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Partial Algebras of Formulas Under Generalized Superpositions

T. Kumduang [1], **K. Wattanatripop** [2]✉

[1] Rajamangala University of Technology Rattanakosin, Nakhon Pathom, Thailand
[2] Rajamangala University of Technology Lanna, Chiang Mai, Thailand
✉ khwancheewa.wat@rmutl.ac.th

Abstract: Formulas which are tools for describing algebraic systems are formal expressions arising from terms, relation symbols, and logical connectives. Under composition of the generalized superposition operation, the set of all terms forms a unitary superassociative algebra. This paper deals with construction of the partial generalized superposition on the set of all terms and formulas satisfying the superassociativity as a weak identity. Partial binary operations induced by such partial generalized superpositions are given and the fact that these operations are weak associative are proved.

Keywords: partial algebra, formula, term, superassociativity, partial semigroup, weak identity, superposition, operation, embeddability, weak monomorphism

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Научная статья

Частичные алгебры формул в обобщенных суперпозициях

Т. Кумдуанг [1], **К. Ватанатрипоп** [2]✉

[1] Технологический университет Раджамангала, Накхонпатхом, Таиланд
[2] Технологический университет Раджамангала Ланна, Чиангмай, Таиланд
✉ khwancheewa.wat@rmutl.ac.th

Аннотация: Формулы, которые являются инструментами для описания алгебраических систем, являются формальными выражениями, возникающими из термов, символов отношений и логических связок. При композиции обобщенной операции суперпозиций множество всех термов образует унитарную суперассоциативную алгебру. Рассматривается построение частичной обобщенной суперпозиции на множестве всех термов и формул, удовлетворяющих суперассоциативности как слабому тождеству. Приводятся частичные бинарные операции, индуцированные такими частичными обобщенными суперпозициями, и доказывается тот факт, что эти операции являются слабоассоциативными.

Ключевые слова: частичная алгебра, формула, терм, суперассоциативность, частичная полугруппа, слабое тождество, суперпозиция, операция, вложимость, слабый мономорфизм

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1. Introduction

Although our paper is contributed to the study of partial algebras as well as the axiom of superassociativity on partial algebras, we first assume that it should be helpful to get a brief excursion to partial algebras and then apply these tools to partial algebras of formulas. See [1;9;15] for more backgrounds on partial algebras, partial operations and weak identities. Extensions of partial lattices were described by I. Chajda and H. Langer in the paper [3]. Recently, a comprehensive monograph on partial algebras is collected by K. Denecke in [4].

In 2021, partial Menger algebras generalizing Menger algebras were introduced by K. Denecke in [6]. By definition, it is a pair of a nonempty set and a partial operation of type $n+1$ defined on that set which satisfies the axiom of superassociativity. It is worth noticing that this structure can be considered as an extension of any partial semigroup, i. e., a partial Menger algebra of rank 1 and a partial semigroup are the same thing. One of the most important developments of Menger algebras is the Menger algebra of terms, a triple consisting of the set of terms of type τ, generalized superpositon S^n and a family of infinitely many nullary operations. This is a main tool to classify any algebra into subclasses called strongly solid varieties. There are many works describing properties of the Menger algebra of terms, for example, [14; 22].

Following the suggestion of A. I. Malcev in [16], formulas which are formal expressions extending the concept of both first and second-order languages are given. To demonstrate an importance of formulas, we consider the formula $\forall z[x + y \cdot z = x]$ of the usual addition and multiplication on the set $\mathbb{N} := \{1, 2, \ldots\}$ of all nutural numbers. It can be calculated that this formula is true if $x = 1$ and $y = 0$, but false in the case when $x = y = 1$. In view of algebrization, structures of formulas with respect to different operations have been widely studied by many authors [2; 10; 19]. Particularly, in the paper [20], the generalized superposition operation, denoted by R^n, of type $n+1$ defined on the set of all formulas of arbitrary type is mentioned. The fact that this operation is superassociative is also proved and characterizations for any element of some algebras induced by such operation to be idempotent and regular in sense of the theory of semigroups are presented. Normally, the interaction between formulas and model theory are given. Algebras of binary formulas in sence of realizations are mentioned in [11]. The concept of pseudofinite formulas and their properties are revealed in [12]. Formulas in first order logic over a given language are also studied in [13].

While a generalized superposition R^n of formulas has been established, its computation relies on specific choices of terms of type τ. In the domain of R^n with type $n+1$, this implies that the first position derives from the set of formulas, while other positions in the domain come from the set $W_\tau(X)$ of terms. This leads to the question: Can we define the superposition for the Cartesian product of the formula set using S^n_g and R^n_g? If so, does this operation fulfill the property of superassociativity? To address these questions, this paper primarily aims to establish a partial operation for the set of formulas, where its domain is defined by the Cartesian product of the formula set. We demonstrate that this partial operation satisfies weak superassociativity. Additionally, partial binary operations induced by this generalized partial superposition are introduced for the set of formulas, and partial semigroups corresponding to these binary operations are constructed.

2. Some preliminary results

This section provides some essential backgrounds concerning terms, formulas, partial algebras, and related topics that need in the paper. See the references [1; 3; 9; 17] for more details.

Normally, a superassociative algebra (a Menger algebra) is a pair of a nonempty G together with one operation of type $n+1$ defined on G which satisfies the axiom of superassociativity, i.e.,

$$o(o(a, b_1, \ldots, b_n), c_1, \ldots, c_n) = o(a, o(b_1, c_1, \ldots, c_n), \ldots, o(b_n, c_1, \ldots, c_n))$$

for all $a, b_j, c_j \in G$ and $j = 1, \ldots, n$. Furthermore, superassociative algebras have been investigated in various directions. For example, superassociative algebras of multiplace functions were deeply considered in the papers [8]. By a unitary superassociative algebra, we mean a superassociative algebra (G, o) that has special elements such that $o(e, a_1, \ldots, a_n) = e$ and $o(a, e_1, \ldots, e_n) = a$ for elements a, e, a_i, e_i in G and $i = 1, \ldots, n$. In this case, this algebra has the type $(n+1, 0, \ldots, 0)$. An excursion of the theory of superassociative algebras or algebras of functions can be found in [7].

A term t of type τ is constructed from an alphabet $X_n = \{x_1, \ldots, x_n\}$ whose elements are called *variables* for all n in $\mathbb{N}$ and operation symbols $\{f_i \mid i \in I\}$ of type τ indexed by the set I. The type is the family $\tau = (n_i)_{i \in I}$ of the natural numbers that correspond to the arities of the operation symbols f_i. In fact, the set $W_\tau(X_n)$ of all n-ary terms of type τ consists of the following elements: Every variable $x_i \in X_n$ and $f_i(t_1, \ldots, t_{n_i})$ where n-ary terms $t_1, \ldots, t_{n_i}$ of type τ are already known. Indeed, $W_\tau(X_n)$ is the smallest set closed under finite application of composition by each operation symbol f_i. In general, if variables come from an infinite set of alphabets $X = \{x_1, x_2, \ldots\}$, we write $W_\tau(X)$ instead of $W_\tau(X_n)$. Moreover, by $\mathrm{var}(t)$ we denote the set of all variables that occur in a term t. For details, one can refer the reader to [5; 14; 21].

One of the most important operations defined on the set of terms is the generalized superposition operation [20]. Basically, a new term is obtained after substituting all variables occuring in a former term by the other terms. This can be described by the $(n+1)$-generalized superpostion S_g^n, $n \geq 1$,

$$S_g^n : W_\tau(X)^{n+1} \to W_\tau(X)$$

defined inductively by the following steps: for $t, t_1, \ldots, t_n \in W_\tau(X)$

1. If $t = x_i$; $1 \leq i \leq n$, then $S_g^n(x_i, t_1, \ldots, t_n) := t_i$.
2. If $t = x_i$; $n < i$, then $S_g^n(x_i, t_1, \ldots, t_n) := x_i$.
3. If $t = f_i(s_1, \ldots, s_{n_i})$, then $S_g^n(t, t_1, \ldots, t_n)$ is equal to $f_i(S_g^n(s_1, t_1, \ldots, t_n), \ldots, S_g^n(s_{n_i}, t_1, \ldots, t_n))$.

We can form the algebra $(W_\tau(X), S_g^n, (x_j)_{j \geq 1})$ of type $(n+1, 0, 0, 0, \ldots)$ consisting of the universe $W_\tau(X)$ together with one $(n+1)$-ary operation S_g^n and the variable terms acting as infinitely many nullary operations. We call this algebra the generalized clone of terms with infinitely many nullary operations.

Another structure that generalizes algebras is an algebraic system, a triple of a nonempty set A equipped with a family of n_i-ary operations defined on A, and a family of n_j-ary relations on A. By the type (τ, τ'), we refer to the families of the arities of operations and relations, respectively. See [16; 19]. It is obvious that a partially ordered semigroup is an example of algebraic systems of type $((2),(2))$. However, if there is no a family of relation symbols, thus an algebraic system and an algebra are identical. Thus, the notion that needs in the investigation of algebraic systems of type (τ, τ') is a formula.

Recall from [5; 16; 17] that for $n \in \mathbb{N}$ an *n-ary formula of type* (τ, τ') is defined in the following way:

1. If t_1, t_2 are n-ary terms of type τ, then the equation $t_1 \approx t_2$ is an n-ary formula of type (τ, τ').
2. If $j \in J$ and $t_1, \ldots, t_{n_j}$ are n-ary terms of type τ and γ_j is an n_j-ary relation symbol, then $\gamma_j(t_1, \ldots, t_{n_j})$ is an n-ary formula of type (τ, τ').
3. If F is an n-ary formula of type (τ, τ'), then $\neg F$ is an n-ary formula of type (τ, τ').
4. If F_1 and F_2 are n-ary formulas of type (τ, τ'), then $F_1 \vee F_2$ is an n-ary formula of type (τ, τ').
5. If F is an n-ary formula of type (τ, τ') and $x_i \in X_n$, then $\exists x_i(F)$ is an n-ary formula of type (τ, τ').

By *atomic formulas of type* (τ, τ'), we refer to the formulas of the form 1. and 2. The formulas of the forms 1. to 4. are called quantifier free formulas. In this paper, for short, we call a formula F in stead of a quantifier free formula F. Thus, the set of all n-ary formulas of type (τ, τ') and the set of all formulas of type (τ, τ') are denoted by $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $\mathcal{F}_{(\tau,\tau')}(W_\tau(X)) := \bigcup_{n\in\mathbb{N}} \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$, respectively.

In [18], the operation

$$R_g^n : \big(W_\tau(X) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X))\big) \times (W_\tau(X))^n \to W_\tau(X) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$$

is defined in the following way:

1. If $t \in W_\tau(X)$, then $R_g^n(t, s_1, \ldots, s_n)$ is equal to $S_g^n(t, s_1, \ldots, s_n)$.
2. If $t_1 \approx t_2 \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$, then $R_g^n(t_1 \approx t_2, s_1, \ldots, s_n)$ is the formula
$$R_g^n(t_1, s_1, \ldots, s_n) \approx R_g^n(t_2, s_1, \ldots, s_n).$$
3. If $\gamma_j(t_1, \ldots, t_{n_j}) \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$, then $R_g^n(\gamma_j(t_1, \ldots, t_{n_j}), s_1, \ldots, s_n)$ is the formula $\gamma_j(R_g^n(t_1, s_1, \ldots, s_n), \ldots, R_g^n(t_{n_j}, s_1, \ldots, s_n))$.
4. If $F \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$, then $R_g^n(\neg F, s_1, \ldots, s_n)$ equals
$$\neg R_g^n(F, s_1, \ldots, s_n).$$
5. If $F_1, F_2 \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$, then $R_g^n(F_1 \vee F_2, s_1, \ldots, s_n)$ is the formula
$$R_g^n(F_1, s_1, \ldots, s_n) \vee R_g^n(F_2, s_1, \ldots, s_n).$$

Thus, the algebra $(W_\tau(X), \mathcal{F}_{(\tau,\tau')}(W_\tau(X)), S^n_g, R^n_g)$ is formed. Furthermore, it was shown in [18] that the operations S^n_g and R^n_g are superassociative in sense of many-sorted algebras. Adding a family of variables $(x_i)_{i\geq 1}$ in this algebra, we obtain a new algebra

$$(W_\tau(X), \mathcal{F}_{(\tau,\tau')}(W_\tau(X)), S^n_g, R^n_g, (x_i)_{i\geq 1})$$

which can be regarded as a unitrary superassociative algebra.

3. Construction of partial unitary algebras of formulas

Let $\tau = (n_i)_{i\in I}$ be a type, $(A, \{f_i \mid i \in I\})$ and $(B, \{f_i \mid i \in I\})$ partial algebras of type τ and p, q terms of type τ. We say that a partial algebra $(A, \{f_i \mid i \in I\})$ satisfies a weak identity $p \approx q$ if the following holds: If $a_1, \ldots, a_n \in A$ and $p(a_1, \ldots, a_n)$ and $q(a_1, \ldots, a_n)$ are defined, then $p(a_1, \ldots, a_n) = q(a_1, \ldots, a_n)$. Additionally, we say that a partial algebra $(A, \{f_i \mid i \in I\})$ is a weak subalgebra of $(B, \{f_i \mid i \in I\})$ if $A \subseteq B$ and if for all $i \in I$ and all $a_1, \ldots, a_{n_i} \in A$, if $f_i(a_1, \ldots, a_{n_i})$ is defined in $(A, \{f_i \mid i \in I\})$, then it is defined in $(B, \{f_i \mid i \in I\})$ and has the same value.

We begin our study in this section with providing the concept of a partial operation on the set of all formulas induced by an alphabet $X = \{x_1, x_2, \ldots\}$. In fact, we let

$$W\mathcal{F}_{(\tau,\tau')}(X) := W_\tau(X) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X)).$$

We now define the partial generalized superposition

$$\overline{R}^n_g : (W\mathcal{F}_{(\tau,\tau')}(X))^{n+1} \multimap\!\!\rightarrow W\mathcal{F}_{(\tau,\tau')}(X)$$

by

$$\overline{R}^n_g(a, b_1, \ldots, b_n) := \begin{cases} S^n_g(a, b_1, \ldots, b_n) & \text{if } a, b_1, \ldots, b_n \in W_\tau(X), \\ R^n_g(a, b_1, \ldots, b_n) & \text{if } a \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X)), b_1, \ldots, b_n \in W_\tau(X), \\ \text{not defined} & \text{otherwise.} \end{cases}$$

To understand this operation in depth, we consider the following example. Let $|I| = 2, |J| = 1$, and $(\tau, \tau') = ((2, 2), (2))$ be a type with two binary operation symbols $\oplus$ and $\otimes$ and one binary relation symbol ∇. We consider the following elements belonging to the set $W\mathcal{F}_{((2,2),(2))}(X)$: a_1 is a variable x_4, a_2 is a term $\oplus(x_2, x_5), a_3$ is a term $\otimes(x_2, \oplus(x_3, x_6)), b_1$ is a formula $\oplus(x_1, x_5) \approx x_3, b_2$ is a formula $\nabla(\otimes(x_1, x_7), x_2), b_3$ is a formula

$\neg(x_2 \approx \otimes(x_8, x_3)), d_1$ is a term $\oplus(\oplus(x_2, x_1), \otimes(x_1, x_3))$ and d_2 is a variable x_1. Then we have

$$\begin{aligned}
\overline{R}_g^3(a_1, a_1, a_2, a_3) &= a_1,\\
\overline{R}_g^3(a_2, a_1, a_2, a_3) &= \oplus(\oplus(x_2, x_5), x_5),\\
\overline{R}_g^3(b_1, a_1, a_2, a_3) &= \oplus(x_4, x_5) \approx \otimes(x_2, \otimes(x_3, x_6)),\\
\overline{R}_g^3(b_2, a_1, d_2, a_3) &= \nabla(\otimes(x_4, x_7), x_1),\\
\overline{R}_g^3(b_3, d_1, d_2, a_2) &= \neg(x_1 \approx \otimes(x_8, \oplus(x_2, x_5))).
\end{aligned}$$

On the other hand, $\overline{R}_g^3(a_1, a_2, b_1, b_2, b_3)$ and $\overline{R}_g^3(d_2, b_1, a_2, a_3)$ are not defined.

Hence, on the set $W\mathcal{F}_{(\tau,\tau')}(X)$, we can form the following partial algebras:

1. The partial algebra $(W\mathcal{F}_{(\tau,\tau')}(X), (\overline{R}_g^n)_{n\in\mathbb{N}})$ of type $(2, 3, 4, \ldots)$,
2. The partial algebra $(W\mathcal{F}_{(\tau,\tau')}(X), (\overline{R}_g^n)_{n\in\mathbb{N}}, (x_i)_{i\in\mathbb{N}})$ of type $(2, 3, 4, \ldots, 0, 0, \ldots)$.

Our next purpose is to show that these two algebras satisfy certain axioms.

Theorem 1. *The partial algebra $(W\mathcal{F}_{(\tau,\tau')}(X), (\overline{R}_g^n)_{n\in\mathbb{N}})$ satisfies, for all $n \in \mathbb{N}$, the identity*

$$\overline{R}_g^n(\overline{R}_g^n(a_1, b_1, \ldots, b_n), d_1, \ldots, d_n) = \overline{R}_g^n(a_1, h_1, \ldots, h_n)$$

as a weak identity where $h_j = \overline{R}_g^n(b_j, d_1, \ldots, d_n)$ for all $j = 1, \ldots, n$.

Proof. Suppose first that $a_1, b_1, \ldots, b_n, d_1, \ldots, d_n$ belong to $W\mathcal{F}_{(\tau,\tau')}(X)$. We aim to prove that each partial operation in a family $(\overline{R}_g^n)_{n\in\mathbb{N}}$ is superassociative, which means that the following identity:

$$\overline{R}_g^n(\overline{R}_g^n(a_1, b_1, \ldots, b_n), d_1, \ldots, d_n) = \overline{R}_g^n(a_1, h_1, \ldots, h_n) \tag{3.1}$$

is weak for $h_j = \overline{R}_g^n(b_j, d_1, \ldots, d_n)$ and $j = 1, \ldots, n$. To do this, assume that the left-hand side of (3.1) is defined. Thus, we have the following two cases: $a_1, b_1, \ldots, b_n, d_1, \ldots, d_n$ are terms of type τ in the first case and a is a formula of type (τ, τ') but $b_1, \ldots, b_n, d_1, \ldots, d_n$ are terms of type τ in the second case. If $a_1, b_1, \ldots, b_n, d_1, \ldots, d_n$ belong to the set $W_\tau(X)$, we have that $\overline{R}_g^n(a_1, b_1, \ldots, b_n)$ is equal to $S_g^n(a_1, b_1, \ldots, b_n)$. Besides, for each $j = 1, \ldots, n, \overline{R}_g^n(b_j, d_1, \ldots, d_n)$ is also defined and is equal to $S_g^n(b_j, d_1, \ldots, d_n)$, which implies that the right-hand side of (3.1) is defined. Applying the fact that the generalized superposition of terms satisfies the superassociativity given in [20], we conclude that the equation

(3.1) is weak. On the other hand, we now consider the case when a belongs to $\mathcal{F}_{(\tau,\tau')}(W_\tau(X))$ but $b_1, \ldots, b_n, d_1, \ldots, d_n$ are in $W_\tau(X)$. Clearly, $\overline{R}^n_g(\overline{R}^n_g(a_1, b_1, \ldots, b_n), d_1, \ldots, d_n)$ is defined and equals to $R^n_g(R^n_g(a_1, b_1, \ldots, b_n), d_1, \ldots, d_n)$. For each $j = 1, 2, \ldots, n$, $\overline{R}^n_g(b_j, d_1, \ldots, d_n)$ is defined and equals to $R^n_g(b_j, d_1, \ldots, d_n)$, which belongs to $W_\tau(X)$. It implies that the right-hand side of (3.1), i.e., $\overline{R}^n_g(a, \overline{R}^n_g(b_1, d_1, \ldots, d_n), \ldots, \overline{R}^n_g(b_n, d_1, \ldots, d_n))$ is defined and equals to the formula $R^n_g(a, R^n_g(b_1, d_1, \ldots, d_n), \ldots, R^n_g(b_n, d_1, \ldots, d_n))$. By the definition of generalized superposition R^n_g of formulas, it was proved in [18] that R^n_g is superassociative. As a consequence, the identity (3.1) is weak for this case. $\square$

The following theorem shows that the partial algebra

$$(W\mathcal{F}_{(\tau,\tau')}(X), (\overline{R}^n_g)_{n\in\mathbb{N}}, (x_i)_{i\in\mathbb{N}})$$

satisfies certain identities which are more complicated than the axiom in Theorem 1.

Theorem 2. $(W\mathcal{F}_{(\tau,\tau')}(X), (\overline{R}^n_g)_{n\in\mathbb{N}}, (x_i)_{i\in\mathbb{N}})$ *is a generalized unitary partial superassociative system.*

Proof. It is left to show that the satisfaction of superassociativity of each partial operation in $(\overline{R}^n_g)_{n\in\mathbb{N}}$ due to a direct verification of Theorem 1. Now we show that for $1 \leq j \leq n$, the equation $\overline{R}^n_g(x_j, b_1, \ldots, b_n) = b_j$ is a weak identity. Assume that the left hand-side of this equation is defined. We obtain that $b_1, \ldots, b_n$ are terms of type τ and thus $\overline{R}^n_g(x_j, b_1, \ldots, b_n) = S^n_g(x_j, b_1, \ldots, b_n) = b_j$, consequently, our claimed is obtained. For $j > n$, we now show that $\overline{R}^n_g(x_j, b_1, \ldots, b_n) = x_j$ is a weak identity. Suppose that $\overline{R}^n_g(x_j, b_1, \ldots, b_n)$ is defined. Then all of $b_1, \ldots, b_n$ belong to the set $W_\tau(X)$. As a result, $\overline{R}^n_g(x_j, b_1, \ldots, b_n) = S^n_g(x_j, b_1, \ldots, b_n) = x_j$. Finally, we prove that for a in $W\mathcal{F}_{(\tau,\tau')}(X)$ the weak identity $\overline{R}^n_g(a, x_1, \ldots, x_n) = a$ holds. Obviously, $\overline{R}^n_g(a, x_1, \ldots, x_n)$ is defined and equal $S^n_g(a, x_1, \ldots, x_n)$ if a is a term of type τ. Otherwise, $R^n_g(a, x_1, \ldots, x_n)$ if a is a formula of type (τ, τ'). It was shown in [20] that if a is a term of type τ, $S^n_g(a, x_1, \ldots, x_n) = a$. For a formula a, we give a proof by the following step. If a is an equation $s \approx t$, then $R^n_g(s \approx t, x_1, \ldots, x_n)$ is equal to $S^n_g(s, x_1, \ldots, x_n) \approx S^n_g(t, x_1, \ldots, x_n)$, subsequently, $s \approx t$. If a has a form $\gamma_j(t_1, \ldots, t_{n_j})$, then by the definition of the operation S^n_g, we obtain $R^n_g(\gamma_j(t_1, \ldots, t_{n_j}), x_1, \ldots, x_n) = \gamma_j(t_1, \ldots, t_{n_j})$. Assume that a is satisfied as a weak identity already. Then we obtain $R^n_g(\neg a, x_1, \ldots, x_n) = \neg R^n_g(a, x_1, \ldots, x_n) = \neg a$. Suppose that F_1 and F_2 are satisfied. Then we have $R^n_g(F_1 \vee F_2, x_1, \ldots, x_n) = R^n_g(F_1, x_1, \ldots, x_n) \vee R^n_g(F_2, x_1, \ldots, x_n) = F_1 \vee F_2$. Therefore, the proof is completed. $\square$

We now discuss some subsets of $W\mathcal{F}_{(\tau,\tau')}(X)$. Recall from [21] that a term t of type τ is called a *term with a fixed variable* if t constructed from only one variable from X, which means that in an inductive step, if $t_1,\ldots,t_{n_i}$ are terms with a fixed variable of type τ and $\operatorname{var}(t_j)=\operatorname{var}(t_k)$ for $1\leq j<k\leq n_i$, then $f_i(t_1,\ldots,t_{n_i})$ is term with a fixed variable of type τ. The set of all terms with a fixed variable is denoted by $W_\tau^{fix}(X)$. For example, let $\tau=(3)$ be a type with a ternary operation symbol f. Thus, the following lists are examples of terms with a fixed variable of type (3): $x_1, x_{10}, f(x_1,x_1,x_1), f(x_5,f(x_5,x_5,x_5),x_5), f(f(x_9,x_9,x_9),f(x_9,x_9,x_9),x_9)$. Nevertheless, terms $f(x_3,x_2,x_{11}), f(x_5,x_5,x_6), f(x_1,f(x_4,x_4,x_4),x_4)$ are not terms with a fixed variable.

Applying terms with a fixed variable, formulas with a fixed variable are considered, which are familiar with equation with one variable in our real-life. In fact, an equation $x^2+x=5x-x^3$ can be viewed as a formula with a fixed variable because both sides of such equation are terms with a fixed variable. However, an equation $2x+y=x^2-y^2$ with two variables is not a formula with a fixed variable. By definition, a formula F is called a *formula with a fixed variable.* The symbol $\mathcal{F}^{fix}_{(\tau,\tau')}(W_\tau^{fix}(X))$ stands for the set of all formulas with a fixed variable.

Thus, we prove the following result.

Theorem 3. *The set $W\mathcal{F}^{fix}_{(\tau,\tau')}(W_\tau^{fix}(X))$ of all formulas with a fixed variable is a partial subalgebra of $(W\mathcal{F}_{(\tau,\tau')}(X),(\overline{R}_g^n)_{n\in\mathbb{N}},(x_i)_{i\in\mathbb{N}})$.*

Proof. Clearly, $W\mathcal{F}^{fix}_{(\tau,\tau')}(W_\tau^{fix}(X))\subseteq W\mathcal{F}_{(\tau,\tau')}(X)$. Assume now that for $a,b_1,\ldots,b_n$ are elements in $W\mathcal{F}^{fix}_{(\tau,\tau')}(W_\tau^{fix}(X))$, $\overline{R}_g^n(a,b_1,\ldots,b_n)$ is defined. We separate our proof into two cases. If $\overline{R}_g^n(a,b_1,\ldots,b_n)$ equals to the term $S_g^n(a,b_1,\ldots,b_n)$, then by Lemma 1.3 given in [21] we have that $S_g^n(a,b_1,\ldots,b_n)$ is a term with a fixed variable and hence belongs to $W\mathcal{F}_{(\tau,\tau')}(X)$ and has the same term. On the other hand, if $\overline{R}_g^n(a,b_1,\ldots,b_n)$ is equal to $R_g^n(a,b_1,\ldots,b_n)$, then by [18], $R_g^n(a,b_1,\ldots,b_n)$ is a formula with a fixed variable, as a result, $W\mathcal{F}_{(\tau,\tau')}(X)$ contains this formula. $\square$

4. Partial semigroups of formulas

The goals of this section are to define three partial binary operations induced by the partial operation $\overline{R}_g^n$ on the set $W\mathcal{F}_{(\tau,\tau')}(X)$ and prove that these operations are associative.

Let a and b be elements in $W\mathcal{F}_{(\tau,\tau')}(X)$. The partial binary operation

$$+_F^n : W\mathcal{F}_{(\tau,\tau')}(X)\times W\mathcal{F}_{(\tau,\tau')}(X)\multimap W\mathcal{F}_{(\tau,\tau')}(X)$$

can be defined by

$$a +_F^n b = \overline{R}_g^n(a, \underbrace{b, \ldots, b}_{n \text{ times}}).$$

Thus, we prove the following result.

Theorem 4. *A tuple $(W\mathcal{F}_{(\tau,\tau')}(X), +_F^n)$ is a partial semigroup.*

Proof. Let a, b and d be arbitrary elements in $W\mathcal{F}_{(\tau,\tau')}(X)$. We aim to show that $(a +_F^n b) +_F^n d = a +_F^n (b +_F^n d)$ is a weak identity. Assume that $(a +_F^n b) +_F^n d$ is defined. Then we have that a belongs to $W\mathcal{F}_{(\tau,\tau')}(X)$ and both b and d are terms of type τ. It implies that $a +_F^n (b +_F^n d)$ is also defined. In order to show that both sides are equal, we devide our consideration into a few cases. If a and b are in $W_\tau(X)$, then we have $(a +_F^n b) +_F^n d = S_g^n(a +_F^n b, d, \ldots, d) = S_g^n(S_g^n(a, b, \ldots, b), d, d, \ldots, d)$ and $a +_F^n (b +_F^n d) = S_g^n(a, b +_F^n d, \ldots, b +_F^n d) = S_g^n(a, S_g^n(b, d, \ldots, d), \ldots, S_g^n(b, d, \ldots, d))$. Due to the satisfaction of S_g^n with the superassociativity, we conclude that $(a +_F^n b) +_F^n d = a +_F^n (b +_F^n d)$. In the second case, we mean the conditions when a is formula of type (τ, τ') but both b and d belong to the set $W_\tau(X)$. It follows from the definition of $+_F^n$ that $(a +_F^n b) +_F^n d$ equals $R_g^n(a +_F^n b, d, \ldots, d)$ and thus $R_g^n(R_g^n(a, b, \ldots, b), d, \ldots, d)$. On the other hand, we also have that $a +_F^n (b +_F^n d)$ is equal to $R_g^n(a, b +_F^n d, \ldots, b +_F^n d)$ and that $R_g^n(a, R_g^n(b, d, \ldots, d), \ldots, R_g^n(b, d, \ldots, d))$. Again by the superassociativity of R_g^n proved in [18], as a consequence, $(a +_F^n b) +_F^n d = a +_F^n (b +_F^n d)$. $\square$

Example 1. Consider the type $(\tau, \tau') = ((2), (2))$ with one binary operation symbol f, one binary relation symbol Δ and a subset

$$A = \{x_2, f(x_3, x_3), \neg(x_3 \approx x_4), \Delta(f(x_4, x_3), x_5)\}$$

of $W\mathcal{F}_{((2),(2))}(X)$ with respect to a partial binary operation $+_F^2$ which is defined by the following table.

$+_F^2$	x_2	$f(x_3, x_3)$	$\neg(x_3 \approx x_4)$	$\Delta(f(x_4, x_3), x_5)$
x_2	x_2	$f(x_3, x_3)$	not defined	not defined
$f(x_3, x_3)$	$f(x_3, x_3)$	$f(x_3, x_3)$	not defined	not defined
$\neg(x_3 \approx x_4)$	$\neg(x_3 \approx x_4)$	$\neg(x_3 \approx x_4)$	not defined	not defined
$\Delta(f(x_4, x_3), x_5)$	$\Delta(f(x_4, x_3), x_5)$	$\Delta(f(x_4, x_3), x_5)$	not defined	not defined

It is not difficult to show that the partial binary operation $+_F^2$ defined on A satisfies an associative law as a weak identity. To illustrate some examples, we consider elements $x_2, f(x_3, x_3)$ and $\Delta(f(x_4, x_3), x_5)$ in A. In order to show that an equation $(\Delta(f(x_4, x_3), x_5) +_F^2 x_2) +_F^2 f(x_3, x_3) \approx \Delta(f(x_4, x_3), x_5) +_F^2 (x_2 +_F^2 f(x_3, x_3))$ is a weak identity, assume first that the left-hand side is defined. Hence, we get $(\Delta(f(x_4, x_3), x_5) +_F^2 x_2) +_F^2 f(x_3, x_3) = \Delta(f(x_4, x_3), x_5) +_F^2 f(x_3, x_3) = \Delta(f(x_4, x_3), x_5)$. Then the right-hand side is defined and equals $\Delta(f(x_4,x_3),x_5) +^2_F x_2$, subsequently, $\Delta(f(x_4,x_3),x_5)$. This shows that the above equation satisfies an associative law as a weak identity. Consequently, $(A,+^2_F)$ forms a partial semigroup. Furthermore, it is also a weak subsemigroup of $(W\mathcal{F}_{((2),(2))}(X),+^2_F)$.

It was mentioned in [9] that an element a of a partial semigroup $(S,\cdot)$ is said to be *idempotent* if a weak identity $a\cdot a=a$ holds. We now apply this concept to characterize any element in the partial semigroup $(W\mathcal{F}_{(\tau,\tau')}(X),+^n_F)$ to be idempotent.

Theorem 5. *An element a in the partial semigroup $(W\mathcal{F}_{(\tau,\tau')}(X),+^n_F)$ is idempotent if and only if it is a term of type τ for which one of the following conditions is satisfied:*

1. $\mathrm{var}(a)\subseteq X\setminus X_n$,
2. $a=x_i$ *for some* $1\leq i\leq n$.

Proof. For a in S, suppose first that a is not a term of type τ under which it does not satisfy both conditions of the theorem. We aim to show that if $a+^n_F a$ and a are defined, then $a+^n_F a\neq a$. Assume that $a+^n_F a$ is defined. Form this, we have two cases: $\overline{R}^n_g(a,a,\ldots,a)$ is the term $S^n_g(a,a,\ldots,a)$ of type τ and $\overline{R}^n_g(a,a,\ldots,a)$ is the formula $R^n_g(a,a,\ldots,a)$ of type (τ,τ'). It is impossible to see that $R^n_g(a,a,\ldots,a)=a$ because the operation R^n_g does not allow a formula a in other places of the domain except the first position. Thus, a is a term in $W_\tau(X)\setminus X$ satisfying $\mathrm{var}(a)\cap X_n\neq\emptyset$. Without loss of generality, we may assume that $a=f_i(t_1,\ldots,t_{n_i})$ and $\mathrm{var}(t_1)\subset X_n$. It can be seen that if $\mathrm{var}(t_1)\subset X_n$, then $S^n_g(a,a,\ldots,a)\neq a$ because there are at least one variable from X_n occurring in t_1 and by the process of computation the resulting term has a term a inside the position of t_1, which proves that a weak identity of idempotency does not true. As a result, a is not idempotent in $W\mathcal{F}_{(\tau,\tau')}(X)$ with respect to $+^n_F$. Following Corollary 2.8 in the paper [7], we can conclude that our claim is obtained. The converse is obvious. $\square$

According to Example 1 and Theorem 5, it can be seen that elements x_2 and $f(x_3,x_3)$ are examples of idempotent elements under the binary operation $+^2_F$. On the other hand, $f(x_1,x_3),x_2\approx f(x_2,x_3),\Delta(x_1,f(x_3,x_1))$ are some examples of elements in $W\mathcal{F}_{(2,2)}(X)$ which are not idempotent.

For each $i=1,\ldots,n$, we define the partial binary operation

$$\cdot^{n,i}_F : W\mathcal{F}_{(\tau,\tau')}(X)\times W\mathcal{F}_{(\tau,\tau')}(X)\multimap W\mathcal{F}_{(\tau,\tau')}(X) \tag{4.1}$$

by

$$a\cdot^{n,i}_F b=\overline{R}^n_g(a,x_1,\ldots,x_{i-1},b,x_{i+1},\ldots,x_n)$$

for every $a,b\in W\mathcal{F}_{(\tau,\tau')}(X)$. Alternatively, for short, we may write

$$a\cdot^{n,i}_F b=\overline{R}^n_g(a,x_1^{i-1},b,x_{i+1}^n)$$

instead of the equation (4.1).

Theorem 6. *The set $W\mathcal{F}_{(\tau,\tau')}(X)$ forms a partial semigroup with respect to the partial operation $\cdot_F^{n,i}$.*

Proof. Let a, b and d be elements in $W\mathcal{F}_{(\tau,\tau')}(X)$. To prove that $\cdot_F^{n,i}$ satisfies an associative law as a weak identity, we need to show that for each $i = 1, \ldots, n$, if $(a \cdot_F^{n,i} b) \cdot_F^{n,i} d$ is defined, then $a \cdot_F^{n,i} (b \cdot_F^{n,i} d)$ is defined and $(a \cdot_F^{n,i} b) \cdot_F^{n,i} d = a \cdot_F^{n,i} (b \cdot_F^{n,i} d)$. Assume that $(a \cdot_F^{n,i} b) \cdot_F^{n,i} d$ is defined. Then we obtain that a belongs to $W\mathcal{F}_{(\tau,\tau')}(X)$ but b and d are in $W_\tau(X)$. From this assumption, $a \cdot_F^{n,i} (b \cdot_F^{n,i} d)$ is also defined. To show that $(a \cdot_F^{n,i} b) \cdot_F^{n,i} d$ and $a \cdot_F^{n,i} (b \cdot_F^{n,i} d)$ are equal, some cases are considered. We start with the case when all a, b and d are terms of type τ. Then $(a \cdot_F^{n,i} b) \cdot_F^{n,i} d$ is equal to $S_g^n(S_g^n(a, x_1^{i-1}, b, x_n^{i+1}), x_1^{i-1}, d, x_n^{i+1})$ and $a \cdot_g^F (b \cdot_g^F d)$ equals $S_g^n(a, x_1^{i-1}, S_g^n(b, x_1^{i-1}, b, x_n^{i+1}), x_n^{i+1})$. As a result, by the fact that S_g^n is superassociative, we conclude that $(a \cdot_F^{n,i} b) \cdot_F^{n,i} d = a \cdot_F^{n,i} (b \cdot_F^{n,i} d)$. For another case, we also have that $(a \cdot_F^{n,i} b) \cdot_F^{n,i} d = R_g^n(a, x_1^{i-1}, b, x_n^{i+1}) \cdot_F^{n,i} d = R_g^n(R_g^n(a, x_1^{i-1}, b, x_n^{i+1}), x_1^{i-1}, d, x_n^{i+1})$ and in another side $a \cdot_g^F (b \cdot_g^F d) = R_g^n(a, x_1, \ldots, x_{i-1}, b \cdot_g^F d, x_{i+1}, \ldots, x_n)$, which is equal to
$R_g^n(a, x_1, \ldots, x_{i-1}, R_g^n(b, x_1^{i-1}, d, x_n^{i+1}), x_{i+1}, \ldots, x_n)$. Due to the superassociativity of R_g^n, in this case, $(a \cdot_F^{n,i} b) \cdot_F^{n,i} d$ and $a \cdot_F^{n,i} (b \cdot_F^{n,i} d)$ are equal. $\square$

The symbol $(a_j)_{j=1}^n$ denotes an n-tuple of the form $(a_1, \ldots, a_n)$. On the Cartesian product $(W\mathcal{F}_{(\tau,\tau')}(X))^n$ of $W\mathcal{F}_{(\tau,\tau')}(X)$, the partial binary operation

$$*_F^n : (W\mathcal{F}_{(\tau,\tau')}(X))^n \times (W\mathcal{F}_{(\tau,\tau')}(X))^n \multimap\!\!\rightarrow W\mathcal{F}_{(\tau,\tau')}(X)$$

is defined by

$$(a_j)_{j=1}^n *_F^n (b_j)_{j=1}^n = (\overline{R}_g^n(a_j, b_1, \ldots, b_n))_{j=1}^n$$

for every $(a_j)_{j=1}^n, (b_j)_{j=1}^n \in (W\mathcal{F}_{(\tau,\tau')}(X))^n$.

Theorem 7. *A system $((W\mathcal{F}_{(\tau,\tau')}(X))^n, *_F^n)$ is a partial semigroup.*

Proof. Assume that $(a_j)_{j=1}^n, (b_j)_{j=1}^n$ and $(d_j)_{j=1}^n$ are elements in

$$(W\mathcal{F}_{(\tau,\tau')}(X))^n.$$

To show that an associativity, i.e.,

$$((a_j)_{j=1}^n *_F^n (b_j)_{j=1}^n) *_F^n (d_j)_{j=1}^n = (a_j)_{j=1}^n *_F^n ((b_j)_{j=1}^n *_F^n (d_j)_{j=1}^n) \quad (4.2)$$

is weak, suppose that the left-hand side of (4.2) is defined. Thus, we have that $(a_j)_{j=1}^n$ belongs to the set $(W\mathcal{F}_{(\tau,\tau')}(X))^n$ but a pair of $(b_j)_{j=1}^n$ and

$(d_j)_{j=1}^n$ belongs to the set $(W_\tau(X))^n$. From this, obviously, the right-hand side of (4.2) is also defined. In order to show that both sides of (4.2) are equal, we devide our consideration into some cases. If tuples $(a_j)_{j=1}^n, (b_j)_{j=1}^n$ and $(d_j)_{j=1}^n$ come from the product of terms of type τ, we have that left hand-side of equation (4.2) equals $S_g^n(S_g^n(a_j, b_1, \ldots, b_n), d_1, \ldots, d_n)_{j=1}^n$ and the right hand-side of (4.2) is equal to

$$(S_g^n(a_j, S_g^n(b_1, d_1, \ldots, d_n), \ldots, S_g^n(b_n, d_1, \ldots, d_n)))_{j=1}^n.$$

Because the superposition S_g^n on the set $W_\tau(X)$ satisfies the superassociative law, we conclude that, in this case, both side of the equation (4.2) are equal. For the case when $(a_j)_{j=1}^n$ is an n-tuple of formulas of type (τ, τ') while $(b_j)_{j=1}^n$ and $(d_j)_{j=1}^n$ are n-tuples of terms of type τ, we get that $((a_j)_{j=1}^n *_F^n (b_j)_{j=1}^n) *_F^n (d_j)_{j=1}^n$ equals $(R_g^n(R_g^n(a_j, b_1, \ldots, b_n), d_1, \ldots, d_n))_{j=1}^n$ and $(a_j)_{j=1}^n *_F^n ((b_j)_{j=1}^n *_F^n (d_j)_{j=1}^n)$ is equal to

$$(R_g^n(a_j, R_g^n(b_1, d_1, \ldots, d_n), \ldots, R_g^n(b_n, d_1, \ldots, d_n)))_{j=1}^n.$$

It follows from [18] that $(R_g^n(R_g^n(a_j, b_1, \ldots, b_n), d_1, \ldots, d_n))_{j=1}^n$ and

$$(R_g^n(a_j, R_g^n(b_1, d_1, \ldots, d_n), \ldots, R_g^n(b_n, d_1, \ldots, d_n)))_{j=1}^n$$

are coincide.

Finally, suppose that there are two subsets $\{i_1, \ldots, i_k\}$ and $\{i'_1, \ldots, i'_k\}$ of an infinite set $\{1, 2, \ldots\}$ such that

1. $\{i_1, \ldots, i_k\} \cap \{i'_1, \ldots, i'_k\} = \emptyset$,
2. $a_{i_l} \in W_\tau(X)$, for all $l = 1, \ldots, k$,
3. $a_{i'_l} \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$, for all $l = 1, \ldots, k$.

Thus, we consider in the case when a_{i_l} belongs to $W_\tau(X_n)$ and a'_{i_l} is a formula in $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ and both $(b_j)_{j=1}^n, (d_j)_{j=1}^n$ are n-tuples in

$$(W_\tau(X_n))^n.$$

From the left-hand side of the equation (4.2), we obtain that $((a_j)_{j=1}^n *^F (b_j)_{j=1}^n) *^F (d_j)_{j=1}^n$ is equal to $(e_j)_{j=1}^n *^F (d_j)_{j=1}^n$ where

$$e_{i_l} = S_g^n(a_{i_l}, b_1, \ldots, b_n) \text{ and } e_{i'_l} = R_g^n(a_{i'_l}, b_1, \ldots, b_n)$$

for all $l = 1, \ldots, k$, consequently, $(p_j)_{j=1}^n$ where

$$p_{i_l} = S_g^n(S_g^n(a_{i_l}, b_1, \ldots, b_n), d_1, \ldots, d_n)$$

and

$$p_{i'_l} = R^n(R^n(a_{i'_l}, b_1, \ldots, b_n), d_1, \ldots, d_n$$

for all $l = 1, \ldots, k$. For the right-hand side of the equation (4.2), we have that $(a_j)_{j=1}^n *^F ((b_j)_{j=1}^n *^F (d_j)_{j=1}^n)$ is equal to

$$(a_j)_{j=1}^n *^F (S^n(b_j, d_1, \ldots, d_n))_{j=1}^n,$$

subsequently, $(\overline{p}_j)_{j=1}^n$ where for all $l = 1, \ldots, k$,

$$\overline{p}_{i_l} = S^n(a_{i_l}, S^n(b_1, d_1, \ldots, d_n), \ldots, S^n(b_n, d_1, \ldots, d_n))$$

and $\overline{p}_{i'_l} = R^n(a_{i'_l}, S^n(b_1, d_1, \ldots, d_n), \ldots, S^n(b_n, d_1, \ldots, d_n))$. Therefore, we have $(p_j)_{j=1}^n = (\overline{p}_j)_{j=1}^n$ where $p_{i_l} = \overline{p}_{i_l}$ and $p_{i'_l} = \overline{p}_{i'_l}$ for all $l = 1, \ldots, k$. This shows that the equation (4.2) of associativity holds as a weak identity. $\square$

This section is closed with a discussion of connections between partial semigroups $(W\mathcal{F}_{(\tau,\tau')}(X), +_F^n)$ and $((W\mathcal{F}_{(\tau,\tau')}(X))^n, *_F^n)$ through the concept of an embedding theorem. To attain this, the notion of weak homomorphisms is required. Normally, a function ψ from a partial semigroup $(S, \cdot)$ to $(H, \odot)$ is called a weak homomorphism if $a \cdot b$ is defined, then $\psi(a) \odot \psi(b)$ is also defined, and then $\psi(a \cdot b) = \psi(a) \odot \psi(b)$ for all $a, b \in S$. If a weak homomorphism ψ is injective, we call ψ a weak monomorphism. Alternatively, we say that a partial semigroup $(S, \cdot)$ can be weakly embedded into $(H, \odot)$.

Theorem 8. *The partial semigroup $(W\mathcal{F}_{(\tau,\tau')}(X), +_F^n)$ is weakly embeddable into $((W\mathcal{F}_{(\tau,\tau')}(X))^n, *_F^n)$.*

Proof. For every $a \in W\mathcal{F}_{(\tau,\tau')}(X)$, we define the function

$$\psi : W\mathcal{F}_{(\tau,\tau')}(X) \to (W\mathcal{F}_{(\tau,\tau')}(X))^n$$

by $\psi(a) = (a, \ldots, a)$. Clearly, ψ is an injection. To show that ψ is a weak homomorphism, let a and b be two elements in the set $W\mathcal{F}_{(\tau,\tau')}(X)$. Suppose that $a +_F^n b$ is defined, which means that $\overline{R}_g^n(a, b, \ldots, b)$ is defined, and equals $S_g^n(a, b, \ldots, b)$ for terms a, b and $R_g^n(a, b, \ldots, b)$ for a formula a and a term b. If $a, b \in W_\tau(X)$, then $S_g^n(a, b, \ldots, b) \in W_\tau(X)$, and then $\psi(a) *_F^n \psi(b)$ is also defined and equals $(a, \ldots, a) *_F^n (b, \ldots, b)$. It follows that $\psi(a +_F^n b) = \psi(S_g^n(a, b, \ldots, b)) = (S_g^n(a, b, \ldots, b), \ldots, S_g^n(a, b, \ldots, b)) = (a, \ldots, a) *_F^n (b, \ldots, b) = \psi(a) *_F^n \psi(b)$. Otherwise, we have $\psi(a +_F^n b) = \psi(R_g^n(a, b, \ldots, b)) = (R_g^n(a, b, \ldots, b), \ldots, R_g^n(a, b, \ldots, b)) = \psi(a) *_F^n \psi(b)$. From these preparations, we conclude that an injective mapping ψ is a weak homomorphism. $\square$

5. Conclusion

The paper addresses the construction of a partial superassociative algebra of formulas under generalized superpositions, where formulas are generated by terms from an infinite set of alphabets, logical connectives, and relation symbols. The method employs weak identities and partial algebras. Three associative binary operations on the union set of terms and formulas, induced by the partial operation $\overline{R}^n$ for every $n \in \mathbb{N}$ are discussed.

For future work, the concept of many-sorted algebras associated with partial algebras could be explored. Alternatively, representations of these partial structures through partial functions offer an interesting direction.

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Об авторах

Кумдуанг Т., канд. физ.-мат. наук, Технологический университет Раджамангала, Накхонпатхом, 73170, Таиланд, thodsaporn.kum@rmutr.ac.th, https://orcid.org/0000-0001-9848-133X

Ваттанатрипоп К., канд. физ.-мат. наук, Технологический университет Раджамангала Ланна, Чиангмай, 50300, Таиланд, khwancheewa.wat@rmutl.ac.th, https://orcid.org/0000-0002-6496-4015

About the authors

T. Kumduang, Ph. D. (Math.), Lecturer, Rajamangala University of Technology Rattanakosin, Nakhon Pathom, 73170, Thailand, thodsaporn.kum@rmutr.ac.th, https://orcid.org/0000-0001-9848-133X

K. Wattanatripop, Ph. D. (Math.), Lecturer, Rajamangala University of Technology Lanna, Chiang Mai, 50300, Thailand, khwancheewa.wat@rmutl.ac.th, https://orcid.org/0000-0002-6496-4015

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