Серия «Математика»
2026. Т. 55. С. 110—122

Онлайн-доступ к журналу:
http://mathizv.isu.ru

И З В Е С Т И Я
Иркутского государственного университета

Research article

УДК 512.542
MSC 20B05, 20B30
DOI https://doi.org/10.26516/1997-7670.2026.55.110

N-recognizability of Groups $Alt_p \times Alt_5$, Where $p > 1361$ Is a Prime Number

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Abstract: Given a finite group L, let $N(L)$ denote the set of its conjugacy class sizes. Let X and Y be sets of natural numbers, G be a finite group such that $N(G) = X \times Y$. In the article [16] the question is formulated: for which sets X and Y is it true that $G \simeq A \times B$, where $N(A) = X$ and $N(B) = Y$? More than 30 years ago, J. Thompson formulated a conjecture that any finite simple group is uniquely determined by its set of sizes of conjugacy classes in the class of finite groups with trivial center. In 2019, the validity of this conjecture was proven. In 2020, it was noted that in addition to simple groups, some direct products of simple groups are also determined by this set.
We prove that if $N(G) = N(Alt_p \times Alt_5)$, where p is a prime greater than 1361 and the group G has a trivial center, then $G \simeq Alt_5 \times Alt_p$.

Keywords: finite groups, conjugacy class, alternating groups

Acknowledgements: The work was supported by Russian Science Foundation (project № 24-41-10004).

For citation: Gorshkov I. B., Shepelev V. D. N-recognizability of Groups $Alt_p \times Alt_5$, Where $p > 1361$ Is a Prime Number. *The Bulletin of Irkutsk State University. Series Mathematics*, 2026, vol. 55, pp. 110–122.
https://doi.org/10.26516/1997-7670.2026.55.110

Научная статья

N-распознаваемость групп $Alt_p \times Alt_5$, где $p > 1361$ – простое число

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Аннотация: Показывается, что для конечной группы G обозначено через $N(G)$ множество размеров классов сопряженности в G. Пусть X и Y множества натуральных чисел, G — конечная группа, что $N(G) = X \times Y$. Отмечается, что в [16] сформулирован вопрос: для каких множеств X и Y верно, что $G = A \times B$, где $N(A) = X$ и $N(B) = Y$? Указывается, что более 30 лет назад Дж. Томпсон сформулировал гипотезу о том, что любая конечная простая группа однозначно определяется своим множеством размеров классов сопряженности в классе конечных групп с тривиальным центром. В 2019 г. была доказана справедливость этой гипотезы, а в 2020 г. замечено, что помимо простых групп по данному множеству определяются и некоторые прямые произведения простых групп. Доказывается, что если $N(G) = N(Alt_p \times Alt_5)$, где p – простое число большее 1361, и группа G имеет тривиальный центр, то $G \simeq Alt_5 \times Alt_p$.

Ключевые слова: конечные группы; классы сопряженности, знакопеременные группы

Благодарности: Работа выполнена при поддержке Российского научного фонда (проект № 24-41-10004).

Ссылка для цитирования: Gorshkov I. B., Shepelev V. D. N-recognizability of Groups $Alt_p \times Alt_5$, Where $p > 1361$ Is a Prime Number // Известия Иркутского государственного университета. Серия Математика. 2026. Т. 55. С. 110–122.
https://doi.org/10.26516/1997-7670.2026.55.110

1. Introduction

The sizes of conjugacy classes are important in the study of finite groups. In 1904, Burnside [6] showed that a group with a conjugacy class of primary order is not simple. Much later, Kazarin [19] proved that the subgroup generated by a conjugacy class of primary order is solvable. In [18] Ito first introduced the concept of a conjugate type vector. This is the set of conjugacy classes sizes ordered in increasing order. The latest survey of results on the relationship between the structure of a finite group and the set of sizes of conjugacy classes can be found in [8; 21].

Let G be a finite group. In 1987, John G. Thompson (see [20][Question 12.38]) conjectured that if L is a finite simple non-abelian group and G is a finite group with a trivial center such that $N(G) = N(L)$, then $G \simeq L$. Since then, significant progress has been made for various families of non-abelian simple groups (see [1–5; 9]), culminating in the complete confirmation of Thompson's conjecture in 2019 [15].

We say that a finite group L is N-recognizable in the class of finite groups with a trivial center (or simply N-recognizable) if for any finite group G with a trivial center, the equality $N(G) = N(L)$ implies the isomorphism $G \simeq L$. It is easy to show that Sym_3 is N-recognizable. Thus, the condition of non-solvability is not a necessary condition for N-recognizability. As an example of a non-recognizable group we can take a Frobenius group of order 18. There exist two non-isomorphic Frobenius groups of order 18 with the same set of conjugacy class sizes. However non-recognizable groups with a trivial center are very rare. In particular all known non-recognizable groups have non-trivial solvable radicals. Let G^n denote the direct product of n copies of the group G. The following question generalizes Thompson's conjecture.

Question 1. [12, Question 2] *Let S be a non-abelian simple group. Is it true that for any $n \in \mathbb{N}$, the group S^n is N-recognizable?*

This question has not been resolved for any non-abelian simple group. However in the articles [12;13;17;23] N-recognizability of the groups Alt_5^2, Alt_6^2, $L_2(q)^2$, where q is a power of a prime number satisfying certain conditions, has been proven. In this work we prove N-recognizability of the groups $Alt_p \times Alt_5$, where p is a prime number greater than 1361.

Theorem. *Let p be a prime number greater than 1361. The group $Alt_5 \times Alt_p$ is N-recognizable.*

In [16] the following question was formulated.

Question 2. [16, Question 0.1] *Let G be a group such that $N(G) = \Omega \times \Delta$. Which Ω and Δ guarantee that $G \simeq A \times B$, where A and B are subgroups such that $N(A) = \Omega$ and $N(B) = \Delta$?*

Let $\Omega = N(Alt_5)$ and $\Delta = N(Alt_p)$. It follows from Theorem that a finite group G such that $N(G) = \Omega \times \Delta$ and $Z(G) = 1$ is isomorphic to $A \times B$, where $N(A) = \Omega$ and $N(B) = \Delta$.

2. Preliminaries

In this article only finite groups will be considered. We introduce the following notation: $Ind(G, x)$ is the index of the centralizer of element x in group G. Note that if $x \in G$, then $Ind(G, x) = |x^G|$.

Lemma 1. *[7, Lemma 1] If for some prime p every p'-element of a group G has index prime to p, then the Sylow p-subgroup of G is a direct factor of G.*

Lemma 2. *[14, Lemma 1.4] Let G be a finite group, $K \trianglelefteq G$ and $\overline{G} = G/K$. Take $x \in G$ and $\overline{x} = xK \in G/K$. Then the following conditions hold:*

(i) $|x^K|$ and $|\overline{x}^{\overline{G}}|$ divide $|x^G|$.

(ii) If L and M are consequent members of a composition series of G, $L < M$, $S = M/L$, $x \in M$ and $\widetilde{x} = xL$ is an image of x, then $|\widetilde{x}^S|$ divides $|x^G|$.

(iii) If $y \in G, xy = yx$, and $(|x|, |y|) = 1$, then $C_G(xy) = C_G(x) \cap C_G(y)$.

(iv) If $(|x|, |K|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$.

(v) $\overline{C_G(x)} \leq C_{\overline{G}}(\overline{x})$.

Lemma 3. *[11, Theorem 5.2.3] Let A be a p'-group of the automorphism group of an abelian p-group P. Then $P = C_P(A) \times [P, A]$.*

Lemma 4. *[14, Lemma 1.6] Let S be a non-abelian simple group. If $p \in \pi(S)$, then there exists $a \in N(S)$ such that $|a|_p = |S|_p$, and an element $g \in S$ of prime order such that $|g^S| = a$.*

Lemma 5. *Let S be a non-abelian simple group, and let p be the largest prime in $\pi(G)$. Then for any $g \in S$, $|g^S| \geq p$.*

Proof. Since S is a simple group, it follows that S acts on $|g^S|$ faithfully. Therefore S embeds in $Sym_{|g^S|}$. If $|g^S| < p$, then $|Sym_{|g^S|}|$ is not divisible by p; a contradiction. □

Lemma 6. *[10, Theorem 1] Let G be a finite group acting transitively on a set Ω with $|\Omega| > 1$. Then there exists a prime r and an r-element $g \in G$ such that g acts without fixed points on Ω.*

For a given natural number n we denote by $\pi(n)$ the set of all prime divisors of n. Everywhere we denote by $\mathbb{P}$ the set of all prime numbers. If $\pi \subseteq \mathbb{P}$, then for a given integer n, we denote by n_π its π-part, that is the largest number k such that $\pi(k) \subseteq \pi$. In particular for $p \in \mathbb{P}$ we denote the p-part of the number n by n_p. Let H be a group. Put $\pi(H) = \pi(|H|)$, $N(H)_p = \{|x^H|_p \,|\, x \in H\}$, $|H||_p = \max\{|x^H|_p \,|\, x \in H\}$, and $|H|| = \prod_{p \in \pi(H)} |H||_p$.

Let p be a prime number greater than 1361, $L = Alt_p \times Alt_5$, and G be a group with a trivial center such that $N(G) = N(L)$.

The next Lemma is a simple exercise.

Lemma 7. $N(L) = \{\alpha \cdot \beta \,|\, \alpha \in N(Alt_5) \quad \textit{and} \quad \beta \in N(Alt_p)\}$.

Lemma 8. $\pi(G) = \pi(p!)$.

Proof. The statement of the lemma follows from Lemma 1 and the fact that the center of the group G is trivial. □

Lemma 9. *The number $|L|$ divides $|G|$.*

Proof. Let $g \in L$ such that $|g| = tp$, where $t \in \{3, 5\}$. Then $|g^L| = |L|/tp$. Since $|g^L|$ divides $|G|$, it follows that $|G|$ is divisible by $|L|/p$. From Lemma 8, it follows that p divides $|G|$. Therefore, $|L|$ divides $|G|$. □

We denote by Ω the set of prime numbers contained in the interval $[\frac{p}{2}; p]$.

Lemma 10. *[14, Lemma 2.3] Let $t \in \Omega$, $\alpha \in N(Alt_n)$, and α is not divisible by t. Then $\alpha = |Alt_n|/(t|C|)$ or $\alpha = |Alt_n|/(|Alt_{t+i}||B|)$, where $C = C_{Alt_{n-t}}(g)$ for some element $g \in Alt_{n-t}$, $t + i \leq n$, and $B = C_{Alt_{n-t-i}}(h)$ for some $h \in Alt_{n-t-i}$.*

Let $\Phi_{(t,n)} = \{\alpha \in N(Alt_n) \mid \alpha = |Alt_n|/(t|C|)$, where $C = C_{Alt_{n-t}}(g)$ for some $g \in Alt_{n-t}\}$, and $\Psi_{(t,n)} = \{\alpha \in N(Alt_n) \mid \alpha = |Alt_n|/(|Alt_{t+i}||B|)$, where $i \geq 0, t + i < n - 1, B = C_{Alt_{n-t-i}}(g), g \in Alt_{n-t-i}\}$.

Lemma 11. *[14, Lemma 2.4] The set $\Psi_{(t_i,n)} \setminus \Psi_{(t_{i+1},n)}$, where $t_i \in \Omega$, $1 \leq i < |\Omega|$, is non-empty.*

3. Proof of the Theorem

Lemma 12. *Let $t, l \in \Omega$ be distinct numbers. If $T < G$ is a $\{t, l\}$-group, then $T = S_t \times S_l$, where $S_t \in \mathrm{Syl}_t(T), S_l \in \mathrm{Syl}_l(T)$.*

Proof. Suppose that the statement of the Lemma is false. Let $H < G$ be a $\{t, l\}$-subgroup of minimal order, which does not decompose into a direct product of its Sylow t- and l-subgroups. Let $X \lhd G$ be the largest subgroup among those for which HX/X is not a direct product of Sylow subgroups. We denote $\bar{\ } : G \to G/X$ is the natural homomorphism, and $Y \lhd \overline{G}$ is the minimal normal subgroup.

Assume that Y intersects trivially with $\overline{H}$. In this case, $\overline{H}Y/Y$ is also not a direct product of its Sylow subgroups; this contradicts the definition of the subgroup X. Thus $\overline{H}$ non-trivially intersects with every normal subgroup in $\overline{G}$. Let $H_Y = \overline{H} \cap Y$. We can assume that l divides $|H_Y|$.

Assume that Y is non-solvable. We have $Y = Y_1 \times \ldots \times Y_n$, where Y_i are isomorphic simple groups. By Lemma 2, it follows that for any $\alpha \in N(Y)$, there exists $\beta \in N(G)$ such that α divides β. By Lemma 4, we have that $|Y|| = |Y|$. If $n > 1$, then $|Y|$ is divisible by l^2. Therefore l^2 divides some number from $N(G)$; a contradiction. We conclude that $n = 1$, Y is a simple group, and $|Y|_{\{t,l\}} \in \{t, l, tl\}$. Notice that H_Y is either a Sylow l-subgroup or a Hall $\{t, l\}$-subgroup of Y. Since H is a $\{t, l\}$-group, we have $\pi(H_Y) \subseteq \pi(Y) \subseteq \{t, l\}$. If $|\pi(H_Y)| = 1$, then H_Y is a cyclic group of prime order. Suppose that $\pi(H_Y) = \{t, l\}$. Then $|H_Y| = tl$. From the fact that t does not divide $l - 1$ and l does not divide $t - 1$, it follows that H_Y is cyclic.

From the maximality of X, it follows that $\overline{H}Y/Y$ is a direct product of Sylow subgroups. From the homomorphism theorem, it follows that $\overline{H}Y/Y = \overline{H}/H_Y$. Let R be a Sylow l-subgroup of the group H_Y, and T be a Sylow t-subgroup of the group $\overline{H}$. Then $T \lhd N_{\overline{G}}(R)$. Since R is a cyclic group of order l and $l - 1$ does not divide t, it follows that T centralizes

R. Similarly, it can be shown that TR/R centralizes the Sylow l-subgroup of $\overline{H}/R$. Thus T centralizes the Sylow l-subgroup of $\overline{H}$ and hence H is a direct product of its Sylow t- and l-subgroups; a contradiction.

Thus we have shown that Y is an elementary abelian l-subgroup. Suppose that every t-element of the group $\overline{H}$ acts trivially on H_Y. Since $\overline{H}/H_Y$ is a direct product of its Sylow subgroups, it follows that $\overline{H}$ is also a direct product of its Sylow subgroups; a contradiction. Therefore, there exists a t-element x that acts non-trivially on H_Y. By Lemma 3, we have that $Y = C_Y(x) \times [\langle x\rangle, Y]$. Since $|[\langle x\rangle, Y]|$ divides $Ind(Y,x)$, and $l-1$ does not divide t, we have that l^2 divides $Ind(Y,x)$. Since Y is normal in $\overline{G}$, we conclude that $Ind(Y,x)$ divides $|x^{\overline{G}}|$. Therefore l^2 divides some number from $N(\overline{G})$; a contradiction. □

Lemma 13. *Let $t \in \Omega$. Then the Sylow t-subgroup of G is elementary abelian.*

Proof. The assertion of the Lemma follows from the fact that for any $\alpha \in N(G)$ we have $\alpha_t \in \{1, t\}$ and [24, Lemma 4]. □

It follows from Lemmas 12 and 13 that if $T < G$ and $\pi(T) \subseteq \Omega$, then T is an abelian group.

Proposition 1. *The group G does not contain a non-abelian composition factor S isomorphic to a group of Lie type such that $\Omega \subseteq \pi(S)$.*

Proof. The idea of the proof of this statement is similar to the idea of the proof of the main theorem from [14].

Suppose there exists a non-abelian composition factor S isomorphic to a group of Lie type such that $\Omega \subseteq \pi(S)$. Let $g \in G$ be a p-element such that its image $\overline{g} \in S$ is non-trivial. Since the Sylow p-subgroup of G is elementary abelian (see Lemma 13), we have that g has order p, and $|g^G|$ is not divisible by p. Hence $|\overline{g}^S|$ is also not divisible by p. Therefore $|g^G| = \sigma\delta$, where $\sigma \in \{1\} \cup \Phi_{(p,p)}$ and $\delta \in \{1, 12, 15, 20\}$. By Lemma 5, we have that $\sigma \neq 1$. The rest of the proof of the Proposition is divided into 2 lemmas.

Lemma 14. *Let $t \in \Omega$ be such that there exists an element $h \in G$ such that $|h^G| = \phi\alpha$, where $\phi \in \Psi(t,p)$ and $\alpha \in \{1, 12, 15, 20\}$. Then for any $r \in \Omega$ greater than t, there exists an element $l \in G$ such that $|l^G| = \phi'\beta$, where $\phi' \in \Psi(r,p)$ and $\beta \in \{1, 12, 15, 20\}$.*

Proof. Note that the group G does not contain a Hall $\{t, r\}$-subgroup. Therefore, there exists a non-abelian composition factor T which does not contain a Hall $\{t, r\}$-subgroup. Hence, there exists an element $\bar{l} \in T$ such that $|\bar{l}^S|$ is divisible by t. Let $l' \in G$ be the preimage of the element $\bar{l}$ of order r. We have that $|l'^G|$ is not divisible by r and is divisible by t. Therefore, $|l'^G| = \phi'\beta$, where $\phi' \in \Psi(r,p)$ and $\beta \in \{1, 12, 15, 20\}$. □

Let $t \in \Omega$ be the largest number such that there does not exist a t-element $h \in G$ with the property $|h^G| = \psi\beta$, where $\psi \in \Psi(t,p)$ and $\beta \in \{1, 12, 15, 20\}$.

Lemma 15. *$t < \frac{3}{4}p$.*

Proof. Suppose that $t \geq \frac{3}{4}p$.Let Δ be a subset of Ω containing numbers less than or equal to t. Note that for any $r \in \Delta$ and any r-element a, the number $|a^G|$ is not divisible by t. Therefore $|a^G| = \psi'\beta$, where $\psi' \in \Psi(t,p)$ and $\beta \in \{1, 12, 15, 20\}$. In particular $\pi(|a^G|) \cap \Delta = \emptyset$. Thus S contains an abelian Hall Δ-subgroup. This means that S contains an element g such that $\pi(g) = \Delta$. Similarly, as in Lemmas [14, 2.10, 2.11, 2.12, 2.13], we obtain a contradiction. □

Thus for any $r \in \Omega$, greater than $\frac{3}{4}p$, there exists an r-element $h \in G$ such that $|h^G| = \phi\alpha$, where $\phi \in \Phi(r,p)$ and $\alpha \in \{1, 12, 15, 20\}$. In particular, for any pair a, b of distinct numbers from $\Omega \setminus \Delta$, S does not contain an element of order ab. Therefore S cannot be isomorphic to an exceptional group of Lie type. Similarly, as in Lemma [14, 2.14], it is proved that S is not isomorphic to a classical group of Lie type. □

Lemma 16. *G does not include a non-abelian composition factor S isomorphic to a sporadic simple group such that $\Omega \cap \pi(S) \neq \emptyset$.*

Proof. The statement of the Lemma follows from the fact that the orders of sporadic groups are not divisible by primes greater than 71. □

Proposition 2. *Let $t \in \Omega$. There is a unique composition factor S such that $|S|_t = |G|_t$.*

Proof. Let $1 \lhd K_1 \lhd K_2 \lhd \ldots \lhd K_l \lhd G$ be some chief series. We denote $S_j = K_j/K_{j-1}$, $G_j = G/K_{j-1}$, and i is the smallest index such that $t \in \pi(S_i)$. Suppose that $t \in \pi(S_i) \cap \Omega$. We have $S_i = R_1 \times \ldots \times R_r$, where R_j is a simple group for $1 \leq j \leq r$. Since series $K_1, \ldots, K_l$ is chief series, we have that R_j are isomorphic subgroups for every $j = 1, \ldots, r$, and G_i acts transitively on the set $\Delta = R_1, \ldots, R_r$. We will divide the further proof of the proposition into 5 steps. In steps 1, 2, 3, and 4, we assume that $t \neq p$. If $t = p$ we can replace p by any number from Ω in all statements of steps 1, 2, 3, and 4.

Step 1. Suppose that $r > 2$. By Lemma 6 we have that G_i contains an element g of prime power order such that g acts without fixed points on the index set $\{1, 2, \ldots, r\}$. Up to renaming we can assume that $\{1, 2, \ldots, s\}$ is the orbit of 1 under the action of g. We have $C_{R_1 \times R_2 \times \ldots \times R_s}(g) \leq \{r_1 \cdot r_1^g \cdot r_1^{g^2} \cdot \ldots \cdot r_1^{g^l} \mid r_1 \in R_1,$ where l is the smallest integer such that $R_1^{g^l} = R_1\}$. Clearly, the set $\{r_1 \cdot r_1^g \cdot r_1^{g^2} \cdot \ldots \cdot r_1^{g^l}\}$ is a subgroup, which is isomorphic to

R_1. Thus, we have $\mathrm{Ind}(S_i, g)_t \geq |R_1|_t^{r-1}$. The number $\mathrm{Ind}(S_i, g)_t$ divides $|g^G|_t$. We have $|g^{G_i}|_t > t$; a contradiction.

Step 2. Suppose that $r = 2$. Assume that R_i is a non-abelian simple group. By Lemma 4, there exists an element $g \in S_i$ such that $|g^{S_i}|_t > t$. So, S_i and hence G contains an element whose conjugacy class size is divisible by t^2; a contradiction.

Therefore S_i is an elementary abelian group of order t^2. Suppose that G_i is non-solvable. Since $Aut(S_i) = \mathrm{GL}_2(t)$, it follows that G_i has the unique non-abelian composition factor L isomorphic to $L_2(t)$. Let $G_i \geq \bar{L}$ be a subgroup of minimal order such that $\bar{L}/K(\bar{L}) \simeq L$, where $K(\bar{L})$ is the solvable radical of $\bar{L}$. Let $g \in \bar{L}$ be an element of order $\frac{t+1}{2}$. We will show that $Ind(G_i, g)_t > t$. By Lemma 3, it follows that $S_i = C_{S_i}(g) \times [S_i, g]$, where $|[S_i, g]| - 1$ divides $|g|$. If $|g| = t+1$, then $|[S_i, g]| = t^2$, and therefore $|g^{G_i}|_t > t$; a contradiction. Thus G_i is a solvable group.

From the fact that $|K_i|_t = 1$ and G_i is solvable, it follows that for any $l \in \Omega$, the group G has a Hall $\{t, l\}$-subgroup H. In particular G contains a Hall $\{t, p\}$-subgroup H. By Lemmas 12 and 13, we have that H is an abelian group. Thus for any $h \in H$, $|h^G|_{\{t,p\}} = 1$. Therefore $|h^G| \in \{12, 15, 20\}$. Let $g \in G$ be a t-element. Suppose that g acts non-trivially on K_{i-1}. We have $|g^G| > t$; a contradiction with the fact that $l > \frac{1361}{2} > 20$. Thus the preimage of the group S_i in G acts trivially on K_{i-1}. Therefore, we can assume that S_i is a normal subgroup of G. Let $T = G/C_G(S_i)$. Therefore T is isomorphic to some subgroup of $\mathrm{GL}_2(t)$. Since $Z(G) = 1$, and in particular, $S_i \cap Z(G) = 1$, the group T acts without fixed points on S_i. Suppose that T contains an element h, the order of which does not divide $t - 1$. Since $h \in \mathrm{Aut}(S_i)$, we have that h acts non-trivially. Therefore $S_i = C_{S_i}(h) \times [S_i, h]$. Since $|h|$ does not divide $t - 1$, we have $|[S_i, h]| > t$, and therefore $|h^G|_t > t$; a contradiction. The Hall $\pi(\frac{t-1}{2})$-subgroup of $\mathrm{GL}_2(t)$ is a direct product of two cyclic subgroups of order $\frac{t-1}{2}$. Since the group T acts without fixed points on $S_i \simeq \mathbb{Z}_t \times \mathbb{Z}_t$, in this case, there is an element whose conjugacy class size is divisible by t^2; a contradiction. Therefore $r = 1$ and S_i is a simple group. We have $|S_i|_t = t$.

Step 3. Let $g \in S_i$ be a t-element. We will prove that $|g^{G_i}|$ is one of the numbers $12, 15, 20$. Suppose that $|g^{G_i}|$ is not divisible by any of the numbers $12, 15, 20$. If p does not divide $|S_i|$, then by Frattini's argument, $N_{G_i}(\langle g \rangle)$ contains a Sylow p-subgroup of G_i. By Lemma 12, it follows that p does not divide $|g^{G_i}|$. In particular, $|g^{G_i}|$ must divide one of the numbers 12, 15, or 20; a contradiction. Thus, p divides $|S_i|$.

Now we will prove that $\Omega \subseteq \pi(S_i)$. Let $h \in S_i$ be a p-element. It is clear that $|h^{G_i}|$ is not divisible by p. If there exists an $l \in \Omega$ such that $|h^{S_i}|_l = 1$, then, similarly to the previous argument, we can show that $|h^{G_i}|_l = 1$, and hence $|h^{G_i}|$ is not divisible by p or l. Since any $\{p, l\}$-subgroup of G is abelian, it follows that $|h'^G|_{\{p,l\}} = 1$, where $h' \in G$ is the preimage of the element h of prime order. Therefore $|h'^G| \in \{12, 15, 20\}$, and h centralizes some Sylow t-subgroup of S_i. We have that $|g^{S_i}|$ is not divisible by p, and hence $|g^{G_i}|$ is not divisible by p. Thus $|g^{G_i}|$ divides one of the numbers 12, 15, or 20; a contradiction. Therefore $\pi(h^{S_i}) \cap \Omega = \Omega \setminus \{p\}$, and $\Omega \subseteq \pi(S_i)$.

From Proposition 1 and Lemma 16, it follows that S_i is isomorphic to Alt_n. Since $p \in \pi(S_i)$, we have $n \geq p$. If $n > p$, then there exists a conjugacy class in S_i whose order does not divide any number from $N(G)$. Thus $S_i \simeq Alt_p$. Suppose that there is an element of G_i acting as an outer automorphism on S_i. We have $|h^{G_i}| = (p-1)!$. Let $h' \in G$ be the preimage of the element h of order p. If h' acts non-trivially on K_{i-1}, then $|h'^G| = (p-1)!k$, where $k > p$. In particular, $|h'^G|$ is greater than any number of $N(G)$. Thus $|h'^G| = (p-1)!$; a contradiction with the fact that there is no such number in $N(G)$. Therefore $G_i = S_i \times A$, and the size of any conjugacy class of A divides one of the numbers $12, 15, 20$.

If $|A|$ is not divisible by t, the lemma is proved, and S_i is the desired factor. Let $b \in A$ be a t-element. Suppose that b acts non-trivially on the socle C of the group A. We have $|b^A| > b > 20$; a contradiction. Hence t divides $|C|$. We can assume that $b \in C$. It is clear that the Sylow t-subgroup T of the group C is normal subgroup of A. Suppose that there is an element $c \in A$ that acts non-trivially on T. We have $T = C_T(c) \times [c, T]$ and $|[c, T]| > r$ divides $|c^A|$; a contradiction. Therefore $|b^A| = 1$. Let $b' \in G$ be the preimage of the element b. If b' acts trivially on K_{i-1}, then $|b'^G| = 1$; a contradiction. Thus b' acts non-trivially on K_{i-1}. In this case, $|b'^G| > t$.

Since $|K_{i-1}|$ is not divisible by t, it follows that $C_G(b')/C_{K_{i-1}}(b') = C_{G_i}(b)$. Let $d \in S_i$ be such that $|d|$ is not divisible by t and $|d^{S_i}|$ is maximal, and let $d' \in G$ be the preimage of the element d. Then $|(d'b')^G| = |d'^G||b'^G| > |d'^G|t$; a contradiction with the fact that $20|d'^G|$ is maximal in $N(G)$. Thus $|g^{G_i}|$ is one of the numbers $12, 15, 20$.

Step 4. Suppose that S_i is a non-abelian group. In the alternating group of degree greater than 10, there are no conjugacy classes of size less than 30. Since $t > \frac{1361}{2}$ and t divides $|S_i|$, S_i cannot be isomorphic to an alternating group. From Proposition 1 and Lemma 16, it follows that S_i is not isomorphic to any of the sporadic simple groups or groups of Lie type.

Step 5. Thus S_i is a group of order t. Since the Sylow t-subgroup is an elementary abelian group, it follows that $C_{G_i}(S_i) = S_i \times H$. Since the composition series was chosen arbitrarily, we can assume that $O_{t'}(H)$ is the trivial group. Thus, the order of any minimal normal subgroup of H divides t. By the arbitrariness of the composition series and already proven results, it follows that any minimal normal subgroup of H is a group of order t. Thus, the Sylow t-subgroup T of G_i is normal, and the order of the conjugacy class of any t-element in G_i is one of the numbers 12, 15, or 20. Let $g \in S_i$. Since $Ind(G_i, g) > 1$, there exists an element $a \in G_i \setminus C_{G_i}(g)$ of prime order. We have $T = S_i \times C_T(a)$. Let $h \in C_T(a)$. Then there exists an element $b \in G_i \setminus C_{G_i}(h)$ whose order is coprime to a. Since a lies in a normal subgroup not containing b, and b lies in a normal subgroup not containing a, it follows that a and b commute. Thus $Ind(G_i, ab)$ divides t^2; a contradiction. □

Lemma 17. *The group G includes a composition factor S such that $|S|_\Omega = |G|_\Omega$.*

Proof. By Proposition 2, it follows that for any $t \in \Omega$ there exists the unique composition factor S_t such that $|S_t|_t = t$ and $|G|_t/|S_t|_t = 1$. Let $K = O_{\Omega'}(G)$ and $\bar{\ } : G \to G/K$ be the natural homomorphism. Order of any minimal normal subgroup of the group $\overline{G}$ divides some number of Ω. Let S be a minimal normal subgroup of the group $\overline{G}$. Suppose that $\pi(S) \cap \Omega \neq \Omega$. Let l be the maximum number of $\Omega \setminus \pi(S)$ and $g \in \overline{G}$ be an element of order l. By Lemma 12 and Frattini's argument, we have that $|g^{\overline{G}}|$ is not divisible by numbers from $\pi(S) \cap \Omega$ and l. In particular, $|g^{\overline{G}}|$ is not divisible by p. Therefore $|g^{\overline{G}}|$ divides one of the numbers 12, 15, or 20. Let r be the maximum number of $\pi(S) \cap \Omega$ and $h \in S$ be an element of order r. We have $|h^{\overline{G}}|$ is not divisible by r and l. In particular, $|h^{\overline{G}}|$ is not divisible by p. Therefore $|h^{\overline{G}}|$ divides one of the numbers 12, 15, or 20. By Lemma 4, we have that S is a group of order r. Thus, the socle C of the group $\overline{G}$ is an abelian Hall Ω-subgroup of the group $\overline{G}$, and for any $g \in C$, it holds that $|g^{\overline{G}}|$ divides one of the numbers 12, 15, or 20.

Note that $C_{\overline{G}}(C) = C$. Let t be the maximum number of $\Omega \setminus \{p\}$. We have $C_{\overline{G}}(S_p) \cap C_{\overline{G}}(S_t) = 1$. Let $x \in C_{\overline{G}}(S_p)$ be a p'-element. The number $|x^{\overline{G}}|$ cannot divide 60, since in that case $x \in C_{\overline{G}}(C)$. Thus, $|x^{\overline{G}}|$ divides all the numbers from $\Omega \setminus \{p\}$. In particular, the p-complement T in the group $C_{\overline{G}}(C)$ acts non-trivially on S_l for any $l \in \Omega \setminus \{p\}$. Let $a \in S_t$. Since S_t is a cyclic group of order t and $|a^{\overline{G}}|$ does not exceed 20, we have that T is a cyclic group of order at most 20. Let $b \in S_p$. We have $|b^{\overline{G}}| = |\overline{G}|/C_{\overline{G}}(S_p) \leq 20$. Thus $|\overline{G}| \leq 20^2|C|$.

For $l \in \Omega$, define an element $y_l \in G$ such that $|y_l^G| = p!/(t(p-t)!)$. It is easy to see that size of the conjugacy class of any element from K is not divisible by numbers of Ω. Therefore $\overline{y_l}$ is non-trivial, and $\pi(|\overline{y_l}^G|) \cap \Omega = \Omega \setminus \{l\}$. Hence $\overline{y_l} \in C_{\overline{G}}(S_l)$ and $\overline{y_l}$ does not lie in $C_{\overline{G}}(S_k)$ for any $k \in \Omega \setminus \{l\}$. Therefore, $C_{\overline{G}}(S_l)$ does not lie in $C_{\overline{G}}(S_r)$. Thus, the Ω-complement O in $\overline{G}$ contains more than $|\Omega|$ proper normal subgroups, and therefore $|O| \geq 2^{|\Omega|}$; a contradiction. □

Lemma 18. *The group G includes a composition factor that is isomorphic to Alt_p.*

Proof. Lemma 17 implies that G contains a factor S such that $|G|_\Omega = |S|_\Omega$. It follows from the Proposition 1 and Lemma 16 that S is isomorphic to the alternating group Alt_n, where $n \geq p$. For $n > p$, there is a conjugacy class of S whose order does not divide any number of $N(G)$. □

We have $G = K.S.A$, where K is a solvable radical, S is the socle of the group $\bar{G} = G/K$, $A = \bar{G}/S$. Let T is the factor of G, isomorphic to Alt_p.

Lemma 19. *$T \leq S$.*

Proof. Assume that T is not a subgroup of the group S. Let $\overline{T} < \overline{G}$ be a subgroup of minimal order, one of whose factors is isomorphic to T. Note that $\overline{T}$ is generated by elements of order p. Let $g \in \overline{T}$ be an element of order p. We have $|g^{\overline{G}}| \geq |g^{\overline{T}}| \geq (p-1)!/2$. Note that the conjugacy class of the maximum order in the group G has size $10(p-1)!$. Suppose that g acts non-trivially on S. We will show that in this case, $Ind(\overline{G}, g) > 10(p-1)!$. Since the number $|S|$ is coprime to $|g|$, we have $C_A(gS) = C_{\overline{G}}(g)/C_S(g)$. In particular, $Ind(\overline{G}, g) = Ind(S, g)Ind(A, gS)$. Therefore, $Ind(S, g) \leq 20$. The element g acts non-trivially on the set of conjugacy classes by $C_S(g)$, and in particular, $|S|/|C_S(g)| \geq p > 20$, which implies that $Ind(S, g) > 20$; a contradiction.

Thus, $\overline{T}$ acts trivially on S; a contradiction with the fact that S is a self-centralizing subgroup. □

Lemma 20. *$G \simeq T \times X$.*

Proof. Let $\widetilde{T} \leq G$ be a subgroup of minimal order, whose image in $\overline{G}$ is T, $g \in \widetilde{T}$ be an element of order p. Arguing as in Lemma 19, we can show that $\widetilde{T}$ acts trivially on K. In particular, $\widetilde{T} \cap K$ lies in the center of the group $\widetilde{T}$. Therefore, $\widetilde{T}$ is the normal subgroup of G, and $\widetilde{T} \cap K$ is normal in G. From the fact that the Schur multiplier of an alternating group of degree greater than 6 has order 2, it follows that $\widetilde{T} \cap K$ is either trivial or a subgroup of order 2. Therefore, $\widetilde{T} \cap K$ lies in the center of the group G. Since the center of the group G is trivial, it follows that $\widetilde{T} \cap K = 1$. Hence G contains a normal subgroup isomorphic to T.

Assume that there is an element $x \in G$ that acts on $\widetilde{T}$ as an outer automorphism. We have $G/C_G(\widetilde{T}) \simeq Sym_p$. Let $g \in G$ be an element of order p. We have $Ind(G, g) = (p-1)!$, but there is no such a number in $N(G)$; a contradiction. Therefore, $\widetilde{T}$ is a direct factor of the group G. □

Lemma 21. *$X \simeq Alt_5$.*

Proof. We have $N(G) = N(T \times X) = N(T) \cdot N(X)$. We will show that $N(X) = \{12, 15, 20\}$. Let $g \in G$ be an element of order p, $x \in X$. We have $Ind(G, gx) = Ind(T, g)Ind(X, x) = (p-1)!/2 \cdot Ind(X, x)$. Since $|X|$ is not divisible by p, it follows that $Ind(X, x)$ is not divisible by p. Therefore, $Ind(G, gx)$ is one of the numbers $(p-1)!/2 \cdot 12, (p-1)!/2 \cdot 15, (p-1)!/2 \cdot 20$. This means that $Ind(X, x)$ is one of the numbers $12, 15, 20$. Let

$t \in \{1, 12, 15, 20\}$ and $y \in G$ such that $Ind(G, y) = (p-1)!/2 \cdot t$. Notice that $y \in C_G(g)$. Since $C_T(g) = \langle g \rangle$, we have $y^p \in X$. Therefore, $Ind(G, gy^p) = Ind(T, g)Ind(X, y^p) = (p-1)!/2 \cdot t$. This implies that $Ind(X, y^p) = t$ and $N(X) = \{1, 12, 15, 20\}$. Thus, from the N-recognizability of the group Alt_5 it follows that $X \simeq Alt_5$. □

Thus $G \simeq Alt_p \times Alt_5$.

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Поступила в редакцию / Received 26.03.2025
Поступила после рецензирования / Revised 18.05.2025
Принята к публикации / Accepted 17.06.2025