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Algebras of Binary Formulas for Weakly Circularly Minimal Theories with Trivial Definable Closure: Monotonic-to-right Case

Aizhan B. Altayeva[1], **Beibut Sh. Kulpeshov**[2,3], **Sergey V. Sudoplatov**[4,5]✉

[1] International Information Technology University, Almaty, Kazakhstan
[2] Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[3] Kazakh British Technical University, Almaty, Kazakhstan
[4] Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russian Federation
[5] Novosibirsk State Technical University, Novosibirsk, Russian Federation
✉ vip.altayeva@mail.ru, b.kulpeshov@kbtu.kz, sudoplat@math.nsc.ru

Abstract: This article concerns the notion of weak circular minimality being a variant of o-minimality for circularly ordered structures. We consider the binary level of these structures forming algebras of binary isolating formulas, which are based on families of labels and compositions of related formulas. These algebras are studied for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank greater than 1 with a trivial definable closure having a non-trivial monotonic-to-right function to the definable completion of a structure. On the basis of the study, the authors present a description of these algebras. It is shown that for this case there exist only commutative algebras. A strict s-deterministicity of such algebras for some natural number s is also established.

Keywords: algebra of binary formulas, $\aleph_0$-categorical theory, weak circular minimality, circularly ordered structure, convexity rank

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Научная статья

Алгебры бинарных формул для слабо циклически минимальных теорий с тривиальным определимым замыканием: случай монотонности вправо

А. Б. Алтаева[1], **Б. Ш. Кулпешов**[2,3], **С. В. Судоплатов**[4,5]✉

[1] Международный университет информационных технологий, Алматы, Казахстан
[2] Институт математики и математического моделирования МНВО РК, Алматы, Казахстан
[3] Казахстанско-Британский технический университет, Алматы, Казахстан
[4] Институт математики им. С. Л. Соболева СО РАН, Новосибирск, Российская Федерация
[5] Новосибирский государственный технический университет, Новосибирск, Российская Федерация
✉ kulpesh@mail.ru, sudoplat@math.nsc.ru

Аннотация: Изучается понятие слабой циклической минимальности как вариант o-минимальности для циклически упорядоченных структур. Рассматривается бинарный уровень этих структур, образующий алгебры бинарных изолирующих формул, которые основаны на семействах меток и композициях относящихся к ним формул. Эти алгебры изучаются для $\aleph_0$-категоричных 1-транзитивных непримитивных слабо циклически минимальных теорий ранга выпуклости больше 1 с тривиальным определимым замыканием, имеющим нетривиальную монотонную вправо функцию к определимому пополнению структуры. На основе исследования представлено описание этих алгебр. Показано, что для данного случая существуют только коммутативные алгебры. Также установлена строгая s-детерминированность таких алгебр для некоторого натурального числа s.

Ключевые слова: алгебра бинарных формул, $\aleph_0$-категоричная теория, слабая циклическая минимальность, циклически упорядоченная структура, ранг выпуклости

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1. Preliminaries

Algebras of binary formulas are a tool for describing relationships between elements of the sets of realizations of an one-type at the binary level with respect to the superposition of binary definable sets [17; 18]. A *binary isolating formula* is a formula of the form $\varphi(x, y)$ such that for some parameter a the formula $\varphi(a, y)$ isolates a complete type in $S(\{a\})$. In recent years, algebras of binary formulas have been studied intensively and have been continued in the works [4]– [9], [14], [15].

Let L be a countable first-order language. Throughout we consider L-structures and assume that L contains a ternary relational symbol K, interpreted as a circular order in these structures (unless otherwise stated).

Let $M = \langle M, \leq\rangle$ be a linearly ordered set. If we connect two endpoints of M (possibly, $-\infty$ and $+\infty$), then we obtain a circular order. More formally, the *circular order* is described by a ternary relation K satisfying the following conditions:

(co1) $\forall x \forall y \forall z (K(x, y, z) \to K(y, z, x))$;

(co2) $\forall x \forall y \forall z (K(x, y, z) \wedge K(y, x, z) \Leftrightarrow x = y \vee y = z \vee z = x)$;

(co3) $\forall x \forall y \forall z (K(x, y, z) \to \forall t[K(x, y, t) \vee K(t, y, z)])$;

(co4) $\forall x \forall y \forall z (K(x, y, z) \vee K(y, x, z))$.

The following observation relates linear and circular orders.

Fact 1. [1] *(i) If $\langle M, \leq\rangle$ is a linear ordering and K is the ternary relation derived from $\leq$ by the rule $K(x, y, z) :\Leftrightarrow (x \leq y \leq z) \vee (z \leq x \leq y) \vee (y \leq z \leq x)$ then K is a circular order relation on M.*

(ii) If $\langle N, K\rangle$ is a circular ordering and $a \in N$, then the relation $\leq_a$ defined on $M := N \setminus \{a\}$ by the rule $y \leq_a z :\Leftrightarrow K(a, y, z)$ is a linear order.

Thus, any linearly ordered structure is circularly ordered, since the relation of circular order is $\emptyset$-definable in an arbitrary linearly ordered structure. However, the opposite is not true. The following example shows that there are circularly ordered structures not being linearly ordered (in the sense that a linear ordering relation is not $\emptyset$-definable in an arbitrary circularly ordered structure).

Example 1. [2; 3] Let $\mathbb{Q}_2^* := \langle \mathbb{Q}_2, K, L\rangle$ be a circularly ordered structure, where $L = \{\sigma_0^2, \sigma_1^2\}$, for which the following conditions hold:

(i) its domain $\mathbb{Q}_2$ is a countable dense subset of the unit circle, no two points making the central angle π;

(ii) for distinct $a, b \in \mathbb{Q}_2$

$$(a, b) \in \sigma_0 \Leftrightarrow 0 < \arg(a/b) < \pi,$$

$$(a, b) \in \sigma_1 \Leftrightarrow \pi < \arg(a/b) < 2\pi,$$

where $\arg(a/b)$ means the value of the central angle between a and b clockwise.

Indeed, one can check that the linear order relation is not $\emptyset$-definable in this structure.

The notion of *weak circular minimality* was studied initially in [10]. Let $A \subseteq M$, where M is a circularly ordered structure. The set A is called *convex* if for any $a, b \in A$ the following property is satisfied: for any $c \in M$ with $K(a, c, b)$, $c \in A$ holds, or for any $c \in M$ with $K(b, c, a)$, $c \in A$ holds. A *weakly circularly minimal structure* is a circularly ordered structure $M = \langle M, K, \ldots \rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in M.

Let M be an $\aleph_0$-categorical weakly circularly minimal structure, $G := \mathrm{Aut}(M)$. Following the standard group theory terminology, the group G is called *k-transitive* if for any pairwise distinct $a_1, a_2, \ldots, a_k \in M$ and pairwise distinct $b_1, b_2, \ldots, b_k \in M$ there exists $g \in G$ such that $g(a_1) = b_1, g(a_2) = b_2, \ldots, g(a_k) = b_k$. A *congruence* on M is an arbitrary G-invariant equivalence relation on M. The group G is called *primitive* if G is 1-transitive and there are no non-trivial proper congruences on M.

Notation 2. (1) $K_0(x, y, z) := K(x, y, z) \wedge y \neq x \wedge y \neq z \wedge x \neq z$.

(2) $K(u_1, \ldots, u_n)$ denotes a formula saying that all subtuples of the tuple $\langle u_1, \ldots, u_n \rangle$ having the length 3 (in ascending order) satisfy K; similar notations are used for K_0.

(3) Let A, B, C be disjoint convex subsets of a circularly ordered structure M. We write $K(A, B, C)$ if for any $a, b, c \in M$ with $a \in A$, $b \in B$, $c \in C$ we have $K(a, b, c)$. We extend naturally that notation using, for instance, the notation $K_0(A, d, B, C)$ if $d \notin A \cup B \cup C$ and $K_0(A, d, B) \wedge K_0(d, B, C)$ holds.

Further we need the notion of the definable completion of a circularly ordered structure, introduced in [10]. Its linear analog was introduced in [16]. A *cut* $C(x)$ in a circularly ordered structure M is maximal consistent set of formulas of the form $K(a, x, b)$, where $a, b \in M$. A cut is said to be *algebraic* if there exists $c \in M$ that realizes it. Otherwise, such a cut is said to be *non-algebraic*. Let $C(x)$ be a non-algebraic cut. If there is some $a \in M$ such that either for all $b \in M$ the formula $K(a, x, b) \in C(x)$, or for all $b \in M$ the formula $K(b, x, a) \in C(x)$, then $C(x)$ is said to be *rational*. Otherwise, such a cut is said to be *irrational*. A *definable cut* in M is a cut $C(x)$ with the following property: there exist $a, b \in M$ such that $K(a, x, b) \in C(x)$ and the set $\{c \in M \mid K(a, c, b) \text{ and } K(a, x, c) \in C(x)\}$ is definable. The *definable completion* $\overline{M}$ of a structure M consists of M together with all definable cuts in M that are irrational (essentially $\overline{M}$ consists of endpoints of definable subsets of the structure M).

Notation 3. [10] Let $F(x,y)$ be an L-formula such that $F(M,b)$ is convex infinite co-infinite for each $b \in M$. Let $F^{\ell}(y)$ be the formula saying y is a left endpoint of $F(M,y)$:

$$\exists z_1 \exists z_2 [K_0(z_1,y,z_2) \wedge \forall t_1 (K(z_1,t_1,y) \wedge t_1 \neq y \to \neg F(t_1,y)) \wedge$$
$$\forall t_2 (K(y,t_2,z_2) \wedge t_2 \neq y \to F(t_2,y))].$$

We say that $F(x,y)$ is *convex-to-right* if

$$M \models \forall y \forall x [F(x,y) \to F^l(y) \wedge \forall z (K(y,z,x) \to F(z,y))].$$

If $F_1(x,y), F_2(x,y)$ are arbitrary convex-to-right formulas we say F_2 is *bigger than* F_1 if there is $a \in M$ with $F_1(M,a) \subset F_2(M,a)$. If M is 1-transitive and this holds for some a, it holds for all a. This gives a total ordering on the (finite) set of all convex-to-right formulas $F(x,y)$ (viewed up to equivalence modulo $Th(M)$).

Consider $F(M,a)$ for arbitrary $a \in M$. In general, $F(M,a)$ has no the right endpoint in M. For example, if $dcl(\{a\}) = \{a\}$ holds for some $a \in M$ then for any convex-to-right formula $F(x,y)$ and any $a \in M$ the formula $F(M,a)$ has no the right endpoint in M. We write $f(y) := \text{rend } F(M,y)$, assuming that $f(y)$ is the right endpoint of the set $F(M,y)$ that lies in general in the definable completion $\overline{M}$ of M. Then f is a function mapping M in $\overline{M}$.

Let $F(x,y)$ be a convex-to-right formula. We say that $F(x,y)$ is *equivalence-generating* if for any $a,b \in M$ such that $M \models F(b,a)$ the following holds:

$$M \models \forall x (K(b,x,a) \wedge x \neq a \to [F(x,a) \leftrightarrow F(x,b)]).$$

Lemma 4. [12] *Let M be an $\aleph_0$-categorical 1-transitive weakly circularly minimal structure, $F(x,y)$ be a convex-to-right formula that is equivalence-generating. Then $E(x,y) := F(x,y) \vee F(y,x)$ is an equivalence relation partitioning M into infinite convex classes.*

Notation 5. Let $E(x,y)$ be an $\emptyset$–definable equivalence relation partitioning M into infinite convex classes. Suppose that y lies in $\overline{M}$ (non-obligatory in M). Then

$$E^*(x,y) := \exists y_1 \exists y_2 [y_1 \neq y_2 \wedge \forall t (K(y_1,t,y_2) \to E(t,x)) \wedge K_0(y_1,y,y_2)].$$

Let M, N be circularly ordered structures. The 2-*reduct* of M is a circularly ordered structure with the same universe of M and consisting of predicates for each $\emptyset$-definable relation on M of arity ≤ 2 as well as of the ternary predicate K for the circular order, but does not have other predicates of arities more than two. We say that the structure M is *isomorphic* to N *up to binarity* or *binarily isomorphic to* N if the 2-reduct of M is isomorphic to the 2-reduct of N.

Let f be a unary function from M to $\overline{M}$. We say that f is *monotonic-to-right (left) on* M if it preserves (reverses) the relation K_0, i.e. for any $a, b, c \in M$ such that $K_0(a, b, c)$, we have $K_0(f(a), f(b), f(c))$ $(K_0(f(c), f(b), f(a)))$.

The following definition can be used in a circular ordered structure as well.

Definition 1. [13] Let T be a weakly o-minimal theory, M be a sufficiently saturated model of T, $A \subseteq M$. *The rank of convexity of the set* A $(RC(A))$ is defined as follows:

1) $RC(A) = -1$ if $A = \emptyset$.
2) $RC(A) = 0$ if A is finite and non-empty.
3) $RC(A) \geq 1$ if A is infinite.
4) $RC(A) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:
   – For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;
   – For every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A.
5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α, we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A)) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, i.e. $RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$.

The rank of convexity of an 1-type p is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

In particular, a theory has convexity rank 1 if there is no definable (with parameters) equivalence relations with infinitely many infinite convex classes.

The following theorem characterizes up to binarity $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structures M of convexity rank greater than 1 having both a trivial definable closure and a convex-to-right formula $R(x, y)$ such that $r(y) := \mathrm{rend}R(M, y)$ is monotonic-to-right on M:

Theorem 1. [11] *Let M be an $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank greater than* 1, *$dcl(\{a\}) = \{a\}$ for some $a \in M$. Suppose that there exists a convex-to-right formula $R(x, y)$ such that $r(y) := \mathrm{rend}R(M, y)$ is monotonic-to-right on M. Then M is isomorphic up to binarity to*

$$M'_{s,m,k} := \langle M, K^3, E_1^2, E_2^2, \ldots, E_s^2, E_{s+1}^2, R^2 \rangle,$$

where M is a circularly ordered structure, M is densely ordered, $s \geq 1$; E_{s+1} is an equivalence relation partitioning M into m infinite convex classes without endpoints; E_i for every $1 \leq i \leq s$ is an equivalence relation partitioning every E_{i+1}-class into infinitely many infinite convex E_i-subclasses without endpoints so that the induced order on E_i-subclasses is dense without endpoints; $R(M,a)$ has no right endpoint in M and $r^k(a) = a$ for all $a \in M$ and some $k \geq 2$, where $r^k(y) := r(r^{k-1}(y))$; for every $1 \leq i \leq s+1$ and any $a \in M$

$$M'_{s,m,k} \models \neg E_i^*(a, r(a)) \wedge \forall y(E_i(y,a) \to \exists u[E_i^*(u, r(a)) \wedge E_i^*(u, r(y))]),$$

$m = 1$ or k divides m.

In [9] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories with a primitive automorphism group. Here we describe algebras of binary isolating formulas for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and a trivial definable closure having a monotonic-to-right function to the definable completion of a structure.

2. Results

Let M be an 1-transitive structure. We denote every binary isolating formula acting in M by a label $u \in \rho_M$, where ρ_M denotes the set of all labels for the algebra $\mathcal{P}_M$ of binary isolating formulas of the structure M.

Definition 2. The algebra $\mathcal{P}_M$ is said to be *deterministic* if $u_1 \cdot u_2$ is a singleton for any labels $u_1, u_2 \in \rho_M$.

Generalizing the last definition, we say that the algebra $\mathcal{P}_M$ is *m-deterministic* if the product $u_1 \cdot u_2$ consists of at most m elements for any labels $u_1, u_2 \in \rho_M$. We also say that an m-deterministic algebra $\mathcal{P}_M$ is *strictly m-deterministic* if it is not $(m-1)$-deterministic.

Example 2. Consider the structure $M'_{1,1,2} := \langle M, K^3, E_1^2, R^2 \rangle$ from Theorem 1 with the condition that the function $r(y) := \text{rend}R(M,y)$ is monotonic-to-right on M.

We assert that $Th(M'_{1,1,2})$ has seven binary isolating formulas:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := K_0(x,y,r(x)) \wedge E_1(x,y),$$

$$\theta_2(x,y) := K_0(x,y,r(x)) \wedge \neg E_1(x,y) \wedge \neg E_1^*(y, r(x)),$$

$$\theta_3(x,y) := K_0(x,y,r(x)) \wedge \neg E_1(x,y) \wedge E_1^*(y, r(x)),$$

$$\theta_4(x,y) := K_0(r(x),y,x) \wedge \neg E_1(x,y) \wedge E_1^*(y,r(x)),$$
$$\theta_5(x,y) := K_0(r(x),y,x) \wedge \neg E_1(x,y) \wedge \neg E_1^*(y,r(x)),$$
$$\theta_6(x,y) := K_0(r(x),y,x) \wedge E_1(x,y),$$

and the following holds for any $a \in M$:

$$K_0(\theta_0(a,M),\theta_1(a,M),\theta_2(a,M),\theta_3(a,M),\theta_4(a,M),\theta_5(a,M),\theta_6(a,M)).$$

Define labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y), \text{ where } 0 \le k \le 6.$$

It easy to check that for the algebra $\mathfrak{P}_{M'_{1,1,2}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3	4	5	6
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{3,4\}$	$\{4\}$	$\{5\}$	$\{0,1,6\}$
2	$\{2\}$	$\{2\}$	$\{2,3,4,5\}$	$\{5\}$	$\{5\}$	$\{0,1,2,5,6\}$	$\{2\}$
3	$\{3\}$	$\{3,4\}$	$\{5\}$	$\{6\}$	$\{0,1,6\}$	$\{2\}$	$\{3\}$
4	$\{4\}$	$\{4\}$	$\{5\}$	$\{0,1,6\}$	$\{1\}$	$\{2\}$	$\{3,4\}$
5	$\{5\}$	$\{5\}$	$\{0,1,2,5,6\}$	$\{2\}$	$\{2\}$	$\{2,3,4,5\}$	$\{5\}$
6	$\{6\}$	$\{0,1,6\}$	$\{2\}$	$\{3\}$	$\{3,4\}$	$\{5\}$	$\{6\}$

By the Cayley table the algebra $\mathfrak{P}_{M'_{1,1,2}}$ is commutative and strictly 5-deterministic.

Example 3. Consider the structure $M'_{2,1,2} := \langle M, K^3, E_1^2, E_2^2, R^2 \rangle$ from Theorem 1 with the condition that the function $r(y) := \mathrm{rend}R(M,y)$ is monotonic-to-right on M.

We assert that $Th(M'_{2,1,2})$ has eleven binary isolating formulas:

$$\theta_0(x,y) := x = y,$$

$$\theta_1(x,y) := K_0(x,y,r(x)) \wedge E_1(x,y),$$
$$\theta_2(x,y) := K_0(x,y,r(x)) \wedge \neg E_1(x,y) \wedge E_2(x,y),$$
$$\theta_3(x,y) := K_0(x,y,r(x)) \wedge \neg E_2(x,y) \wedge \neg E_2^*(y,r(x)),$$
$$\theta_4(x,y) := K_0(x,y,r(x)) \wedge E_2^*(y,r(x)) \wedge \neg E_1^*(y,r(x)),$$
$$\theta_5(x,y) := K_0(x,y,r(x)) \wedge E_1^*(y,r(x)),$$
$$\theta_6(x,y) := K_0(r(x),y,x) \wedge E_1^*(y,r(x)),$$
$$\theta_7(x,y) := K_0(r(x),y,x) \wedge E_2^*(y,r(x)) \wedge \neg E_1^*(y,r(x)),$$
$$\theta_8(x,y) := K_0(r(x),y,x) \wedge \neg E_2(x,y) \wedge \neg E_2^*(y,r(x)),$$
$$\theta_9(x,y) := K_0(r(x),y,x) \wedge \neg E_2(x,y) \wedge \neg E_1(x,y),$$

$$\theta_{10}(x,y) := K_0(r(x),y,x) \wedge E_1(x,y).$$

Obviously, the following holds for any $a \in M$:

$$K_0(\theta_0(a,M),\theta_1(a,M),\theta_2(a,M),\theta_3(a,M),\ldots,\theta_9(a,M),\theta_{10}(a,M)).$$

Define labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y), \text{ where } 0 \le k \le 10.$$

It easy to check that for the algebra $\mathfrak{P}_{M'_{2,1,2}}$ the following equalities hold:
$0 \cdot k = k \cdot 0 = \{k\}$ for every $0 \le k \le 10$,
$1 \cdot 1 = \{1\}$, $1 \cdot 2 = \{2\}$, $1 \cdot 3 = \{3\}$, $1 \cdot 4 = \{4\}$, $1 \cdot 5 = \{5,6\}$, $1 \cdot 6 = \{6\}$, $1 \cdot 7 = \{7\}$, $1 \cdot 8 = \{8\}$, $1 \cdot 9 = \{9\}$, and $1 \cdot 10 = \{10,0,1\}$,
$2 \cdot 1 = \{2\}$, $2 \cdot 2 = \{2\}$, $2 \cdot 3 = \{3\}$, $2 \cdot 4 = \{4,5,6,7\}$, $2 \cdot 5 = \{7\}$, $2 \cdot 6 = \{7\}$, $2 \cdot 7 = \{7\}$, $2 \cdot 8 = \{8\}$, $2 \cdot 9 = \{9,10,0,1,2\}$, and $2 \cdot 10 = \{2\}$,
$3 \cdot 1 = \{3\}$, $3 \cdot 2 = \{3\}$, $3 \cdot 3 = \{3,4,5,6,7,8\}$, $3 \cdot 4 = \{8\}$, $3 \cdot 5 = \{8\}$, $3 \cdot 6 = \{8\}$,
$3 \cdot 7 = \{8\}$, $3 \cdot 8 = \{8,9,10,0,1,2,3\}$, $3 \cdot 9 = \{3\}$, and $3 \cdot 10 = \{3\}$,
$4 \cdot 1 = \{4\}$, $4 \cdot 2 = \{4,5,6,7\}$, $4 \cdot 3 = \{8\}$, $4 \cdot 4 = \{9\}$, $4 \cdot 5 = \{9\}$, $4 \cdot 6 = \{9\}$, $4 \cdot 7 = \{9,10,0,1,2\}$, $4 \cdot 8 = \{3\}$, $4 \cdot 9 = \{4\}$, and $4 \cdot 10 = \{4\}$,
$5 \cdot 1 = \{5,6\}$, $5 \cdot 2 = \{7\}$, $5 \cdot 3 = \{8\}$, $5 \cdot 4 = \{9\}$, $5 \cdot 5 = \{10\}$, $5 \cdot 6 = \{10,1,2\}$, $5 \cdot 7 = \{2\}$, $5 \cdot 8 = \{3\}$, $5 \cdot 9 = \{5\}$, and $5 \cdot 10 = \{5\}$,
$6 \cdot 1 = \{6\}$, $6 \cdot 2 = \{7\}$, $6 \cdot 3 = \{8\}$, $6 \cdot 4 = \{9\}$, $6 \cdot 5 = \{10,0,1\}$, $6 \cdot 6 = \{1\}$, $6 \cdot 7 = \{2\}$, $6 \cdot 8 = \{3\}$, $6 \cdot 9 = \{4\}$, and $6 \cdot 10 = \{5,6\}$,
$7 \cdot 1 = \{7\}$, $7 \cdot 2 = \{7\}$, $7 \cdot 3 = \{8\}$, $7 \cdot 4 = \{9,10,0,1,2\}$, $7 \cdot 5 = \{2\}$, $7 \cdot 6 = \{2\}$, $7 \cdot 7 = \{2\}$, $7 \cdot 8 = \{3\}$, $7 \cdot 9 = \{4,5,6,7\}$, and $7 \cdot 10 = \{7\}$,
$8 \cdot 1 = \{8\}$, $8 \cdot 2 = \{8\}$, $8 \cdot 3 = \{8,9,10,0,1,2,3\}$, $8 \cdot 4 = \{3\}$, $8 \cdot 5 = \{3\}$, $8 \cdot 6 = \{3\}$, $8 \cdot 7 = \{3\}$, $8 \cdot 8 = \{3,4,5,6,7,8\}$, $8 \cdot 9 = \{8\}$, and $8 \cdot 10 = \{8\}$,
$9 \cdot 1 = \{9\}$, $9 \cdot 2 = \{9,10,0,1,2\}$, $9 \cdot 3 = \{3\}$, $9 \cdot 4 = \{4\}$, $9 \cdot 5 = \{4\}$, $9 \cdot 6 = \{4\}$, $9 \cdot 7 = \{4,5,6,7\}$, $9 \cdot 8 = \{8\}$, $9 \cdot 9 = \{9\}$ and $9 \cdot 10 = \{9\}$,
$10 \cdot 1 = \{10,0,1\}$, $10 \cdot 2 = \{2\}$, $10 \cdot 3 = \{3\}$, $10 \cdot 4 = \{4\}$, $10 \cdot 5 = \{5\}$, $10 \cdot 6 = \{5,6\}$, $10 \cdot 7 = \{7\}$, $10 \cdot 8 = \{8\}$, $10 \cdot 9 = \{9\}$ and $10 \cdot 10 = \{9\}$.
By these equalities the algebra $\mathfrak{P}_{M'_{2,1,2}}$ is commutative and strictly 7-deterministic.

Theorem 2. *The algebra $\mathfrak{P}_{M'_{s,m,k}}$ of binary isolating formulas with monotonic-to-right function r has $2sk+m+k+1$ labels, is commutative and strictly $(2s+3)$-deterministic for all valid values s,m and k.*

Proof of Theorem 2. We assert that the algebra $\mathfrak{P}_{M'_{s,m,k}}$ has $2sk+m+k+1$ binary isolating formulas: $\theta_0(x,y) := x = y$, $\theta_{(2s+m/k+1)i+1}(x,y) := K_0(r^i(x),y,r^{i+1}(x)) \wedge E^*_1(y,r^i(x))$, where $0 \le i \le k-1$, $\theta_{(2s+m/k+1)i+w}(x,y) := K_0(r^i(x),y,r^{i+1}(x)) \wedge E^*_w(y,r^i(x)) \wedge \neg E^*_{w-1}(y,r^i(x))$, where $0 \le i \le k-1, 2 \le w \le s+1$, $\theta_{(2s+m/k+1)(i+1)+1-w}(x,y) := K_0(r^i(x),y,r^{i+1}(x)) \wedge$

$E^*_w(y, r^{i+1}(x)) \wedge \neg E^*_{w-1}(y, r^{i+1}(x))$, where $0 \leq i \leq k-1, 2 \leq w \leq s+1$, $\theta_{(2s+m/k+1)(i+1)}(x, y) := K_0(r^i(x), y, r^{i+1}(x)) \wedge E^*_1(y, r^{i+1}(x))$, where $0 \leq i \leq k-1$, $\theta_{(2s+m/k+1)i+s+j}(x, y) := K_0(r^i(x), y, r^{i+1}(x)) \wedge \neg E^*_{s+1}(y, r^i(x)) \wedge \neg E^*_{s+1}(y, r^{i+1}(x)) \wedge \exists t_1 \exists t_2 \ldots \exists t_{j-1} [K_0(r^i(x), t_1, t_2, \ldots, t_{j-1}, r^{i+1}(x)) \wedge \neg E^*_{s+1}(t_1, r^i(x)) \wedge \wedge_{l=1}^{j-2} \neg E_{s+1}(t_l, t_{l+1}) \wedge E_{s+1}(t_{j-1}, y) \wedge \forall u (K_0(r^i(x), u, t_{j-1}) \rightarrow E^*_{s+1}(u, r^i(x)) \vee \vee_{l=1}^{j-1} E_{s+1}(u, t_l))]$, where $0 \leq i \leq k-1, 2 \leq j \leq m/k$.

Thus, we have $1+k+sk+sk+k+k(m/k-1) = 2sk+m+k+1$ binary isolating formulas. Moreover, we have defined the formulas so that for any $a \in M$ the following holds: $K_0(\theta_0(a, M), \theta_1(a, M), \theta_2(a, M), \ldots, \theta_{2sk+m+k-1}(a, M), \theta_{2sk+m+k}(a, M))$.

Prove now that the algebra $\mathfrak{P}_{M'_{s,1,k}}$ is commutative and strictly $(2s+3)$-deterministic for all valid values s, m and k.

Firstly, obviously that $0 \cdot l = l \cdot 0 = \{l\}$ for any $0 \leq l \leq 2sk+m+k$. Suppose further that $l_1 \neq 0$ and $l_2 \neq 0$.

Consider the following formula $\exists t[\theta_{l_1}(x, t) \wedge \theta_{l_2}(t, y)]$.

Case 1: $l_1 = (2s+m/k+1)i_1 + 1$ for some $0 \leq i_1 \leq k-1$.

We have: $K_0(r^{i_1}(x), t, r^{i_1+1}(x))$ and $E^*_1(t, r^{i_1}(x))$.

Let also $l_2 = (2s+m/k+1)i_2 + 1$ for some $0 \leq i_2 \leq k-1$, i.e. $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$ and $E^*_1(y, r^{i_2}(t))$. Whence we obtain: $K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x))$ and $E^*_1(y, r^{i_1+i_2}(x))$.

Let $q = (i_1+i_2)$[mod k]. Clearly, $0 \leq i_1+i_2 \leq (k-1)+(k-1) = 2k-2$, whence $0 \leq q \leq k-2$. If $i_1+i_2 < k$ then $l_1 \cdot l_2 = \{(2s+m/k+1)(i_1+i_2)+1\}$. If $i_1+i_2 \geq k$ then we have $K_0(r^q(x), y, r^{q+1}(x))$ and $E^*_1(y, r^q(x))$. Thus, $l_1 \cdot l_2 = \{(2s+m/k+1)q+1\}$.

Let now $l_2 = (2s+m/k+1)i_2+w$ for some $0 \leq i_2 \leq k-1$ and $2 \leq w \leq s+1$. Then we have: $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $E^*_w(y, r^{i_2}(t))$ and $\neg E^*_{w-1}(y, r^{i_2}(t))$. Whence we obtain: $K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x))$, $E^*_w(y, r^{i_1+i_2}(x))$ and $\neg E^*_{w-1}(y, r^{i_1+i_2}(x))$.

If $i_1+i_2 < k$ then $l_1 \cdot l_2 = \{(2s+m/k+1)(i_1+i_2)+w\}$. If $i_1+i_2 \geq k$ then we have $K_0(r^q(x), y, r^{q+1}(x))$, $E^*_w(y, r^q(x))$ and $\neg E^*_{w-1}(y, r^q(x))$, where $q = (i_1+i_2)$[mod k]. Then $l_1 \cdot l_2 = \{(2s+m/k+1)q+w\}$.

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $E^*_w(t, r^{i_2}(x))$, $\neg E^*_{w-1}(t, r^{i_2}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$ and $E^*_1(y, r^{i_1}(t))$. Whence we obtain: $K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x))$, $E^*_w(y, r^{i_1+i_2}(x))$ and $\neg E^*_{w-1}(y, r^{i_1+i_2}(x))$, and consequently, $l_2 \cdot l_1 = l_1 \cdot l_2$.

Let now $l_2 = (2s+m/k+1)(i_2+1)+1-w$ for some $0 \leq i_2 \leq k-1$ and $2 \leq w \leq s+1$, i.e. $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $E^*_w(y, r^{i_2+1}(t))$ and $\neg E^*_{w-1}(y, r^{i_2+1}(t))$. Whence we obtain: $K_0(r^{i_1+i_2}(t), y, r^{i_1+i_2+1}(t))$, $E^*_w(y, r^{i_1+i_2+1}(t))$ and $\neg E^*_{w-1}(y, r^{i_1+i_2+1}(t))$.

Let $q = (i_1+i_2)$[mod k]. Then $l_1 \cdot l_2 = \{(2s+m/k+1)(q+1)+1-w\}$.

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $E^*_w(t, r^{i_2}(x))$, $\neg E^*_{w-1}(t, r^{i_2+1}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$ and $E^*_1(y, r^{i_1}(t))$.

Whence we obtain:

$$K_0(r^{i_1+i_2}(t), y, r^{i_1+i_2+1}(t)), E^*_w(y, r^{i_1+i_2+1}(t)) \text{ and } \neg E^*_{w-1}(y, r^{i_1+i_2+1}(t)),$$

and consequently, $l_2 \cdot l_1 = l_1 \cdot l_2$.

Let now $l_2 = (2s + m/k + 1)(i_2 + 1)$ for some $0 \leq i_2 \leq k - 1$, i.e. $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$ and $E^*_1(y, r^{i_2+1}(t))$. Whence we obtain: $E^*_1(y, r^{i_1+i_2+1}(x))$ and either $K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x))$ or $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$.

Let $q = (i_1+i_2+1)$[mod k]. Clearly, $0 \leq q \leq k-1$, since $1 \leq i_1+i_2+1 \leq (k-1)+(k-1)+1 = 2k-1$. If $q = 0$ then $l_1 \cdot l_2 = \{0, 1, 2sk + m + k\}$. If $q \neq 0$ then $l_1 \cdot l_2 = \{(2s + m/k + 1)q, (2s + m/k + 1)q + 1\}$.

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $E^*_1(t, r^{i_2+1}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$ and $E^*_1(y, r^{i_1}(t))$. Whence we obtain: $E^*_1(y, r^{i_1+i_2+1}(x))$ and either $K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x))$ or $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, and consequently, $l_2 \cdot l_1 = l_1 \cdot l_2$.

Let now $l_2 = (2s+m/k+1)i_2+s+j$ for some $0 \leq i_2 \leq k-1$ and $2 \leq j \leq m/k$, i.e. $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $\neg E^*_{s+1}(y, r^{i_2}(t))$, $\neg E^*_{s+1}(y, r^{i_2+1}(t))$ and there exist $y_1, y_2, \ldots, y_{j-1}$ such that $K_0(r^{i_2}(t), y_1, y_2, \ldots, y_{j-1}, r^{i_2+1}(t))$, $\neg E^*_{s+1}(y_1, r^{i_2}(t))$, $\wedge_{l=1}^{j-2} \neg E_{s+1}(y_l, y_{l+1})$, $E_{s+1}(y_{j-1}, y)$ and

$$M \models \forall u[K_0(r^{i_2}(t), u, y_{j-1}) \to E^*_{s+1}(u, r^{i_2}(t)) \vee \vee_{l=1}^{j-1} E_{s+1}(u, y_l)].$$

Whence we obtain: $K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x))$, $\neg E^*_{s+1}(y, r^{i_1+i_2}(x))$, $\neg E^*_{s+1}(y, r^{i_1+i_2+1}(x))$ and there exist $y_1, y_2, \ldots, y_{j-1}$ such that $K_0(r^{i_1+i_2}(t), y_1, y_2, \ldots, y_{j-1}, r^{i_1+i_2+1}(t))$, $\neg E^*_{s+1}(y_1, r^{i_1+i_2}(t))$, $\wedge_{l=1}^{j-2} \neg E_{s+1}(y_l, y_{l+1})$, $E_{s+1}(y_{j-1}, y)$ and $M \models \forall u[K_0(r^{i_1+i_2}(t), u, y_{j-1}) \to E^*_{s+1}(u, r^{i_1+i_2}(t)) \vee \vee_{l=1}^{j-1} E_{s+1}(u, y_l)]$.

Let $q = (i_1 + i_2)$[mod k]. Then $l_1 \cdot l_2 = \{(2s + m/k + 1)q + s + j\}$.

By considering the product $l_2 \cdot l_1$ we obtain that $l_2 \cdot l_1 = l_1 \cdot l_2$.

Case 2. $l_1 = (2s + m/k + 1)i_1 + w_1$ for some $0 \leq i_1 \leq k - 1$ and $2 \leq w_1 \leq s + 1$.

We have the following: $K_0(r^{i_1}(x), t, r^{i_1+1}(x))$, $E^*_{w_1}(t, r^{i_1}(x))$ and $\neg E^*_{w_1-1}(t, r^{i_1}(x))$. Let also $l_2 = (2s+m/k+1)i_2+w_2$ for some $0 \leq i_2 \leq k-1$ and $2 \leq w_2 \leq s+1$. Then we have $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $E^*_{w_2}(y, r^{i_2}(t))$ and $\neg E^*_{w_2-1}(y, r^{i_2}(t))$.

Let $q = (i_1 + i_2)$[mod k]. If $w_1 \leq w_2$, we obtain: $K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x))$, $E^*_{w_2}(y, r^{i_1+i_2}(x))$ and $\neg E^*_{w_2-1}(y, r^{i_1+i_2}(x))$, whence $l_1 \cdot l_2 = \{(2s+m/k+1)q+w_2\}$. If $w_1 > w_2$, we obtain: $K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x))$, $E^*_{w_1}(y, r^{i_1+i_2}(x))$ and $\neg E^*_{w_1-1}(y, r^{i_1+i_2}(x))$, whence $l_1 \cdot l_2 = \{(2s + m/k + 1)q + w_1\}$.

Let now $l_2 = (2s+m/k+1)(i_2+1)+1-w_2$ for some $0 \leq i_2 \leq k-1$ and $2 \leq w_2 \leq s + 1$. We have the following: $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $E^*_{w_2}(y, r^{i_2+1}(t))$ and $\neg E^*_{w_2-1}(y, r^{i_2+1}(t))$.

Let $q = (i_1 + i_2)[\text{mod k}]$. If $w_1 < w_2$, we obtain: $K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x))$, $E^*_{w_2}(y, r^{i_1+i_2+1}(x))$ and $\neg E^*_{w_2-1}(y, r^{i_1+i_2+1}(x))$ and consequently, $l_1 \cdot l_2 = \{(2s + m/k + 1)(q + 1) + 1 - w_2\}$.

If $w_1 = w_2$, we obtain: $E^*_1(y, r^{i_1+i_2+1}(x))$ and either $K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x))$ or $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$. If $i_1+i_2+1 = k$ then $l_1 \cdot l_2 = \{0, 1, 2sk + m + k\}$. If $i_1 + i_2 + 1 \neq k$ then $l_1 \cdot l_2 = \{(2s + m/k + 1)(q + 1), (2s + m/k + 1)(q + 1) + 1\}$.

If $w_1 > w_2$, we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x)), E^*_{w_1}(y, r^{i_1+i_2+1}(x))$ and $\neg E^*_{w_1-1}(y, r^{i_1+i_2+1}(x))$, whence $l_1 \cdot l_2 = \{(2s + m/k + 1)(q + 1) + w_1\}$.

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $E^*_{w_2}(t, r^{i_2+1}(x))$, $\neg E^*_{w_2-1}(t, r^{i_2+1}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$, $E^*_{w_1}(y, r^{i_1}(t))$ and $\neg E^*_{w_1-1}(y, r^{i_1}(t))$.

Let $q = (i_1 + i_2)[\text{mod k}]$. If $w_1 < w_2$, we obtain: $K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x))$, $E^*_{w_2}(y, r^{i_1+i_2+1}(x))$ and $\neg E^*_{w_2-1}(y, r^{i_1+i_2+1}(x))$, whence $l_2 \cdot l_1 = \{(2s + m/k + 1)(q + 1) + 1 - w_2\}$.

If $w_1 = w_2$, we obtain: $E^*_1(y, r^{i_1+i_2+1}(x))$ and either $K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x))$ or $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$. If $i_1+i_2+1 = k$ then $l_2 \cdot l_1 = \{0, 1, 2sk + m + k\}$. If $i_1 + i_2 + 1 \neq k$ then $l_1 \cdot l_2 = \{(2s + m/k + 1)(q + 1), (2s + m/k + 1)(q + 1) + 1\}$.

If $w_1 > w_2$, we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $E^*_{w_1}(y, r^{i_1+i_2+1}(x))$ and $\neg E^*_{w_1-1}(y, r^{i_1+i_2+1}(x))$, whence $l_2 \cdot l_1 = \{(2s + m/k + 1)(q + 1) + w_1\}$.

Let now $l_2 = (2s + m/k + 1)(i_2 + 1)$ for some $0 \leq i_2 \leq k - 1$, i.e. $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$ and $E^*_1(y, r^{i_2+1}(t))$. Whence we obtain:

$$K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x)), E^*_{w_1}(y, r^{i_1+i_2+1}(x))$$

and $\neg E^*_{w_1-1}(y, r^{i_1+i_2+1}(x))$.

Let $q = (i_1 + i_2 + 1)[\text{mod k}]$. If $i_1 + i_2 + 1 = k$ then $l_1 \cdot l_2 = \{w_1\}$. If $i_1 + i_2 + 1 \neq k$ then $l_1 \cdot l_2 = \{(2s + m/k + 1)q + w_1\}$.

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $E^*_1(t, r^{i_2+1}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$, $E^*_{w_1}(y, r^{i_1}(t))$ and $\neg E^*_{w_1-1}(y, r^{i_1}(t))$. Whence we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $E^*_{w_1}(y, r^{i_1+i_2+1}(x))$ and $\neg E^*_{w_1-1}(y, r^{i_1+i_2+1}(x))$, and consequently $l_2 \cdot l_1 = l_1 \cdot l_2$.

Let now $l_2 = (2s + m/k + 1)i_2 + s + j$ for some $0 \leq i_2 \leq k - 1$ and $2 \leq j \leq m/k$. Then we have $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $\neg E^*_{s+1}(y, r^{i_2}(t))$, $\neg E^*_{s+1}(y, r^{i_2+1}(t))$ and there exist $y_1, y_2, \ldots, y_{j-1}$ such that $K_0(r^{i_2}(t), y_1, y_2, \ldots, y_{j-1}, r^{i_2+1}(t))$, $\neg E^*_{s+1}(y_1, r^{i_2}(t))$, $\wedge_{l=1}^{j-2} \neg E_{s+1}(y_l, y_{l+1})$, $E_{s+1}(y_{j-1}, y)$ and $M \models \forall u[K_0(r^{i_2}(t), u, y_{j-1}) \to E^*_{s+1}(u, r^{i_2}(t)) \vee \vee_{l=1}^{j-1} E_{s+1}(u, y_l)]$. Whence we obtain:

$$K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x)) \neg E^*_{s+1}(y, r^{i_1+i_2}(x)), \neg E^*_{s+1}(y, r^{i_1+i_2+1}(x))$$

and there exist $y_1, y_2, \ldots, y_{j-1}$ such that $K_0(r^{i_1+i_2}(x), y_1, y_2, \ldots, y_{j-1}, r^{i_1+i_2+1}(x))$, $\neg E^*_{s+1}(y_1, r^{i_1+i_2}(x))$, $\wedge_{l=1}^{j-2}\neg E_{s+1}(y_l, y_{l+1})$, $E_{s+1}(y_{j-1}, y)$ and $M \models \forall u[K_0(r^{i_1+i_2}(x), u, y_{j-1}) \to E^*_{s+1}(u, r^{i_1+i_2}(x)) \vee \vee_{l=1}^{j-1} E_{s+1}(u, y_l)]$.

Let $q = (i_1 + i_2)[\text{mod k}]$. Then $l_1 \cdot l_2 = \{(2s + m/k + 1)q + s + j\}$.

By considering the product $l_2 \cdot l_1$, we obtain that $l_2 \cdot l_1 = l_1 \cdot l_2$.

Case 3. $l_1 = (2s + m/k + 1)(i_1 + 1) + 1 - w_1$ for some $0 \leq i_1 \leq k - 1$ and $2 \leq w_1 \leq s + 1$.

We have the following: $K_0(r^{i_1}(x), t, r^{i_1+1}(x))$, $E^*_{w_1}(t, r^{i_1+1}(x))$ and $\neg E^*_{w_1-1}(t, r^{i_1+1}(x))$.

Let also $l_2 = (2s + m/k + 1)(i_2 + 1) + 1 - w_2$ for some $0 \leq i_2 \leq k - 1$ and $2 \leq w_2 \leq s + 1$, i.e. $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $E^*_{w_2}(y, r^{i_2+1}(t))$ and $\neg E^*_{w_2-1}(y, r^{i_2+1}(t))$.

Let $q = (i_1 + i_2 + 1)[\text{mod k}]$. If $w_1 \leq w_2$, we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $E^*_{w_2}(y, r^{i_1+i_2+2}(x))$ and $\neg E^*_{w_2-1}(y, r^{i_1+i_2+2}(x))$, whence $l_1 \cdot l_2 = \{(2s+m/k+1)(q+1)+1-w_2\}$. If $w_1 > w_2$, we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $E^*_{w_1}(y, r^{i_1+i_2+2}(x))$ and $\neg E^*_{w_1-1}(y, r^{i_1+i_2+2}(x))$, whence $l_1 \cdot l_2 = \{(2s + m/k + 1)(q + 1) + 1 - w_1\}$.

Let now $l_2 = (2s + m/k + 1)(i_2 + 1)$ for some $0 \leq i_2 \leq k - 1$, i.e. $K_0(r^{i_2}(t), y, r^{i_1+1}(t))$ and $E^*_1(y, r^{i_2+1}(t))$. Whence we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $E^*_{w_1}(y, r^{i_1+i_2+2}(x)$ and $\neg E^*_{w_1-1}(y, r^{i_1+i_2+2}(x))$.

Let $q = (i_1+i_2+1)[\text{mod k}]$. Then $l_1 \cdot l_2 = \{(2s+m/k+1)(q+1)+1-w_1\}$.

Consider the product $l_2 \cdot l_1$. We have the following:

$$K_0(r^{i_2}(x), t, r^{i_2+1}(x)), E^*_1(t, r^{i_2+1}(x)), K_0(r^{i_1}(t), y, r^{i_1+1}(t)), E^*_{w_1}(y, r^{i_1+1}(t))$$

and $\neg E^*_{w_1-1}(y, r^{i_1+1}(t))$. Whence we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $E^*_{w_1}(y, r^{i_1+i_2+2}(x))$ and $\neg E^*_{w_1-1}(y, r^{i_1+i_2+2}(x))$.

Let $q = (i_1 + i_2 + 1)[\text{mod k}]$. Then also $l_2 \cdot l_1 = \{(2s + m/k + 1)(q + 1) + 1 - w_1\}$.

Let now $l_2 = (2s + m/k + 1)i_2 + s + j$ for some $0 \leq i_2 \leq k - 1$ and $2 \leq j \leq m/k$. Then we have $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $\neg E^*_{s+1}(y, r^{i_2}(t))$, $\neg E^*_{s+1}(y, r^{i_2+1}(t))$ and there exist $t_1, t_2, \ldots, t_{j-1}$ such that $K_0(r^{i_2}(t), t_1, t_2, \ldots, t_{j-1}, r^{i_2+1}(t))$, $\neg E^*_{s+1}(t_1, r^{i_2}(t))$, $\wedge_{l=1}^{j-2}\neg E_{s+1}(t_l, t_{l+1})$, $E_{s+1}(t_{j-1}, y)$ and $M \models \forall u[K_0(r^{i_2}(t), u, t_{j-1}) \to E^*_{s+1}(u, r^{i_2}(t)) \vee \vee_{l=1}^{j-1} E_{s+1}(u, t_l)]$.

Whence we obtain:

$$K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x)), \neg E^*_{s+1}(y, r^{i_1+i_2+1}(x)), \neg E^*_{s+1}(y, r^{i_1+i_2+2}(x))$$

and there exist $y_1, y_2, \ldots, y_{j-1}$ such that

$$K_0(r^{i_1+i_2+1}(x), y_1, y_2, \ldots, y_{j-1}, r^{i_1+i_2+2}(x)), \neg E^*_{s+1}(y_1, r^{i_1+i_2+1}(x)),$$

$\wedge_{l=1}^{j-2}\neg E_{s+1}(y_l, y_{l+1})$, $E_{s+1}(y_{j-1}, y)$ and $M \models \forall u[K_0(r^{i_1+i_2+1}(x), u, y_{j-1}) \to E^*_{s+1}(u, r^{i_1+i_2+1}(x)) \vee \vee_{l=1}^{j-1} E_{s+1}(u, y_l)]$.

Let $q = (i_1 + i_2 + 1)$[mod k]. Then $l_1 \cdot l_2 = \{(2s + m/k + 1)q + s + j\}$.
By considering the product $l_2 \cdot l_1$, we obtain that $l_2 \cdot l_1 = l_1 \cdot l_2$.

Case 4. $l_1 = (2s + m/k + 1)(i_1 + 1)$ for some $0 \le i_1 \le k - 1$.
We have the following: $K_0(r^{i_1}(x), t, r^{i_1+1}(x))$ and $E_1^*(t, r^{i_1+1}(x))$.
Let also $l_2 = (2s + m/k + 1)(i_2 + 1)$ for some $0 \le i_2 \le k - 1$, i.e. $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$ and $E_1^*(y, r^{i_2+1}(t))$. Whence we obtain:

$$K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x)) \text{ and } E_1^*(y, r^{i_1+i_2+2}(x)).$$

Let $q = (i_1 + i_2 + 1)$[mod k]. Then $l_1 \cdot l_2 = \{(2s + m/k + 1)(q + 1)\}$.
Let now $l_2 = (2s + m/k + 1)i_2 + s + j$ for some $0 \le i_2 \le k - 1$ and $2 \le j \le m/k$. Then we have $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $\neg E_{s+1}^*(y, r^{i_2}(t))$, $\neg E_{s+1}^*(y, r^{i_2+1}(t))$ and there exist $t_1, t_2, \ldots, t_{j-1}$ such that $K_0(r^{i_2}(t), t_1, t_2, \ldots, t_{j-1}, r^{i_2+1}(t))$, $\neg E_{s+1}^*(t_1, r^{i_2}(t))$, $\wedge_{l=1}^{j-2} \neg E_{s+1}(t_l, t_{l+1})$, $E_{s+1}(t_{j-1}, y)$ and $M \models \forall u[K_0(r^{i_2}(t), u, t_{j-1}) \to E_{s+1}^*(u, r^{i_2}(t)) \vee \vee_{l=1}^{j-1} E_{s+1}(u, t_l)]$.
Whence we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $\neg E_{s+1}^*(y, r^{i_1+i_2+1}(x))$, $\neg E_{s+1}^*(y, r^{i_1+i_2+2}(x))$ and there exist $y_1, y_2, \ldots, y_{j-1}$ such that $K_0(r^{i_1+i_2+1}(x), y_1, y_2, \ldots, y_{j-1}, r^{i_1+i_2+2}(x))$, $\neg E_{s+1}^*(y_1, r^{i_1+i_2+1}(x))$, $\wedge_{l=1}^{j-2} \neg E_{s+1}(y_l, y_{l+1})$, $E_{s+1}(y_{j-1}, y)$ and $M \models \forall u[K_0(r^{i_1+i_2+1}(x), u, y_{j-1}) \to E_{s+1}^*(u, r^{i_1+i_2+1}(x)) \vee \vee_{l=1}^{j-1} E_{s+1}(u, y_l)]$.
Let $q = (i_1 + i_2 + 1)$[mod k]. Then $l_1 \cdot l_2 = \{(2s + m/k + 1)q + s + j\}$.
By considering the product $l_2 \cdot l_1$, we obtain that $l_2 \cdot l_1 = l_1 \cdot l_2$.

Case 5. $l_1 = (2s + m/k + 1)i_1 + s + j_1$ for some $0 \le i_1 \le k - 1$ and $2 \le j_1 \le m/k$.
We have the following: $K_0(r^{i_1}(x), t, r^{i_1+1}(x))$, $\neg E_{s+1}^*(t, r^{i_1}(x))$ and $\neg E_{s+1}^*(t, r^{i_1+1}(x))$ and there exist $t_1, t_2, \ldots, t_{j_1-1}$ such that $K_0(r^{i_1}(x), t_1, t_2, \ldots, t_{j_1-1}, r^{i_1+1}(x))$, $\neg E_{s+1}^*(t_1, r^{i_1}(x))$, $\wedge_{l=1}^{j_1-2} \neg E_{s+1}(t_l, t_{l+1})$, $E_{s+1}(t_{j_1-1}, t)$ and $M \models \forall u[K_0(r^{i_1}(x), u, t_{j_1-1}) \to E_{s+1}^*(u, r^{i_1}(x)) \vee \vee_{l=1}^{j_1-1} E_{s+1}(u, t_l)]$.
Let now $l_2 = (2s + m/k + 1)i_2 + s + j$ for some $0 \le i_2 \le k - 1$ and $2 \le j_2 \le m/k$. Then we have $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $\neg E_{s+1}^*(y, r^{i_2}(t))$, $\neg E_{s+1}^*(y, r^{i_2+1}(t))$ and there exist $t_1, t_2, \ldots, t_{j_2-1}$ such that $K_0(r^{i_2}(t), t_1, t_2, \ldots, t_{j_2-1}, r^{i_2+1}(t))$, $\neg E_{s+1}^*(t_1, r^{i_2}(t))$, $\wedge_{l=1}^{j_2-2} \neg E_{s+1}(t_l, t_{l+1})$, $E_{s+1}(t_{j_2-1}, y)$ and $M \models \forall u[K_0(r^{i_2}(t), u, t_{j_2-1}) \to E_{s+1}^*(u, r^{i_2}(t)) \vee \vee_{l=1}^{j_2-1} E_{s+1}(u, t_l)]$.
Let $q = (i_1 + i_2)$[mod k]. We obtain: $K_0(r^q(x), y, r^{q+2}(x))$. If $j_1 + j_2 < m/k + 2$ then we have: $K_0(r^q(x), y, r^{q+1}(x))$, $\neg E_{s+1}^*(y, r^q(x))$, $\neg E_{s+1}^*(y, r^{q+1}(x))$ and there exist $y_1, y_2, \ldots, y_{j_1+j_2-2}$ such that $E_{s+1}(t_{j_1+i_2-2}, y)$, $K_0(r^q(x), y_1, y_2, \ldots, y_{j_1+j_2-2}, r^{q+1}(x))$, $\neg E_{s+1}^*(y_1, r^q(x))$, $\wedge_{l=1}^{j_1+j_2-3} \neg E_{s+1}(y_l, y_{l+1})$, and $M \models$

$$\forall u[K_0(r^q(x), u, y_{j_1+j_2-2}) \to E_{s+1}^*(u, r^q(x)) \vee \vee_{l=1}^{j_1+j_2-2} E_{s+1}(u, y_l)],$$

whence $l_1 \cdot l_2 = \{(2s + m/k + 1)q + s + j_1 + j_2 - 1\}$.

Let now $j_1+j_2=m/k+2$. Then we have $E*_1(y,r^{q=1}(x))$. If $i_1+i_2+1=k$ then $l_1\cdot l_2=\{2sk+m+k-s,2sk+m+k-s+1,\ldots,2sk+m+k,0,1,2,\ldots,s,s+1\}$, i.e. the product $l_1\cdot l_2$ consists of $2s+3$ labels. If $i_1+i_2+1\neq k$ then $l_1\cdot l_2=\{(2s+m/k+1)(q+1)-s,(2s+m/k+1)(q+1)-s+1,\ldots,(2s+m/k+1)(q+1),\ (2s+m/k+1)(q+1)+1,\ldots,(2s+m/k+1)(q+1)+s+1\}$, i.e. the product $l_1\cdot l_2$ consists of $2s+2$ labels.

Let now $j_1+j_2>m/k+2$. Also, let $\mu=(j_1+j_2-1)$[mod k]. Then we obtain: $K_0(r^{q+1}(x),y,r^{q+2}(x))$, $\neg E^*_{s+1}(y,r^{q+1}(x))$, $\neg E^*_{s+1}(y,r^{q+2}(x))$ and there exist $y_1,y_2,\ldots,y_{\mu-1}$ such that $K_0(r^{q+1}(x),y_1,y_2,\ldots,y_{\mu-1},r^{q+2}(x))$, $\neg E^*_{s+1}(y_1,r^{q+1}(x))$, $\wedge_{l=1}^{\mu-2}\neg E_{s+1}(y_l,y_{l+1})$, $E_{s+1}(t_{\mu-1},y)$ and

$$M\models\forall u[K_0(r^{q+1}(x),u,y_{\mu-1})\to E^*_{s+1}(u,r^{q+1}(x))\vee\vee_{l=1}^{\mu-1}E_{s+1}(u,y_l)],$$

whence $l_1\cdot l_2=\{(2s+m/k+1)(q+1)+s+\mu\}$.

Thus, we established that the algebra $\mathfrak{P}_{M'_{s,m,k}}$ is commutative and strictly $(2s+3)$-deterministic for all valid values s,m and k.

3. Conclusion

We investigated algebras of binary isolating formulas for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank greater than 1 with a trivial definable closure having a non-trivial monotonic-to-right function acting on the universe of a structure. We also proved their commutativity and established their strict m-deterministicity for some natural m. It would now be interesting to describe the corresponding algebras for theories having a non-trivial piecewise monotonic function.

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Об авторах

Алтаева Айжан Бакаткалиевна, PhD, Международный университет информационных технологий, Алматы, 050040, Казахстан, vip.altayeva@mail.ru, https://orcid.org/0000-0001-9238-7131

About the authors

Aizhan Bakatkalievna Altayeva, PhD, International Information Technology University, Almaty, 050040, Kazakhstan, vip.altayeva@mail.ru, https://orcid.org/0000-0001-9238-7131

Кулпешов Бейбут Шайыкович, д-р физ.-мат. наук, проф., Институт математики и математического моделирования МНВО РК, Алматы, 050010, Казахстан, kulpesh@mail.ru; Казахстанско-Британский технический университет, Алматы, 050000, Казахстан, b.kulpeshov@kbtu.kz, https://orcid.org/0000-0002-4242-0463

Beibut Sh. Kulpeshov, Dr. Sci. (Phys.-Math.), Prof., Institute of Mathematics and Mathematical Modeling, Almaty, 050010, Kazakhstan, kulpesh@mail.ru; Kazakh British Technical University, Almaty, 050000, Kazakhstan, b.kulpeshov@kbtu.kz, https://orcid.org/0000-0002-4242-0463

Судоплатов Сергей Владимирович, д-р физ.-мат. наук, проф., Институт математики им. С. Л. Соболева СО РАН, Новосибирск, 630090, Российская Федерация, sudoplat@math.nsc.ru; Новосибирский государственный технический университет, Новосибирск, 630073, Российская Федерация, sudoplatov@corp.nstu.ru, https://orcid.org/0000-0002-3268-9389

Sergey V. Sudoplatov, Dr. Sci. (Phys.-Math.), Prof., Sobolev Institute of Mathematics SB RAS, Novosibirsk, 630090, Russian Federation, sudoplat@math.nsc.ru; Novosibirsk State Technical University, Novosibirsk, 630073, Russian Federation, sudoplatov@corp.nstu.ru, https://orcid.org/0000-0002-3268-9389

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