H}$ if and only if $p_{n',m'}(urh) = \mathcal{H}(u) \circ p_{n,m}(r) \circ h$, that is,

$$\widehat{urh} = \mathcal{H}(u) \circ \widehat{r} \circ h \tag{6}$$

for all $h \in Hom_N(n', n)$, $u \in Hom_M(m, m')$, $r \in \mathcal{T}(n, m)$. The family $q = \{q_{n,m} : Hom_N(n, \mathcal{H}(m)) \to \mathcal{T}(n, m) \mid n \in Ob(N), m \in Ob(M)\}$ is a natural transformation $q : \mathcal{T}_\mathcal{H} \to \mathcal{T}$ if and only if

$$q_{n',m'}(\mathcal{H}(u) \circ w \circ h) = uq_{n,m}(w)h \tag{7}$$

for all $h \in Hom_N(n', n)$, $u \in Hom_M(m, m')$, $w \in Hom_N(n, \mathcal{H}(m))$.

Let $p : \mathcal{T} \to \mathcal{T}_\mathcal{H}$, $q : \mathcal{T}_\mathcal{H} \to \mathcal{T}$ be mutually inverse natural transformations, so for all n, m the mappings $p_{n,m}$ and $q_{n,m}$ are mutually inverse. We fix the notation

$$\widehat{r} = p_{n,m}(r) \in Hom_N(n, \mathcal{H}(m));\quad r_m = q_{\mathcal{H}(m),m}(1_{\mathcal{H}(m)}) \in \mathcal{T}(\mathcal{H}(m), m).$$

Then

(8) $\widehat{q_{n,m}(h)} = h; q_{n,m}(\widehat{r}) = r; r_1 = r_2 \Leftrightarrow \widehat{r_1} = \widehat{r_2}; q_{n,m}(h) = r_m h; r = r_m \widehat{r}.$

If $u \in Hom_M(m, m')$, $v \in Hom_N(n, n')$, $r \in \mathcal{T}(n, m)$, $r' \in \mathcal{T}(n', m;)$, *then*

(9) $\widehat{ur} = \mathcal{H}(u) \circ \widehat{r}; \quad \widehat{r'v} = \widehat{r'} \circ v; \quad ur = r'v \Leftrightarrow \mathcal{H}(u) \circ \widehat{r} = \widehat{r'} \circ v,$

and

$$ur = r_{m'}v \Leftrightarrow v = \mathcal{H}(u) \circ \widehat{r}; r = r_m v \Leftrightarrow v = \widehat{r}; ur_m = r_{m'}v \Leftrightarrow v = \mathcal{H}(u).$$

Definition (Categories of $\mathcal{T}Chu$ spaces). Let N, M be categories, $\mathcal{T} : N^{op} \times M \to Set$ be a functor. The category $\mathcal{T}Chu$ *of* $\mathcal{T}Chu$ *spaces* is defined as follows:

- $Ob(\mathcal{T}Chu)$ is the class of all triples (n, m, r), where $r \in \mathcal{T}(n, m)$ (the objects of $\mathcal{T}Chu$ are called $\mathcal{T}Chu$ *spaces*; we will often denote the $\mathcal{T}Chu$ space (n, m, r) as r);
- if $(n,m,r), (n', m', r') \in Ob(\mathcal{T}Chu)$, then $Hom_{\mathcal{T}Chu}((n,m,r), (n',m',r'))$ is the set of pairs (v, u) such that $v : n \to n'$, $u : m \to m'$ and

  (10) $ur = r'v, \text{ that is, } \mathcal{T}(1_n, u)(r) = \mathcal{T}(v, 1_{m'})(r');$

  a morphism $(v, u) : r \to r'$ is called *Chu transform* from r to r';
- if $(v, u) : (n, m, r) \to (n', m', r')$, $(v', u') : (n', m', r') \to (n'', m'', r'')$, then $(v', u') \circ (v, u) = (v' \circ v, u' \circ u)$;
- the identity of (n, m, r) is the morphism $1_{(n,m,r)} = (1_n, 1_m)$.

Let us show that the category $\mathcal{T}Chu$ is well defined, i.e., the composition of morphisms is a morphism of $\mathcal{T}Chu$. Indeed, if $(v, u) \in Hom_{\mathcal{T}Chu}(r, r')$, $(v', u') \in Hom_{\mathcal{T}Chu}(r', r'')$, then $ur = r'v$, $r''v' = u'r'$, and by (2), (3) we have $r''(v' \circ v) = (r''v')v = (u'r')v = u'(r'v) = u'(ur) = (u' \circ u)r$, that is $(v' \circ v, u' \circ u) \in Hom_{\mathcal{T}Chu}(r, r'')$.

3. Functors with values in the category $\mathcal{T}Chu$

Let $\mathcal{T} : N^{op} \times M \to Set$ be a functor. The projection functors

$$P_1 : \mathcal{T}Chu \to N, \quad P_2 : \mathcal{T}Chu \to M$$

are defined as follows: $P_1(n, m, r) = n$, $P_2(n, m, r) = m$ for any $\mathcal{T}Chu$ space (n, m, r), $P_1(v, u) = v \in Hom_N(n, n')$, $P_2(v, u) = u \in Hom_M(m, m')$ for any $(v, u) \in Hom_{\mathcal{T}Chu}((n, m, r), (n', m', r'))$.

Let Z be a category, $F : Z \to \mathcal{T}Chu$ be a functor, $F_1 = P_1 \circ F$, $F_2 = P_2 \circ F$. Then for all $z, z' \in Ob(Z)$, $f \in Hom_Z(z, z')$ we have

(11) $F(z) = (F_1(z), F_2(z), r(z)), \text{ where } r(z) \in \mathcal{T}(F_1(z), F_2(z))$

$F(f) = (F_1(f), F_2(f)) \in Hom_{\mathcal{T}Chu}(r(z), r(z'))$, that is,

$$F_2(f)r(z) = r(z')F_1(f). \tag{12}$$

The opposite is also true, namely, if $F_1 : Z \to N$, $F_2 : Z \to M$ be functors, and for each $z \in Ob(Z)$ we fix $r(z) \in \mathcal{T}(F_1(z), F_2(z))$, such that the equality (7) holds for any $f \in Hom_Z(z, z')$, then the mapping $F : Z \to \mathcal{T}Chu$, defined as follows:

$$F(z) = (F_1(z), F_2(z), r(z)), \quad F(f) = (F_1(f), F_2(f)), \tag{13}$$

is a functor.

In Proposition 1 there are the conditions describing functors with values in the category $\mathcal{T}Chu$, where $\mathcal{T}$ is an $\mathcal{H}$-closed functor.

Proposition 1. *Let $\mathcal{H} : M \to N$ be a functor, $\mathcal{T} : N^{op} \times M \to Set$ be an $\mathcal{H}$-closed functor, $p : \mathcal{T} \to \mathcal{T}_{\mathcal{H}}$, $q : \mathcal{T}_{\mathcal{H}} \to T$ be mutually inverse natural transformations.*

1) Let $F_1 : Z \to N$, $F_2 : Z \to M$ be functors and for each $z \in Ob(Z)$ we fix $r(z) \in \mathcal{T}(F_1(z), F_2(z))$. We denote $W = \{W(z) = \widehat{r(z)} \in Hom_N(F_1(z), \mathcal{H}(F_2(z))) \mid z \in Ob(Z)\}$. Then the mapping $F : Z \to \mathcal{T}Chu$ defined by equalities (13), i.e., $F(z) = (F_1(z), F_2(z), r(z))$, $F(f) = (F_1(f), F_2(f))$ is a functor if and only if W is a natural transformation $F_1 \to \mathcal{H} \circ F_2$.

2) If $F_1 : Z \to N$, $F_2 : Z \to M$ are functors, $W : F_1 \to \mathcal{H} \circ F_2$ is a natural transformation, and $r(z) = q_{F_1(z),F_2(z)}(W(z)) \in \mathcal{T}(F_1(z), F_2(z))$, then the mapping defined by the equalities (13) is a functor $F : Z \to \mathcal{T}Chu$.

Proof. 1) By the relations (12), (13), F is a functor iff

$$F_2(f)r(z) = r(z')F_1(f).$$

By (9), this is equivalent to the equality $F_2(f) \circ \widehat{r(z)} = \widehat{r(z')} \circ F_1(f)$, that is, $F_2(f) \circ W(z) = W(z') \circ F_1(f)$. Thus, $W : F_1 \to \mathcal{H} \circ F_2$ is a natural transformation.

2) follows from $\widehat{r(z)} = \widehat{q_{F_1(z),F_2(z)}(W(z))} = W(z)$ and 1). □

Theorem 1. *Let $\mathcal{H} : M \to N$ be a functor, $\mathcal{T} : N^{op} \times M \to Set$ be an $\mathcal{H}$-closed functor, $p : \mathcal{T} \to \mathcal{T}_{\mathcal{H}}$, $q : \mathcal{T}_{\mathcal{H}} \to \mathcal{T}$ be mutually inverse natural transformations.*

1) The mapping H defined as follows: $H(m) = (\mathcal{H}(m), m, r_m)$, $H(u) = (\mathcal{H}(u), u)$, where $m \in Ob(M)$, $u \in Hom_M(m, m')$, is a fully faithful functor $H : M \to \mathcal{T}Chu$, which is right adjoint to the functor $P_2 : \mathcal{T}Chu \to M$.

2) The full subcategory $\widetilde{M} = \{(\mathcal{H}(m), m, r_m) \mid m \in Ob(M)\}$ of the category $\mathcal{T}Chu$ is reflective and isomorphic to the category M.

Proof. 1) Let $H_1 = \mathcal{H} : M \to N$, $H_2 = 1_M : M \to M$, $m \in Ob(M)$ and $r(m) = r_m$. Then $H(m) = (H_1(m), H_2(m), r(m))$, $H(u) = (H_1(u), H_2(u))$, where $r(m) \in \mathcal{T}(H_1(m), H_2(m))$. By (8), (9), we have $H(u) = (\mathcal{H}(u), u) = (H_1(u), H_2(u)) \in Hom_{\mathcal{T}Chu}(r_m, r_{m'})$, so conditions (12) are satisfied, and H is a functor. Consider the mapping

$$H : Hom_M(m, m') \to Hom_{\mathcal{T}Chu}(r_m, r_{m'}), H(u) = (\mathcal{H}(u), u).$$

Clearly, it is injective. Let $(v, u) \in Hom_{\mathcal{T}Chu}(r_m, r_{m'})$ be a morphism. Then, by (9), $v = \mathcal{H}(u)$, that is, the mapping $u \mapsto H(u)$ is bijective, so H is fully faithful.

Proof of adjunction. For arbitrary $r \in \mathcal{T}(n, m)$, $m' \in Ob(M)$, we define the mapping $\psi_q(r, m') : Hom_{\mathcal{T}Chu}(r, H(m')) \to Hom_M(P_2(r), m')$ as follows: $\psi_q(r, m')(v, u) = u$ for any $(v, u) \in Hom_{\mathcal{T}Chu}(r, H(m'))$. One can directly verify that this mapping is natural in r and m'. To prove the adjunction between P_2 and H, it suffices to show that each $\psi_q(r, m')$ is bijective. We have $Hom_{\mathcal{T}Chu}(r, H(m')) = Hom_{\mathcal{T}Chu}(r, r_{m'})$. By relations (8), (9), we have $(v, u) \in Hom_{\mathcal{T}Chu}(r, r_{m'}) \Leftrightarrow v = \mathcal{H}(u) \circ \widehat{r}$. Thus, for any $u \in Hom_M(m, m')$, there exists a unique v such that $(v, u) \in Hom_{\mathcal{T}Chu}(r, H(m'))$. Then $\psi_q(r, m')$ is bijective.

2) Let $in : \widetilde{M} \to \mathcal{T}Chu$ be the inclusion functor. Then $H = in \circ \widetilde{H}$, where $\widetilde{H} : M \to \widetilde{M}$. Since the functor H is fully faithful and the mapping $m \mapsto (\mathcal{H}(m), m, r_m)$ is a bijection between the objects of categories M and $\widetilde{M}$, it is follows that $\widetilde{H}$ is a category isomorphism and $\widetilde{M}$ is a full subcategory. From the adjunction of functors P_2 and H, we have that the functor $\widetilde{H} \circ P_2$ is left adjoint to the functor in. □

4. Limits and colimits in the category $\mathcal{T}Chu$

Let K and L be categories. For each object $l \in Ob(L)$, the *constant functor* $C(l) : K \to L$ is defined as follows: $C(l)(k) = l$, $C(l)(f) = 1_l$ for any $f \in Hom_K(k, k')$. If $h \in Hom_L(l, l')$, then there is a natural transformation $C(h) : C(l) \to C(l')$ given by: $C(h)(k) = h \in Hom_L(C(l)(k), C(l')(k)) = Hom(l, l')$.

Let $G : K \to L$ be a functor. A *cone* over G with vertex l is a natural transformation $\varphi : C(l) \to G$, that is, $\varphi = \{\varphi(k) \in Hom_K(l, G(k)) \mid k \in Ob(K)\}$, such that $G(f) \circ \varphi(k) = \varphi(k')$ for any morphism $f \in Hom(k, k')$. A cone $\varphi : C(l_0) \to G$ is *universal* or *a limit cone* if for any cone $\psi : C(l) \to G$, there is a unique morphism $h \in Hom_L(l, l_0)$ such that $\varphi \circ C(h) = \psi$, i.e. $\varphi(k) \circ h = \psi(k)$ for any $k \in Ob(K)$. The vertex l_0 of a universal cone is called the *limit* of G and is denoted by $l_0 = \lim G$.

Dually, a natural transformation $\varphi : G \to C(l)$ is called a *cocone*. The vertex l_0 of a universal cocone $\varphi : G \to C(l_0)$ is called a *colimit* of G and is denoted by $l_0 = \operatorname{colim} G$.

For a functor $G : Z \to K^{op}$, the equalities $G^{op}(z) = G(z)$, $G^{op}(f) = G(f)$ for all $z \in Ob(z)$, $f \in Hom_Z(k, k')$ define a functor $G^{op} : Z^{op} \to K$. Hence, a (universal) cone over G is a (universal) cocone over G^{op}, and vice versa. Thus, $\lim G = \operatorname{colim} G^{op}$ and $\operatorname{colim} G = \lim G^{op}$.

Let $\mathcal{H} : M \to N$ be a functor, $\mathcal{T} : N^{op} \times M \to Set$ be $\mathcal{H}$-closed, and $F : Z \to \mathcal{T}Chu$ be a functor. By (11), we have $F(z) = (F_1(z), F_2(z), r(z))$, where $r(z) \in \mathcal{T}(F_1(z), F_2(z))$. By Proposition 1, we get description of cones.

1) Let $\varphi_1 : C(n_0) \to F_1$, $\varphi_2 : C(m_0) \to F_2$ be cones, $r_0 \in \mathcal{T}(n_0, m_0)$. The pair $\varphi = (\varphi_1, \varphi_2) : C(n_0, m_0, r_0) \to F$ is a cone iff for any $z \in Ob(z)$

$$\widehat{r(z)} \circ \varphi_1(z) = \mathcal{H}(\varphi_2) \circ \widehat{r_0}. \tag{14}$$

2) The cone φ is universal iff for any cone $\psi = (\psi_1, \psi_2) : C(n, m, r) \to F$, there exist unique morphisms $v \in Hom_N(n, n_0)$ and $u \in Hom_M(m, m_0)$ such that:

$$\mathcal{H}(u) \circ \widehat{r} = \widehat{r_0} \circ v, \quad \psi_1(z) = \varphi_1(z) \circ v, \quad \psi_2(z) = \varphi_2(z) \circ u. \tag{15}$$

Theorem 2 (On limits). *Let $\mathcal{H} : M \to N$ be a functor, $\mathcal{T} : N^{op} \times M \to Set$ be an $\mathcal{H}$-closed functor, Z be a category, $F : Z \to \mathcal{T}Chu$ be a functor, such that $F(z) = (F_1(z), F_2(z), r(z))$ and condition (12) is satisfied. Suppose that the functors $F_1 : Z \to N$, $F_2 : Z \to M$, and $\mathcal{H} \circ F_2 : Z \to N$ have limits, and there exist pullbacks in N. Then the functor F has a limit $\lim F = (n'_0, m_0, r_0)$, where $m_0 = \lim F_2$.*

Proof. We construct a universal cone over F. Let $\varphi_1 : C(n_0) \to F_1$, $\varphi_2 : C(m_0) \to F_2$, $\delta : C(n_1) \to \mathcal{H} \circ F_2$ be the universal cones over F_1, F_2 and $\mathcal{H} \circ F_2$, respectively. By Proposition 1, the family $W = \{\widehat{r(z)} \mid z \in Ob(z)\}$ is a natural transformation $W : F_1 \to \mathcal{H} \circ F_2$. Thus, the composition $W \circ \varphi_1 : C(n_0) \to \mathcal{H} \circ F_2$ is a cone over $\mathcal{H} \circ F_2$ with vertex n_0. Since $\mathcal{H}$ is a functor, $\mathcal{H}(\varphi_2) = \{\mathcal{H}(\varphi_2(z)) \in Hom_N(\mathcal{H}(m_0), \mathcal{H}(F_2(z))) \mid z \in Ob(Z)\}$ is also a cone over $\mathcal{H} \circ F_2$ with vertex $\mathcal{H}(m_0)$.

Since δ is a universal cone, there exist unique morphisms

$$d \in Hom_N(\mathcal{H}(m_0), n_1), \widetilde{w} \in Hom_N(n_0, n_1),$$

such that for all $z \in Ob(Z)$

$$\delta(z) \circ d = \mathcal{H}(\varphi_2(z)), \quad \delta(z) \circ \widetilde{w} = W(z) \circ \varphi_1(z). \tag{16}$$

Let $n'_0 = n_0 \times_{n_1} \mathcal{H}(m_0)$ be the pullback with projections $d_0 \in Hom(n'_0, n_1)$, $w_0 \in Hom_N(n'_0, \mathcal{H}(m_0))$. We have $d \circ w_0 = \widetilde{w} \circ d_0$, and for any $a \in$

$Hom_N(n, n_1)$, $b \in Hom_N(n, \mathcal{H}(m_0))$ such that $\widetilde{w} \circ a = d \circ b$, there exists a unique $v \in Hom_N(n, n_0')$ such that $a = d_0 \circ v$, $b = w_0 \circ v$.

Define $r_0 \in \mathcal{T}(n_0', m_0)$ as the unique element satisfying equality $\widehat{r_0} = w_0$, and let $\varphi_1'(z) = \varphi_1(z) \circ d_0 : n_0' \to F_1(z)$. By conditions (14), (15), (16), the pair $\varphi' = (\varphi_1', \varphi_2) : C(n_0', m_0, r_0) \to F$ is a universal cone over F with vertex r_0. Thus, $(n_0', m_0, r_0) = \lim F$. □

Theorem 3 (On colimits)**.** *Let $\mathcal{H} : M \to N$ be a functor, $\mathcal{T} : N^{op} \times M \to Set$ be an $\mathcal{H}$-closed functor, Z be a category, $F : Z \to \mathcal{T}Chu$ be a functor, such that $F(z) = (F_1(z), F_2(z), r(z))$ and condition (12) is satisfied. Suppose that the functors $F_1 : Z \to N$, $F_2 : Z \to M$, and $\mathcal{H} \circ F_2 : Z \to N$ have colimits. Then the functor F has a colimit* $\operatorname{colim} F = (n_0, m_0, r_0)$*, where* $n_0 = \operatorname{colim} F_1$*,* $m_0 = \operatorname{colim} F_2$*.*

Proof. Let $\varphi_1 : F_1 \to C(n_0)$, $\varphi_2 : F_2 \to C(m_0)$ be universal cocones. We construct a universal cocone over F. Since φ_2 is a cocone and $\mathcal{H}$ is a functor, $\mathcal{H}(\varphi_2) = \{\mathcal{H}(\varphi_2(z)) \in Hom_N(\mathcal{H}(F_2(z)), \mathcal{H}(m_0)) \mid z \in Ob(Z)\}$ is a cocone over $\mathcal{H} \circ F_2$ with vertex $\mathcal{H}(m_0)$. By Proposition 1, the family $W = \{\widehat{r(z)} \mid z \in Ob(Z)\}$ is a natural transformation $W : F_1 \to \mathcal{H} \circ F_2$. Thus, the composition $\mathcal{H}(\varphi_2) \circ W = \{\mathcal{H}(\varphi_2(z)) \circ \widehat{r(z)} : F_1(z) \to \mathcal{H}(m_0)\}$ is a cocone over F_1 with vertex $\mathcal{H}(m_0)$. Since φ_1 is a universal cocone, there exists a unique morphism $w_0 \in Hom_N$ such that $w_0 \circ \varphi_1(z) = \mathcal{H}(\varphi_2(z)) \circ \widehat{r(z)}$ for any $z \in Ob(Z)$. Define $r_0 \in \mathcal{T}(n_0, m_0)$ as the unique element satisfying equality $\widehat{r_0} = w_0$. By (9), the pair $(\varphi_1(z), \varphi_2(z)) \in Hom_{\mathcal{T}Chu}(r(z), r_0)$ is a cocone $\varphi = (\varphi_1, \varphi_2) : F \to C(n_0, m_0, r_0)$. Since φ_1 and φ_2 are universal cocones, φ is universal. Thus, $(n_0, m_0, r_0) = \operatorname{colim} F$. □

We consider some examples of $\mathcal{T}Chu$ categories.

Example 1. Let N, K, L be categories, $\otimes : N \times K \to L$ be a functor, $l \in Ob(L)$, $M = K^{op}$. We define a functor $\mathcal{T}^{\otimes,l} : N^{op} \times M \to Set$ as follows: $\mathcal{T}^{\otimes,l}(n, k) = Hom_L(n \otimes k, l)$, where $n \in Ob(N)$, $k \in Ob(M) = Ob(K)$. Then, by (1), for $(r : n \otimes k \to l) \in \mathcal{T}^{\otimes,l}(n, k)$, $h \in Hom_N(n', n)$, $u \in Hom_M(k, k') = Hom_K(k', k)$ we have $urh = \mathcal{T}^{\otimes,l}(h, u)(r) = r \circ (h \otimes u)$. By the definition of $\mathcal{T}Chu$, a $\mathcal{T}^{\otimes,l}Chu$ space is any triple (n, k, r), where $r : n \otimes k \to l$. If $r' : n' \otimes k' \to l$, then a morphism (or a Chu transform) is a pair (v, u) such that $ur = r'v$, that is, $r \circ (1_n \otimes u) = r' \circ (v \otimes 1_{k'})$, where $v \in Hom_N(n, n'), u \in Hom_{K^{op}}(k, k') = Hom_K(k', k)$. If $N = K = L$ is a monoidal category, then the category $\mathcal{T}^{\otimes,l}Chu$ coincides with the category $Chu(K, l)$ introduced in [2].

Example 2. Let N, K, L be categories, $M = K^{op} \times L$. If $m, m' \in Ob(M)$, i.e., $m = (k, l)$ and $m' = (k', l')$, where $k, k' \in Ob(K)$, $l, l' \in Ob(L)$, then $Hom_M(m, m') = Hom_K(k', k) \times Hom_L(l, l')$. Hence, if $u \in Hom_M(m, m')$, then $u = (f, g)$, where $f \in Hom_K(k', k)$, $g \in Hom_L(l, l')$. Moreover, if $u' = (f', g') \in Hom_M(m', m'')$, where $m'' = (k'', l'')$, then

$u' \circ u = (f \circ f', g' \circ g) \in Hom_M(m, m'') = Hom_K(k'', k) \times Hom_L(l, l'')$. Since the categories $N^{op} \times M = N^{op} \times (K^{op} \times L)$ and $N^{op} \times K^{op} \times L$ are canonically isomorphic, we will denote pairs (n, m) and (h, u) as (n, k, l) and (h, f, g), respectively.

Let $\otimes : N \times K \to L$, and $\mathcal{H} : K^{op} \times L \to N$ be functors. We denote $H(k, l) = l^k$ and $H(f, g) = g^f$ for all $k \in Ob(k)$, $l \in Ob(L)$, $f \in Hom_K(k', k)$, $g \in Hom_L(l, l')$. Then the functor $\mathcal{T}_{\mathcal{H}} : N^{op} \times K^{op} \times L \to Set$ is given by the equalities: $\mathcal{T}_{\mathcal{H}}(n, k, l) = Hom_N(n, l^k)$, $\mathcal{T}_{\mathcal{H}}(h, f, g)(w) = \mathcal{H}(u) \circ w \circ h = g^f \circ w \circ h$ for any $w \in Hom_N(n, l^k)$.

We define the functor $Q^{\otimes} : N^{op} \times K^{op} \times L \to Set$ as follows: $Q^{\otimes}(n, k, l) = Hom_N(n \otimes l, k)$, $Q^{\otimes}(h, f, g)(r) = g \circ r \circ (h \otimes f) \in Hom_N(n' \otimes l', k')$ for any $r \in Hom_N(n \otimes l, k)$.

We associate the functor $\mathcal{T}^Q : N^{op} \times K^{op} \times L \to Set$ with any functor $Q : N^{op} \times K^{op} \times L \to Set$ as follows: $\mathcal{T}^Q(n, k, l) = Q(n, k, l)$, $\mathcal{T}^Q(h, u) = \mathcal{T}^Q(h, f, g) = Q(h, f, g)$. A $\mathcal{T}^Q Chu$ space is (n, k, l, r), where $r \in Q(n, k, l)$. Let $r' \in Q(n', k', l')$. Then morphism of $\mathcal{T}^Q Chu$ spaces is $(v, u) = (v, f, g) : r \to r'$ such that $ur = r'v$, i.e, $Q(1_n, f, g)(r) = Q(v, 1_{k'}, 1_{l'})(r')$.

If $Q = Q^{\otimes}$, we write $\mathcal{T}^Q = \mathcal{T}^{\otimes}$. For all $\mathcal{T}^{\otimes} Chu$ spaces $r : n \otimes k \to l, r' : n' \otimes k' \to l'$,

(17) $(v, u) = (v, f, g) \in Hom_{\mathcal{T}^{\otimes} Chu}(r, r') \Leftrightarrow g \circ r \circ (1_n \otimes f) = r' \circ (v \otimes 1_{k'})$

Thus, if $N = K = L$ is a monoidal category, then the category $\mathcal{T}^{\otimes} Chu$ coincides with the category $Chu(K)$ introduced in [14]. These categories have been studied for various K in [6,9,11,14].

Let $\mathcal{T}^Q : N^{op} \times K^{op} \times L \to Set$ be an $\mathcal{H}$-closed functor and $p = \{p_{n,k,l} : Q(n, k, l) \to Hom_N(n, l^k) \mid n \in Ob(N), k \in Ob(K)\}, l \in Ob(L)$, $q = \{q_{n,k,l} : Hom_N(n, l^k) \to Q(n, k, l) \mid n \in Ob(N), k \in Ob(K)\}, l \in Ob(L)\}$ mutually inverse natural isomorphisms of functors $p : \mathcal{T}^Q \to \mathcal{T}_{\mathcal{H}}$, $q : \mathcal{T}_{\mathcal{H}} \to \mathcal{T}^Q$. As above, we denote

$$\hat{r} = p_{n,k,l}(r) \in Hom_N(n, l^k), r_{kl} = q_{l^k,k,l}(1^{l^k}) \in Q(l^k, k, l)$$

If $Q = Q^{\otimes}$, we have $r_{kl} : l^k \otimes k \to l$. This morphism is called *the evaluation map* and denoted by ev_{kl}.

The following result is a special case of Theorem 2 and Theorem 3.

Proposition 2. *Let $\mathcal{T}^Q : N^{op} \times K^{op} \times L \to Set$ be an $\mathcal{H}$-closed functor for some $\mathcal{H} : K^{op} \times L \to N$, $F : Z \to \mathcal{T}^Q Chu$ a functor $F(z) = (F_1(z), F_2^1(z), F_2^2(z), r(z))$, where $r(z) \in Q(F_1(z), F_2^1(z), F_2^2(z))$, $F_1 : Z \to N$, $F_2 : Z \to K^{op} \times L$, $F_2^1 : Z \to K^{op}$, and $F_2^2 : Z \to L$.*

1) Suppose that $n_0 = \lim F_1$, $k_0 = \lim F_2^1 = \operatorname{colim} F_2^{1\,op}$, $l_0 = \lim F_2^2$, $n_1 = \lim \mathcal{H} \circ F_2 = \lim {F_2^2}^{F_2^1}$ and in the category N there are pullbacks. Then the functor F has a limit

$$\lim F = (n_0 \times_{n_1} {l_0}^{k_0}, k_0, l_0, r_0),$$

where $r_0 \in Q(n_0 \times_{n_1} {l_0}^{k_0}, k_0, l_0)$. *If* $Q = Q^{\otimes}$, *then* r_0 *is a morphism* $r_0 : (n_0 \times_{n_1} {l_0}^{k_0}) \otimes k_0 \to l_0$.

Thus, if the categories N *and* L *are complete, and* K *is cocomplete, then any functor* $F : Z \to \mathcal{T}^Q Chu$ *has a limit.*

2) Suppose that the $n'_0 = \operatorname{colim} F_1$, $k'_0 = \operatorname{colim} F_2^1 = \lim {F_2^1}^{op}$, $l'_0 = \operatorname{colim} F_2^2$. *Then* $\operatorname{colim} F = (n'_0, k'_0, l'_0, r'_0)$, *where* $r'_0 \in Q(n'_0, k'_0, l'_0)$. *If* $Q = Q^{\otimes}$, *then* r'_0 *is a morphism* $r'_0 : n'_0 \otimes k'_0 \to l'_0$.

Thus, if the categories N *and* L *are cocomplete, and* K *is complete, then any functor* $F : Z \to \mathcal{T}^Q Chu$ *has a colimit.*

The following result is a special case of Theorem 1.

Proposition 3. *Let* $\mathcal{T}^Q : N^{op} \times K^{op} \times L \to Set$ *be an* $\mathcal{H}$*-closed functor.*

1) the functor $H : K^{op} \times L \to \mathcal{T}^Q Chu$ *defined as follows:* $H(k, l) = (l^k, k, l, r_{kl})$, *is fully faithful and right adjoint to the functor* $P_2 : \mathcal{T}^Q Chu \to K^{op} \times L$, *where* $P_2(n, k, l, r) = (k, l)$;

2) the full subcategory $\widetilde{M} = \{(l^k, k, l, r_{kl}) \mid k \in Ob(K), l \in Ob(l)\}$ *of the category* $\mathcal{T}^Q Chu$ *is reflective and isomorphic to the category* $K^{op} \times L$.

5. Monoidal Categories

It is known [2] that if C is a symmetric closed monoidal category, then the category $Chu(C, d)$ for some fixed object $d \in Ob(C)$ is also provided with the structure of a monoidal closed category. In this section we provide the category $Chu(C)$ with a monoidal closed structure.

Let $\otimes : C \times C \to C$ be a functor. A triple $(C, \otimes, I)$ is called *right-monoidal* category, if there are a natural isomorphism $(a \otimes b) \otimes c \to a \otimes (b \otimes c)$ in a, b, c and an object I, which called a *right unit*, such that there is a natural isomorphism $a \to a \otimes I$ in a. If the category $(C, \otimes, I)$ is symmetric, then there exists a unique natural isomorphism $a \otimes b \to b \otimes a$ in a, b.

A right-monoidal category $(C, \otimes, I)$ is called *closed* [4] if there exists a functor $\multimap : C^{op} \times C \to C$ such that for any $y \in Ob(C)$ there exists a natural isomorphism in $x, z \in Ob(C)$

$$adj : Hom(x \otimes y, z) \to Hom(x, y \multimap z).$$

Note that the category $(C, \otimes, I)$ is closed iff the functor $\mathcal{T}^{\otimes} : C^{op} \times C^{op} \times C \to Set$ is an $\mathcal{H}$-closed, where $\mathcal{H} = \multimap : C^{op} \times C \to C$, and the evaluation map is given by

$$ev_{yz} = ev : (y \multimap z) \otimes y \to z = adj^{-1}(1_{y \multimap z}).$$

As above, $Chu(C) = \mathcal{T}^{\otimes} Chu$, hence objects of $Chu(C)$ are morphisms $r : a \otimes b \to c$, and Chu transforms are defined by (17).

Theorem 4. *Let $C = (C, \otimes, I)$ be a closed symmetric monoidal category with finite pullbacks and terminal object $\star$. Then there is a functor $\boxtimes : Chu(C) \times Chu(C) \to Chu(C)$ such that the category $(Chu(C), \boxtimes, T)$ is a closed right-monoidal with a right unit $T : I \otimes \star \to \star$.*

Proof. We fix natural isomorphisms $\alpha : (a_1 \otimes a_2) \otimes a_3 \to a_1 \otimes (a_2 \otimes a_3)$ and $s : a_1 \otimes a_2 \to a_2 \otimes a_1$ in a_1, a_2, a_3. Let $r_1 : a_1 \otimes x_1 \to d_1, r_2 : a_2 \otimes x_2 \to d_2, r_1' : a_1' \otimes x_1' \to d_1', r_2' : a_2' \otimes x_2' \to d_2'$ be objects of $Chu(C)$, $f = (f^+, f^-, f^0) : r_1 \to r_1'$, $g = (g^+, g^-, g^0) : r_2 \to r_2'$, $f' = (f'^+, f'^-, f'^0) : r_1' \to r_1$ be Chu transforms of $Chu(C)$. We define $\boxtimes : Chu(C) \times Chu(C) \to Chu(C)$ as follows:

$$r_1 \boxtimes r_2 : (a_2 \otimes a_1) \otimes ((a_2 \multimap x_1) \times (a_1 \multimap x_2)) \to d_1,$$
$$r_1 \boxtimes r_2 = r_1 \circ (1_{a_1} \otimes ev) \circ \alpha \circ (s \otimes pr_{a_2 \multimap x_1}),$$
$$(f \boxtimes g)^+ = g^+ \otimes f^+, (f \boxtimes g)^- = (g^+ \multimap f^-) \times (f^+ \multimap g^-), (f \boxtimes g)^0 = f^0,$$

where $pr_{a_2 \multimap x_1} : (a_2 \multimap x_1) \times (a_1 \multimap x_2) \to (a_2 \multimap x_1)$ is a projection. Direct computation shows that $\boxtimes$ is a functor. We define the mappings $\widetilde{\alpha} : (r_1 \boxtimes r_2) \boxtimes r_3 \to r_1 \boxtimes (r_2 \boxtimes r_3)$ as follows:

$$\widetilde{\alpha} = (\alpha, (1_{(a_3 \otimes a_2) \multimap x_1} \times \beta^{-1}) \circ (((\gamma^{-1} \times \gamma^{-1}) \circ \beta) \times (\gamma \circ (s \multimap 1_{x_3}))), 1_{d_1}),$$

where $\beta : (a_1 \multimap (a_2 \times a_3)) \to ((a_1 \multimap a_2) \times (a_1 \multimap a_3))$, $\gamma : (a_1 \otimes a_2 \multimap a_3) \to (a_1 \multimap (a_2 \multimap a_3))$ are natural isomorphisms. It is easy to see that these maps are natural isomorphisms in r_1, r_2, r_3. Clearly, the object $T : I \otimes \star \to \star$ is a right unit relative to $\boxtimes$.

We define $\Rightarrow : Chu(C)^{op} \times Chu(C) \to Chu(C)$ as follows:

$$r_1 \Rightarrow r_2 : ((a_1 \multimap a_2) \times (x_2 \multimap x_1)) \otimes (a_1 \otimes x_2) \to d_2,$$
$$(r_1 \Rightarrow r_2) = r_2 \circ ((ev \circ s) \otimes 1_{x_2}) \circ \alpha \circ (pr_{a_1 \multimap a_2} \otimes 1_{a_1 \otimes x_2}),$$
$$(f' \Rightarrow g)^+ = (f'^+ \multimap g^+) \times (g^- \multimap f'^-), (f' \Rightarrow g)^- = f'^+ \otimes g^-, (f' \Rightarrow g)^0 = g^0,$$

where $pr_{a_1 \multimap a_2} : (a_1 \multimap a_2) \times (x_2 \multimap x_1) \to a_1 \multimap a_2$ is a projection. Direct computation shows that $\Rightarrow$ is a functor.

The mappings $p : Hom(r_1 \boxtimes r_3, r_2) \to Hom(r_1, r_3 \Rightarrow r_2)$ and $q : Hom(r_1, r_3 \Rightarrow r_2) \to Hom(r_1 \boxtimes r_3, r_2)$ define as follows:

$$p(u) = (adj(u^+ \circ s) \times adj(adj^{-1}(pr_2 \circ u^-) \circ s), adj^{-1}(pr_1 \circ u^-) \circ s, u^0),$$
$$q(v) = (adj^{-1}(pr_1 \circ v^+) \circ s, adj(v^- \circ s) \times adj(adj^{-1}(pr_2 \circ v^+) \circ s), v^0)$$

for all $u : r_1 \boxtimes r_3 \to r_2$, $v : r_1 \to r_3 \Rightarrow r_2$. Direct computations shows that p and q are mutually inverse bijections natural in r_1, r_2, r_3, which gives a natural isomorphism $Hom(r_1 \boxtimes r_3, r_2) \cong Hom(r_1, r_3 \Rightarrow r_2)$. Thus, $(Chu(C), \boxtimes, T)$ is a closed right-monoidal category. □

Corollary 1. *1) If the category C is complete and cocomplete, then both $Chu(C)$ and $Chu(Chu(C))$ are complete and cocomplete categories.*

2) The category $Chu(C)^{op} \times Chu(C)$ is isomorphic to a reflective full subcategory of the category $Chu(Chu(C))$.

Proof. Since $Chu(C) = \mathcal{T}^{\otimes}Chu$, its completeness and cocompleteness follow from Proposition 2 by settiong $\mathcal{H} = \multimap$. Since $Chu(Chu(C)) = \mathcal{T}^{\boxtimes}Chu$, its completeness and cocompleteness follow from Proposition 2 by setting $\mathcal{H} = \Rightarrow$. Point 2 follows from Proposition 3. □

6. Conclusion

We have introduced a generalized Chu construction that associates with a functor $\mathcal{T} : N^{op} \times M \to Set$ the category $\mathcal{T}Chu$ of generalized Chu spaces. We have shown that this construction encompasses another generalizations of Chu spaces, including those with constant and variable alphabets over monoidal categories. We have proved that for an $\mathcal{H}$-closed functor $\mathcal{T}$ the category M is reflectively embedded in $\mathcal{T}Chu$. For the complete categories N and M we have constructed explicit constructions of limits and colimits in $\mathcal{T}Chu$. Furthermore, we have defined a tensor product on the category $Chu(C)$ of Chu spaces with variable alphabet over a closed monoidal category C, which is associative but not commutative. We have proved that $Chu(C)$ with this tensor product is closed and that the category of Chu spaces over this category is complete. These results establish a foundation for further exploration of noncommutative categorical structures and their applications.

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