Серия «Математика»
2025. Т. 54. С. 143—159

Онлайн-доступ к журналу:
http://mathizv.isu.ru

И З В Е С Т И Я
Иркутского государственного университета

Research article

УДК 510.67
MSC 03C30, 03C45, 03C52
DOI https://doi.org/10.26516/1997-7670.2025.54.143

Expansions and Restrictions of Structures and Theories, Their Hierarchies

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Abstract: The study and description of possibilities of expansions and restrictions of structures and their theories is used to obtain a structural information both in general and for various natural algebraic, geometric, ordered theories and models. The origins for the description are based on known model-theoretic operations of Morleyzation, or Atomization, and Skolemization, allowing to preserve or naturally extend formulaically definable sets of a given structure, and obtain a level of quantifier elimination, where formulaically definable sets are represented as Boolean combinations of definable sets specified by quantifier-free formulae. The operations of Shelahizations, or Namizations, produce both extensions and expansions of a structure giving names or labels for definable sets. In the paper, we introduce and study some general principles and hierarchical properties of expansions and restrictions of structures and their theories. These principles are based on upper and lower cones, lattices, and permutations. The general approach is applied to describe these properties for classes of ω-categorical theories and structures, Ehrenfeucht theories and their models, strongly minimal, ω_1-categorical, and stable ones. Here all these classes are closed under permutations. It is proved that any fusions of strongly minimal structures are strongly minimal, too, whereas the properties of ω-categoricity, Ehrenfeuchtness, ω_1-categoricity, and stability can fail under fusions. It is also shown that the classes of ω-categorical, strongly minimal and stable regular structures are closed under lower cones of all their elements, whereas the classes of Ehrenfeucht and ω_1-categorical structures do not have that property, with some infinite chains of expansions alternating Ehrenfeuchtness and non-Ehrenfeuchtness, and other infinite chains alternating ω_1-categoricity and non-ω_1-categoricity.

Keywords: hierarchy, property, expansion of structure, restriction of structure, theory

Acknowledgements: The work was carried out in the framework of Russian Scientific Foundation, Project No. 24-21-00096. (https://rscf.ru/project/24-21-00096/).

For citation: Sudoplatov S. V. Expansions and Restrictions of Structures and Theories, Their Hierarchies. *The Bulletin of Irkutsk State University. Series Mathematics*, 2025, vol. 54, pp. 143–159.
https://doi.org/10.26516/1997-7670.2025.54.143

Научная статья

Обогащения и обеднения структур и теорий, их иерархии

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Аннотация: Изучение и описание возможностей обогащений и обеднений структур и их теорий используется для получения структурной информации как в целом, так и для различных естественных алгебраических, геометрических, упорядоченных теорий и моделей. Истоки такого описания основаны на известных теоретико-модельных операциях морлизации, или атомизации, и сколемизации, позволяющих сохранять или естественным образом расширять систему формульно определимых множеств данной структуры и получать уровень элиминации кванторов, при котором формульно определимые множества представлены как булевы комбинации определимых множеств, заданных формулами без кванторов. Операции шелахизации, или неймизации, производят как обогащения, так и расширения структуры, при которых задаются имена или метки для определимых множеств. Вводятся и изучаются некоторые общие принципы и иерархические свойства обогащений и обеднений структур и их теорий. Эти принципы основаны на верхних и нижних конусах, решетках и перестановках. Общий подход применяется для описания этих свойств для классов ω-категоричных теорий и структур, эренфойхтовых теорий и их моделей, сильно минимальных, ω_1-категоричных и стабильных теорий и структур. При этом все эти классы замкнуты относительно перестановок. Доказано, что любые слияния сильно минимальных структур также являются сильно минимальными, тогда как свойства ω-категоричности, эренфойхтовости, ω_1-категоричности и стабильности могут нарушаться при слияниях. Также показано, что классы ω-категоричных, сильно минимальных и стабильных регулярных структур замкнуты относительно нижних конусов всех их элементов, тогда как классы эренфойхтовых и ω_1-категоричных структур этим свойством не обладают, причем некоторые бесконечные цепочки расширений чередуют эренфойхтовость и неэренфойхтовость, а другие бесконечные цепочки чередуют ω_1-категоричность и не ω_1-категоричность.

Ключевые слова: иерархия, свойство, обогащение структуры, обеднение структуры, теория

Благодарности: Работа выполнена при финансовой поддержке Российского научного фонда, проект №. 24-21-00096.(https://rscf.ru/project/24-21-00096/).

Ссылка для цитирования: Sudoplatov S. V. Expansions and Restrictions of Structures and Theories, Their Hierarchies // Известия Иркутского государственного университета. Серия Математика. 2025. Т. 54. С. 143–159.
https://doi.org/10.26516/1997-7670.2025.54.143

1. Introduction

The study and description of possibilities of distributions of expansions and restrictions of structures and their theories is of interest both in general and for various natural algebraic, geometric, ordered theories and models. The prerequisites for the description are the original operations of Morleyzation, or Atomization [7; 14], and Skolemization [4; 7; 14], allowing to preserve or non-radically include formulaically defined sets of a given structure, and upon receipt of this elimination of quantifiers, according to which formulaically defined levels arise in the form of Boolean combinations of definable sets specified by quantifier-free formulae. The operations of Shelahizations [2; 13], or Namizations, produce both extensions and expansions of a structure giving names or labels for definable sets. In addition, main types of combinations of theories based on E-combinations and P-combinations of their models are used for various expansions preserving a series of properties [17]. These data allow to introduce topological approximations and develop methods of approximations [18], rank characteristics of a family of theories [19] introduced and studied within the framework of approximating formulae [20] and, in particular, pseudo-finite formulas [9]. In addition, the effects associated with the preservation and violation of properties during expansions and restrictions, with significant formulaicity or non-formulaicity of these properties is investigated. The formulaic properties include known theories with the stability property [12], according to which a theory is stable if each its formula is stable, i.e. it does not have the order property, and a non-formulaic property, for example, is the property of the theory of normality [11], which, as the author's examples show [15], may not be preserved when taking restrictions. Other formulaic properties include the properties of total transcendence and superstability [12] based on the monotonicity of ranks when passing to restricted theories. A semantic characteristic is the value of the number of non-isomorphic countable models of these theories, which can both decrease and increase when the theory is restricted. Using these tools, it is possible to describe the conditions for the preservation and violation of significant properties when constructing theories and their models, both in general and for series of known classes of algebraic, geometric, ordered theories and their models.

The paper is organized as follows. In Section 2, we introduce the notion of regular structure, Boolean algebras for regular expansions and restrictions based on a given universe both for structures and their theories, and characterize these Boolean algebras up to an isomorphism in terms of cardinalities of universes. Kinds of properties on these Boolean algebras with respect to upper and lower cones, lattices, and permutations are studied in Sections 3, 4, 5, respectively. We apply these divisions of properties to the classes of ω-categorical theories and structures, Ehrenfeucht theories and structures (Section 6), strongly minimal ones (Section 7), ω_1-theories and structures (Section 8), stable ones (Section 9). In particular, it is shown that any fusions of strongly minimal structures are strongly minimal, too (Theorem 6), whereas the properties of ω-categoricity, Ehrenfeuchtness, ω_1-categoricity, and stability can fail under fusions.

2. Regular structures, their expansions, restrictions and Boolean algebras

For convenience we consider *regular* structures, i.e. relational structures without repetitions of interpretations of signature symbols. There is no loss of generality with this assumption, since operations can be replaced by their graphs, and multiple names for the same relations are reduced to one of them. The procedure transforming an arbitrary structure $\mathcal{M}$ to a regular one $\mathcal{N}$ is called the *regularization*, and the structure $\mathcal{N}$ is called *regularized*, with respect to $\mathcal{M}$. And the converse procedure transforming $\mathcal{N}$ to the initial $\mathcal{M}$ is called *deregularization*, and $\mathcal{M}$ is called *deregularized*, with respect to $\mathcal{N}$.

Remark 1. Clearly, regularizations and deregularizations are multi-valued with respect to signatures, in general, but they always preserve families of definable sets on the given universe.

Let $\mathcal{M}$ be a regular structure, $\overline{\mathcal{M}}$ be a maximal regular expansion of $\mathcal{M}$ preserving the universe M. We have

$$\left|\Sigma(\overline{\mathcal{M}})\right| = \max\{2^{|M|}, \omega\}, \tag{2.1}$$

where $\Sigma(\overline{\mathcal{M}})$ denotes the signature of $\overline{\mathcal{M}}$. Now we denote by $B(\mathcal{M})$ the set of all restrictions of $\overline{\mathcal{M}}$ preserving the universe M. Clearly, all these restrictions are regular, too. Since each signature relation of $\overline{\mathcal{M}}$ can be independently preserved or removed there are $2^{\left|\Sigma(\overline{\mathcal{M}})\right|} = 2^{\max\{2^{|M|},\omega\}}$ possibilities for these restrictions, i.e. $|B(\mathcal{M})| = 2^{\max\{2^{|M|},\omega\}}$. Since each element of $B(\mathcal{M})$ is uniquely defined by a subset $\Sigma \subseteq \Sigma(\overline{\mathcal{M}})$, the set-theoretic operations on the Boolean $\mathcal{P}(\Sigma(\overline{\mathcal{M}}))$, forming its Cantor algebra, induce the *regular* atomic Boolean algebra $\mathcal{B}(\mathcal{M})$ on a universe $B(\mathcal{M})$, with the greatest element $\overline{\mathcal{M}}$ having a complete, i.e. maximal signature, the least element $\mathcal{M}_0$ having the empty signature, and $|\Sigma(\overline{\mathcal{M}})|$ atoms each of which has exactly one signature symbol. Here unions $\mathcal{N}_1 \cup \mathcal{N}_2$ and intersections $\mathcal{N}_1 \cap \mathcal{N}_2$, for $\mathcal{N}_1, \mathcal{N}_2 \in B(\mathcal{M})$, preserve the universe M and consists of unions of their signature relations, common signature symbols, respectively. The unions can be considered both as *combinations* [17] and *fusions* [6], in a broad sense, of structures.

The algebra $\mathcal{B}(\mathcal{M})$ includes possibilities of regular structures on the universe M, and contains all restrictions of the structure $\overline{\mathcal{M}}$. This algebra is uniquely defined up to the names of signature symbols, and it reflects a hierarchy of regular expansions and restrictions of structures with respect to the names of relational symbols. Moreover, the following theorem gives a characterization of existence of an isomorphism between regular Boolean algebras.

Theorem 1. *For any regular structures $\mathcal{M}$ and $\mathcal{N}$ the following conditions are equivalent:*

(1) $\mathcal{B}(\mathcal{M}) \simeq \mathcal{B}(\mathcal{N})$,

(2) *there is a bijection between sets of atoms for $\mathcal{B}(\mathcal{M})$ and $\mathcal{B}(\mathcal{N})$,*

(3) $\max\{|M|, \omega\} = \max\{|N|, \omega\}$.

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{B}(\mathcal{M}) \simeq \mathcal{B}(\mathcal{N})$ witnessed by an isomorphism f. Then the restriction of f to the set of atoms produces a bijection between sets of atoms for $\mathcal{B}(\mathcal{M})$ and $\mathcal{B}(\mathcal{N})$.

(2) $\Rightarrow$ (1). Since $\mathcal{B}(\mathcal{M})$ and $\mathcal{B}(\mathcal{N})$ are uniquely defined by their atoms and there is a bijection f_a between sets of atoms for $\mathcal{B}(\mathcal{M})$ and $\mathcal{B}(\mathcal{N})$, this bijection f_a has a unique extension till an isomorphism between $\mathcal{B}(\mathcal{M})$ and $\mathcal{B}(\mathcal{N})$.

(2) $\Rightarrow$ (3). Since each atom for $\mathcal{B}(\mathcal{M})$ and for $\mathcal{B}(\mathcal{N})$ is uniquely defined by a signature symbol in $\Sigma(\overline{\mathcal{M}})$ and in $\Sigma(\overline{\mathcal{N}})$, respectively, the bijection between sets of atoms for $\mathcal{B}(\mathcal{M})$ and $\mathcal{B}(\mathcal{N})$ implies $\max\{|M|, \omega\} = \max\{|N|, \omega\}$.

(3) $\Rightarrow$ (2). Since the equality $\max\{|M|, \omega\} = \max\{|N|, \omega\}$ implies $\left|\Sigma(\overline{\mathcal{M}})\right| = \left|\Sigma(\overline{\mathcal{N}})\right|$ in view of (2.1), there is a bijection between sets of atoms for $\mathcal{B}(\mathcal{M})$ and $\mathcal{B}(\mathcal{N})$.

Theorem 1 immediately implies:

Corollary 1. *For any at most countable regular structures $\mathcal{M}$ and $\mathcal{N}$ their Boolean algebras $\mathcal{B}(\mathcal{M})$ and $\mathcal{B}(\mathcal{N})$ are isomorphic.*

Since each algebra $\mathcal{B}(\mathcal{M})$ consists of structures with pairwise distinct signatures all these structures $\mathcal{M}'$ have pairwise distinct theories $\mathrm{Th}(\mathcal{M}')$. Thus all structures in $\mathcal{B}(\mathcal{M})$ can be replaced by their theories obtaining an isomorphic Boolean algebra $\mathcal{B}(\mathrm{Th}(\mathcal{M}))$, which is defined by a partial order $\leq$ with $T_1 \leq T_2$ iff a theory $T_2 \in B(\mathrm{Th}(\mathcal{M}))$ is an expansion of theory $T_1 \in B(\mathrm{Th}(\mathcal{M}))$. Here the algebra $\mathcal{B}(\mathcal{M})$ is composed by semantic objects $\mathcal{M}'$ whereas $\mathcal{B}(\mathrm{Th}(\mathcal{M}))$ consists of syntactic ones: $\mathrm{Th}(\mathcal{M}')$.

Notice that $\mathcal{B}(\mathrm{Th}(\mathcal{M}))$ depends on the cardinality $\lambda = |M|$ and, since models of their theory are uniquely defined up to an isomorphism iff these models are finite, one can reconstruct elements of $\mathcal{B}(\mathcal{M})$ by elements of $\mathcal{B}(\mathrm{Th}(\mathcal{M}))$, up to the bijection of the universe M, iff M is finite. So we denote $\mathcal{B}(\mathrm{Th}(\mathcal{M}))$ by $\mathcal{B}_\lambda(T)$, where $T = \mathrm{Th}(\mathcal{M})$ and $\lambda = |\mathcal{M}|$. Here a finite cardinality λ can be replaced by countable one by Theorem 1.

In view of Theorem 1 we have the following:

Theorem 2. *For any regular theories T_1 and T_2 the following conditions are equivalent:*
(1) $\mathcal{B}_\lambda(T_1) \simeq \mathcal{B}_\mu(T_2)$,
(2) *there is a bijection between sets of atoms for $\mathcal{B}_\lambda(T_1)$ and $\mathcal{B}_\mu(T_2)$,*
(3) $\max\{\lambda, \omega\} = \max\{\mu, \omega\}$.

Corollary 2. *For any countable regular theories T_1 and T_2 their Boolean algebras $\mathcal{B}_\omega(T_1)$ and $\mathcal{B}_\omega(T_2)$ are isomorphic.*

Remark 2. The procedures of (de)regularizations can be naturally transformed from structures to their theories. So by the definition any complete theory is a deregularization of a theory T in appropriate $\mathcal{B}_\lambda(T)$.

3. Kinds of properties with respect to Boolean algebras

Recall that for a lattice $\mathcal{L}$ and its element a the *upper cone*, denoted by $\nabla(a)$ and ∇_a, consists of all elements b in L with $a \leq b$, and the *lower cone*, denoted by $\triangle(a)$ and $\triangle_a$, consists of all elements b in L with $b \leq a$.

Take a Boolean algebra $\mathcal{B}(\mathcal{M})$ and a property $P \subseteq B(\mathcal{M})$. The property P is *closed under expansions* in $\mathcal{B}(\mathcal{M})$, or with respect to $\mathcal{B}(\mathcal{M})$, or briefly the $E_{\mathcal{B}(\mathcal{M})}$*-property*, if all expansions, in $\mathcal{B}(\mathcal{M})$, of any structure $\mathcal{N} \in P$ belong to P. Accordingly, the property P is *closed under restrictions* in $\mathcal{B}(\mathcal{M})$, or with respect to $\mathcal{B}(\mathcal{M})$, or briefly the $R_{\mathcal{B}(\mathcal{M})}$*-property*, if all restrictions, in $\mathcal{B}(\mathcal{M})$, of any structure $\mathcal{N} \in P$ belong to P.

A property P on the class of regular structures is *closed under expansions*, or briefly the *E-property*, if all expansions of any structure $\mathcal{N} \in P$ belong to P. Accordingly, the property P is *closed under restrictions*, or briefly the *R–property*, if all restrictions of any structure $\mathcal{N} \in P$ belong to P.

Remark 3. If a property $P \subseteq B(\mathcal{M})$ is closed both under expansions and restrictions in $B(\mathcal{M})$ then either $P = \emptyset$ or $P = B(\mathcal{M})$.

Remark 4. Using regularizations and deregularizations the notions of E-property and R-property can be naturally spread for arbitrary structures and for theories, not necessary regular.

Remark 5. A series of non-elementary properties separates the closeness under expansions (restrictions) with respect to given Boolean algebras and in general. For instance, if $\mathcal{M}$ is a countable structure without constant symbols then it is expansible by countably many constants and does not expansible in $\mathcal{B}(\mathcal{M})$ by uncountably many constants whereas $\mathcal{M}$ can be arbitrarily many repeated constants.

By the definition we have the following:

Proposition 1. *For any Boolean algebra $\mathcal{B}(\mathcal{M})$ and a property $P \subseteq B(\mathcal{M})$, P is closed under expansions (restrictions) in $\mathcal{B}(\mathcal{M})$ iff for any $\mathcal{N} \in P$, $\nabla_{\mathcal{N}} \subseteq P$ ($\triangle_{\mathcal{N}} \subseteq P$), i.e. $P = \bigcup\limits_{\mathcal{N}\in P} \nabla_{\mathcal{N}}$ $\left(P = \bigcup\limits_{\mathcal{N}\in P} \triangle_{\mathcal{N}}\right)$.*

Corollary 3. *Let P be a property on the class of structures closed under regularizations and deregularizations. Then P is closed under expansions (restrictions) iff for any regular $\mathcal{M} \in P$, $\nabla_{\mathcal{M}} \subseteq P$ ($\triangle_{\mathcal{M}} \subseteq P$) in any $\mathcal{B}(\mathcal{M})$, i.e. P is the deregularization of $\bigcup\limits_{\mathcal{M}\in P} \nabla_{\mathcal{M}}$ $\left(\bigcup\limits_{\mathcal{M}\in P} \triangle_{\mathcal{M}}\right)$.*

Remark 6. Since elements of $\triangle_{\mathcal{N}}$ are obtained from $\mathcal{N}$ just loosing some signature symbols of $\mathcal{N}$ then $\triangle_{\mathcal{N}}$ does not depend on choice of Boolean algebra $\mathcal{B}(\mathcal{M})$ containing $\mathcal{N}$. Similarly $\nabla_{\mathcal{N}}$ does not depend on choice of Boolean algebra $\mathcal{B}(\mathcal{M})$ containing $\mathcal{N}$, since it is obtained from $\mathcal{N}$ by adding arbitrary new signature symbols for relations preserving the regularity. So the assertion of Corollary 3 differs from Proposition 1 just by possibilities of deregularization.

By Proposition 1 any complement $\overline{P}$ of a property $P \subseteq B(\mathcal{M})$, which is closed under expansions (restrictions) in $\mathcal{B}(\mathcal{M})$, is represented by some $\bigcap\limits_{\mathcal{N}} \overline{\nabla_{\mathcal{N}}}$ $\left(\bigcap\limits_{\mathcal{N}} \overline{\triangle_{\mathcal{N}}}\right)$. At the same time, by the duality principle, it is closed under restrictions (expansions). Indeed, let $\overline{P}$ be not closed under restrictions (expansions). Then $\mathcal{B}(\mathcal{M})$ has elements $\mathcal{N}_1$ and $\mathcal{N}_2$ with $\mathcal{N}_1 \in \overline{P}$, $\mathcal{N}_2 \in P$, $\mathcal{N}_2 \in \triangle(\mathcal{N}_1)$ ($\mathcal{N}_2 \in \nabla(\mathcal{N}_1)$). But then $\mathcal{N}_1 \in \nabla(\mathcal{N}_2)$ ($\mathcal{N}_1 \in \triangle(\mathcal{N}_2)$) contradicting the conditions that P is closed under expansions (restrictions).

Thus, applying Proposition 1 we have the following:

Theorem 3. *For any property $P \subseteq B(\mathcal{M})$ the following conditions are equivalent:*

(1) *P is closed under expansions (restrictions) in $\mathcal{B}(\mathcal{M})$;*

(2) $P = \bigcup\limits_{\mathcal{N}\in P} \nabla_{\mathcal{N}}$ $\left(P = \bigcup\limits_{\mathcal{N}\in P} \triangle_{\mathcal{N}}\right)$;

(3) $\overline{P} = \bigcup\limits_{\mathcal{N}\in \overline{P}} \triangle_{\mathcal{N}}$ $\left(\overline{P} = \bigcup\limits_{\mathcal{N}\in \overline{P}} \nabla_{\mathcal{N}}\right)$.

Transforming structures to their theories we obtain the following:

Corollary 4. *For any property $P \subseteq B_\lambda(T)$ the following conditions are equivalent:*

(1) *P is closed under expansions (restrictions) in $\mathcal{B}_\lambda(T)$;*

(2) $P = \bigcup\limits_{T' \in P} \triangledown_{T'}$ $\left(P = \bigcup\limits_{T' \in P} \triangle_{T'}\right)$;

(3) $\overline{P} = \bigcup\limits_{T' \in \overline{P}} \triangle_{T'}$ $\left(\overline{P} = \bigcup\limits_{Y' \in \overline{P}} \triangledown_{T'}\right)$.

4. Properties forming lattices

Definition. A property $P \subseteq B(\mathcal{M})$ is called a *lattice property*, or L-*property*, if P is the universe of a sublattice of the lattice restriction of $\mathcal{B}(\mathcal{M})$.

Since $B(\mathcal{M})$ is distributive, any L-property is distributive, too.

By the definition a nonempty property $P \subseteq B(\mathcal{M})$ is a L-property iff for any $\mathcal{N}_1, \mathcal{N}_2 \in P$, $\mathcal{N}_1 \cap \mathcal{N}_2 \in P$ and $\mathcal{N}_1 \cup \mathcal{N}_2 \in P$. Besides lower cones $\triangle_{\mathcal{N}}$ and upper cones $\triangledown_{\mathcal{N}}$ are closed under intersections and under unions, therefore they are L-properties. Since L-properties preserve the distributivity, we have the following:

Proposition 2. *If $\emptyset \neq P \subseteq B(\mathcal{M})$ is closed under lower (upper) cones then the following conditions are equivalent:*
(1) *P is a L-property,*
(2) *P is a distributive L-property,*
(3) *P is closed under unions (intersections).*

Corollary 5. *If $\emptyset \neq P \subseteq B(\mathcal{M})$ is closed under lower (upper) cones and it is a L-property then the following conditions are equivalent:*
(1) *P is a finite union of lower (upper) cones;*
(2) *P is a lower (upper) cone.*

Proof. Clearly, a lower (upper) cone is a finite union of lower (upper) cones, i.e. (2) $\Rightarrow$ (1) obviously holds. So it suffices to show, using (1) and the L-property for P, that two lower (upper) cones are reduced to one lower (upper) cone. Indeed, if $\triangle(\mathcal{N}_1) \subseteq P$ and $\triangle(\mathcal{N}_2) \subseteq P$ then $\mathcal{N}_1 \cup \mathcal{N}_2 \in P$ by Proposition 2, with $\triangle(\mathcal{N}_1) \cup \triangle(\mathcal{N}_2) \subseteq \triangle(\mathcal{N}_1 \cup \mathcal{N}_2) \subseteq P$. Similarly, if $\triangledown(\mathcal{N}_1) \subseteq P$ and $\triangledown(\mathcal{N}_2) \subseteq P$ then $\mathcal{N}_1 \cup \mathcal{N}_2 \in P$ by Proposition 2, with $\triangledown(\mathcal{N}_1) \cup \triangledown(\mathcal{N}_2) \subseteq \triangledown(\mathcal{N}_1 \cup \mathcal{N}_2) \subseteq P$ completing the proof of (1) $\Rightarrow$ (2).

Remark 7. Clearly, there are many L-properties which are not represented as unions of cones. For instance, closures in $\overline{\mathcal{M}}$ with respect to unions and intersections of sets of finitely many structures $\mathcal{N}_1, \ldots, \mathcal{N}_k \in B(\mathcal{M})$ form finite lattices, moreover, Boolean algebras, as any finitely generated, i.e. finite lattices here. If signatures $\Sigma_1, \ldots, \Sigma_k$ of $\mathcal{N}_1, \ldots, \mathcal{N}_k$, respectively, are infinite and co-infinite and their unions and intersections are infinite and co-infinite, too, then these Boolean algebras do not contain cones at all.

At the same time, if signatures $\Sigma_1, \ldots, \Sigma_k$ are (co)finite then they can be extended by finitely many ones forming cones of the form $\triangle_{\mathcal{N}}$ (respectively, $\nabla_{\mathcal{N}}$), where $\mathcal{N}$ has the signature $\Sigma_1 \cup \ldots \cup \Sigma_k$ ($\Sigma_1 \cap \ldots \cap \Sigma_k$).

5. Structures and properties closed under permutations

Definition. We say that a property $P \subseteq B(\mathcal{M})$ is *closed* or *invariant under permutations* if for any permutation f on M and for any $\mathcal{N} \in B(\mathcal{M})$ the image $f(\mathcal{N})$, with signature relations $f(Q) = \{f(\overline{a}) \mid \overline{a} \in Q\}$ for all given signature relations Q on $\mathcal{N}$, belongs to P. Here the names Q for signature predicates are preserved with respect to f iff $f(Q) = Q$.

A structure $\mathcal{N}$ is *closed* or *invariant under permutations* if the property $P = \{\mathcal{N}\}$ is closed under permutations, i.e. each permutation of the universe of $\mathcal{N}$ is a permutation of all signature relations.

Example 1. If $\mathcal{N}$ consists of all signature singletons $\{a\}$, $a \in N$, then $\mathcal{N}$ is closed under permutations. Moreover, if $\mathcal{N}$ consists of all n-element unary relations, where n is some natural number, then $\mathcal{N}$ is closed under permutations, too. We observe the same effect for relations with n-element complements. In particular, if a structure $\mathcal{N}$ consists of either empty or complete signature relations then $\mathcal{N}$ is closed under permutations.

Proposition 3. *If $P = \triangle_{\mathcal{N}}$ ($P = \nabla_{\mathcal{N}}$) then the following conditions are equivalent:*

(1) *P is closed under permutations;*

(2) *$\mathcal{N}$ is closed under permutations.*

Proof. (1) $\Rightarrow$ (2). Since $P = \triangle_{\mathcal{N}}$ ($P = \nabla_{\mathcal{N}}$) and P is closed under permutations, any permutation f on M preserves the relation "to be a restriction (an expansion)". Then more rich (poor) structures with respect to the signature are transformed to more rich (poor). In particular, the most rich (poor) structure $\mathcal{N}$ in P is transformed to itself, i.e. $\mathcal{N}$ is closed under permutations.

(2) $\Rightarrow$ (1). If $\mathcal{N}$ is closed under permutations then any permutation f of M moves any subsignature (supersignature) in $\mathcal{N}$ again to a subsignature (supersignature) in $\mathcal{N}$. Then $f(\triangle_{\mathcal{N}}) \subseteq \triangle_{\mathcal{N}}$ ($f(\nabla_{\mathcal{N}}) \subseteq \nabla_{\mathcal{N}}$). Since f^{-1} is again a permutation, we have $f(\triangle_{\mathcal{N}}) \supseteq \triangle_{\mathcal{N}}$ ($f(\nabla_{\mathcal{M}}) \supseteq \nabla_{\mathcal{N}}$). Thus P is preserved under permutations.

Remark 8. If a property P on a family $B(\mathcal{M})$ of structures is defined by interactions of definable sets and it is preserved under replacements of universes then P is closed under permutations. We observe the same effect for classes of all models of theories in a given family. These classes are preserved by isomorphisms, in particular, by permutations transforming signature relations.

Example 2. Let $\mathcal{M}_n$ be a structure consisting of all predicate relations with n-element complements, say of given family of arities. Then both $\mathcal{M}_n$ and $\triangle(\mathcal{M}_n)$ are invariant under permutations, with respect to any Boolean algebra $\mathcal{B}(\mathcal{N})$, containing $\mathcal{M}_n$.

The same effect holds in expansions (restrictions) of $\mathcal{M}_n$ by (to) empty and/or complete predicates. We always have this invariance for one-element regular structures.

6. The families of ω-categorical and Ehrenfeucht structures in $\mathcal{B}(\mathcal{M})$

Let $P = P_{\omega\text{-cat}} \subseteq B(\mathcal{M})$ be the property of ω-categoricity. Recall that it is characterized, in view of Ryll-Nardzewski theorem, by finitely many n-types over the empty set, for any natural n. Then this property $P_{\omega\text{-cat}}$ is invariant under permutations.

Since restrictions of theories preserve the finiteness of number of types, P is closed under lower cones: if $\mathcal{N} \in P$ then $\triangle_{\mathcal{N}} \subseteq P$. Thus we have

$$P = \bigcup_{\mathcal{N}\in P} \triangle_{\mathcal{N}}. \tag{6.1}$$

The following two examples show that P is not closed under fusions.

Example 3. Let $\mathcal{N}_i = \langle M; Q_i\rangle$, $i = 1, 2$, be two infinite graphs whose each connected component consists of unique edge. Clearly each $\mathcal{N}_i$ is ω-categorical with finitely many n-types and these types are defined by families of formulae of forms $x \approx x$, $(x \approx y)^\delta$, $Q^\delta(x, y)$, $\delta \in \{0, 1\}$. The structure $\mathcal{N}_1 \cup \mathcal{N}_2 = \langle M; Q_1, Q_2\rangle$ consists of vertices each of which is incident to exactly two edges, of distinct colors Q_1 and Q_2. These edges belong to either cycles of even lengths or to infinite chains, depending on relationship of the relations Q_1 and Q_2. In a case of presence of unbounded lengths of cycles or an infinite chain the fusion $\mathcal{N}_1 \cup \mathcal{N}_2$ looses the ω-categoricity.

Example 4. Let $\mathcal{N}_1 = \langle M; \leq\rangle$ be a countable densely ordered set, $\mathcal{N}_2 = \langle M; R\rangle$ be a structure with a unary predicate R. Obviously, both these structures are ω-categorical. Their fusion $\mathcal{N}_1 \cup \mathcal{N}_2 = \langle M; \leq, R\rangle$ can preserve or loose the ω-categoricity depending on $\leq$-ordering of elements in R. Indeed, if R is a finite union of $\leq$-convex sets the structure is again ω-categorical with densely ordered convex parts. And if R has an infinite $\leq$-discretely ordered part then $\mathcal{N}_1 \cup \mathcal{N}_2$ is not ω-categorical.

Notice that here $\mathcal{N}_1 \cup \mathcal{N}_2$ can be Ehrenfeucht, i.e. with finitely many ($>$ 1) pairwise non-isomorphic elementary equivalent countable structures, if R has finitely many (≥ 1) $\leq$-accumulation points in some saturated elementary extension of $\mathcal{N}_1 \cup \mathcal{N}_2$, and continuum many pairwise non-isomorphic elementary equivalent countable structures, if R has infinitely many $\leq$-accumulation points in some saturated elementary extension of $\mathcal{N}_1 \cup \mathcal{N}_2$.

In view of Examples 3 and 4 the family P does not form a lattice. Besides, it does not have maximal elements since each ω-categorical structure has an ω-categorical proper expansion by new signature singleton. Thus P is formed as a union of infinite increasing chains of lower cones.

Summarizing the arguments above we have the following:

Theorem 4. *The family $P_{\omega\text{-cat}} \subseteq B(\mathcal{M})$ of countably categorical structures is represented as the union of lower cones of all their elements and all these elements are not maximal. This family is closed under permutations and not closed under unions.*

Remark 9. In [16], a machinery described allowing to obtain Ehrenfeucht expansions of some given non-Ehrenfeucht theories, then loosing the Ehrenfeucht property by a suitable expansion, returning it, etc. Thus there is a sequence of expansions of structures in $\mathcal{B}(\mathcal{M})$, with countable $\mathcal{M}$, alternating the Ehrenfeucht property and its complement.

Remind that the Ehrenfeucht property can fail under inessential expansions and restrictions, i.e. under expansions and restrictions by constants [10]. It means that the Ehrenfeucht property is rather different in $\mathcal{B}(\mathcal{M})$ than the property of ω-categoricity.

Let $P = P_{\mathrm{Ehr}} \subseteq B(\mathcal{M})$ be the property of Ehrenfeuchtness, where M is countable. Clearly, P is closed under permutations of structures, and both the least and the greatest elements of $\mathcal{B}(\mathcal{M})$ are not Ehrenfeucht. Therefore P does not contain cones at all. At the same time the property P has atomic elements in the Boolean algebra $\mathcal{B}(\mathcal{M})$, i.e. Ehrenfeucht structures with exactly one signature symbol.

For instance, the classical Ehrenfeucht theory $T = \mathrm{Th}(\langle \mathbb{Q}; \leq, c_n\rangle_{i\in\omega})$, $c_n < c_{n+1}$, $n \in \omega$, [16], with exactly three countable models can be replaced by the theory T' of the binary structure

$$\langle (\mathbb{Q} \setminus \{c_n \mid n \in \omega\} \times \omega) \cup \{(c_n, k) \mid k < n, k, n \in \omega\};$$

$$\{((q, 0), (r, 0)) \mid q, r \in \mathbb{Q}, q \leq r\} \cup \{((q, 0), (q, k)) \mid q \in \mathbb{Q}, k \in \omega \setminus \{0\}\}\rangle.$$

Here finitely many arcs $((q, 0), (q, k))$, where $q = c_n$, code the position of the element c_n among other elements in the linearly ordered part $\{(q, 0) \mid q \in \mathbb{Q}\}$ and produce exactly three non-isomorphic possibilities for countable models of T'.

Summarizing the arguments above we have the following:

Theorem 5. *Any Boolean algebra $\mathcal{B}(\mathcal{M})$ with a countable universe M contains structures with the property P_{Ehr} without the least and the greatest elements of $\mathcal{B}(\mathcal{M})$. This property is closed under permutations and can fail under restrictions and expansions. There are infinite chains alternating the Ehrenfeuchtness and the complement of this property. There are atomic structures $\mathcal{N} \in B(\mathcal{M})$ belonging to P_{Ehr}.*

7. The family of strongly minimal structures in $\mathcal{B}(\mathcal{M})$

Recall [1] that a structure $\mathcal{N}$ is called *strongly minimal* if for any $\mathcal{N}' \equiv \mathcal{N}$ and any formula $\varphi(x, \overline{a})$ in the language of $\mathcal{N}$ with parameters $\overline{a} \in N'$ the set $\varphi(\mathcal{N}', \overline{a}) = \{b \mid \mathcal{N}' \models \varphi(b, \overline{a})\}$ is either finite or cofinite in N.

A theory T is called *strongly minimal* if $T = \mathrm{Th}(\mathcal{N})$ for a strongly minimal structure $\mathcal{N}$.

Let $P = P_{\mathrm{sm}} \subseteq B(\mathcal{M})$ be the property of strong minimality.

Remark 10. By the definition the property P is closed under permutations and intersections of structures, via intersections of signatures, and under naming of new elements by unary singletons. Moreover, unary predicates of finite cardinalities can be added preserving the regularity and the strong minimality, whereas some equivalence relations can be added with finite bounded equivalence classes only, for instance, with unique non-one-element class, since unbounded cardinalities imply infinite ones which is forbidden for strongly minimal structures. The latter condition implies that P does not have maximal elements in $\mathcal{B}(\mathcal{M})$ allowing to add new equivalence classes with $n + 1$ instead of the bound n.

Thus P forms a lower semilattice represented by the equality (6.1). At the same time P admits unions of chains of expansions, where these expansions are obtained by unary predicates with the finiteness condition and by equivalence relations with the bounded finiteness condition. Hence a natural question arises on the existence of a regular strongly minimal structure whose all regular expansions are not strongly minimal. Below we give an answer to this question.

Now we argue to show that any fusion of strongly minimal structures $\mathcal{N}_1, \mathcal{N}_2 \in B(\mathcal{M})$ is again strongly minimal.

Theorem 6. *If structures $\mathcal{N}_1, \mathcal{N}_2 \in B(\mathcal{M})$ are strongly minimal then $\mathcal{N}_1 \cup \mathcal{N}_2$ is strongly minimal, too.*

Proof. By the conjecture any formula $\varphi(x, \overline{a})$ of a theory $T_i = \mathrm{Th}(\mathcal{N}_i)$, $i \in \{1, 2\}$, with parameters $\overline{a} \in M$, has either finitely many or cofinitely many solutions. Now we take a formula $\psi(x, \overline{b})$ of the signature $\Sigma(\mathcal{N}_1 \cup \mathcal{N}_2)$, with parameters $\overline{b} \in M$. Assume that the set $Z = \psi(\mathcal{N}_1 \cup \mathcal{N}_2, \overline{b})$ is both infinite and co-infinite. Since Z can not be obtained by Boolean combinations of finite and cofinite sets, we may assume that $\psi(x, \overline{b})$ has the form $\exists y \chi(x, y, \overline{b})$ and this formula has the minimal length among formulae with that property. Therefore $\chi(x, y, \overline{b})$ witnesses the strong minimality with finite or cofinite sets of solutions for $\chi(x, c, \overline{b})$ and $\chi(c, y, \overline{b})$, $c \in M$. It implies that $\chi(x, c, \overline{b})$ has finitely many solutions for any $c \in M$, since otherwise, if some d produces cofinite $\chi(\mathcal{N}_1 \cup \mathcal{N}_2, d, \overline{b})$, then by

$$\psi(\mathcal{N}_1 \cup \mathcal{N}_2, \overline{b}) = \bigcup_{d' \in M} \chi(\mathcal{N}_1 \cup \mathcal{N}_2, d', \overline{b}) \supseteq \chi(\mathcal{N}_1 \cup \mathcal{N}_2, d, \overline{b})$$

the set $\psi(\mathcal{N}_1 \cup \mathcal{N}_2, \overline{b})$ is cofinite, too, contradicting the assumption of its co-infinity. Similarly for each $e \in M$ the set $\chi(e, \mathcal{N}_1 \cup \mathcal{N}_2, \overline{b})$ is finite. Then there are cofinitely many elements $d, e \in M$ with nonempty $\chi(\mathcal{N}_1 \cup \mathcal{N}_2, d, \overline{b})$ and $\chi(e, \mathcal{N}_1 \cup \mathcal{N}_2, \overline{b})$ producing the co-finiteness of $\psi(\mathcal{N}_1 \cup \mathcal{N}_2, \overline{b})$ which contradicts the assumption. $\square$

The arguments for the proof of Theorem 6 show that any finite family of regular strongly minimal structures has a strongly minimal theory. Since the property of strong minimality is reduced to formulae with special properties and reduced to the family of finite signatures, any family of regular strongly minimal structures $\mathcal{N}_i \in B(\mathcal{M})$, $i \in I$, has a strongly minimal union, too. In particular, there is the greatest strongly minimal structure $\mathcal{SM}$ in $B(\mathcal{M})$, where subsignatures of $\Sigma(\mathcal{SM})$ form strongly minimal restrictions of $\mathcal{SM}$, including the least element $\mathcal{N}_0$ of $\mathcal{B}(\mathcal{M})$ having the empty signature. Here, $P_{\rm sm} = \triangle(\mathcal{SM})$. Moreover, any restriction $\mathcal{N}$ of $\mathcal{SM}$ is strongly minimal, with strongly minimal co-structure of the signature $\Sigma(\mathcal{SM}) \setminus \Sigma(\mathcal{N})$. Thus, the family $B_{\rm sm}(\mathcal{M}) = P_{\rm sm}$ of all strongly minimal structures forms a Boolean algebra whose universe is a proper subset of $B(\mathcal{M})$ which is equal to the lower cone of $\mathcal{SM}$. Hence we have the following:

Theorem 7. *Any Boolean algebra $\mathcal{B}(\mathcal{M})$ with an infinite universe M contains a distributive sublattice $\mathcal{B}_{\rm sm}(\mathcal{M})$ of all strongly minimal structures $\mathcal{N} \in B(\mathcal{M})$. This sublattice closed under permutations and forms a Boolean algebra with the least element $\mathcal{N}_0$ and the greatest element $\mathcal{SM}$ forming $\triangle(\mathcal{SM})$ which is equal to $B_{\rm sm}(\mathcal{M})$.*

8. The family of ω_1-categorical structures in $\mathcal{B}(\mathcal{M})$

Let $P = P_{\omega_1\text{-cat}}$ be the family of infinite structures in $\mathcal{B}(\mathcal{M})$, with countable languages and which are ω_1-categorical, i.e. categorical in some, i.e. any uncountable cardinality. It is known [1; 3; 8] that a countable complete theory T without finite models is ω_1-categorical iff it contains a 1-cardinal formula $\varphi(\overline{x}, \overline{a})$ which is strongly minimal, i.e. both $|\varphi(\mathcal{N}, \overline{a})| = |\mathcal{N}|$ for any $\mathcal{N} \models T$ and each its definable subset, with parameters, of $\varphi(\mathcal{N}, \overline{a})$ is either finite or cofinite.

Clearly, the property P is closed under permutations, contains the least element of $\mathcal{B}(\mathcal{M})$ iff M is infinite, and can fail under expansions. Indeed, if $\mathcal{N} \in B(\mathcal{M})$ is an arbitrary ω_1-categorical structure having infinitely many realizations of a non-algebraic complete type, then its expansion dividing this set of realizations, by new unary predicate, into two infinite parts is already not ω_1-categorical.

The following example shows that restrictions of structures also can violate the ω_1-categoricity.

Example 5. Let Q and R be disjoint infinite unary predicates whose union forms the universe of a structure $\mathcal{N}$ of the signature $\langle Q, R, f\rangle$, where f is s bijection between Q and R. The formulae $Q(x)$ and $R(x)$ are 1-cardinal and strongly minimal, i.e. $\mathcal{N}$ is ω_1-categorical. At the same time restrictions of $\mathcal{N}$ till one or two unary predicates are already not ω_1-categorical since the formulae $Q(x)$ and $R(x)$ in these restrictions do not remain 1-cardinal under these restrictions.

Any expansion of $\mathcal{N}$ by an infinite and co-infinite unary predicate $S \subset Q$ is not ω_1-categorical.

Remark 11. Notice that the behavior of the property $P_{\omega_1\text{-cat}}$ with respect to its non-preservation can be realized by families of unary predicates and bijections responsible for the 1-cardinality and obtaining an infinite chain of expansions alternating the ω_1-categoricity and its negation. For instance, an expansion of the structure $\mathcal{N}$ by the countable and co-countable predicate S is not ω_1-categorical but any expansion of $\langle\mathcal{N}, S\rangle$ by a bijection g between S and $Q \setminus S$ restores the ω_1-categoricity. An appropriate new unary predicate again violates the ω_1-categoricity, a suitable new bijection restores it, etc.

Since the ω_1-categoricity excludes infinite linear orders there are countable structures $\mathcal{N}'$ of finite signatures with $\triangledown(\mathcal{N}') \cap P_{\omega_1\text{-cat}} = \emptyset$.

Summarizing the arguments above we obtain the following:

Theorem 8. *Any Boolean algebra $\mathcal{B}(\mathcal{M})$ with an infinite universe M contains structures with the property $P_{\omega_1\text{-cat}}$ including the least element of $\mathcal{B}(\mathcal{M})$. This property is closed under permutations and can fail under restrictions and expansions. There are infinite chains alternating the ω_1-categoricity and the complement of this property. There are structures $\mathcal{N} \in B(\mathcal{M})$ of finite signatures with $\triangledown_{\mathcal{N}} \cap P_{\omega_1\text{-cat}} = \emptyset$.*

9. The family of stable structures in $\mathcal{B}(\mathcal{M})$

Recall [12] that a formula $\varphi(\overline{x}, \overline{y})$ of a theory T is called *stable* if there are no tuples $\overline{a}_i, \overline{b_i} \in N$, where $i \in \omega$, $\mathcal{N} \models T$, such that

$$\mathcal{N} \models \varphi(\overline{a}_i, \overline{b}_j) \Leftrightarrow i \leq j.$$

The theory T is called *stable* if all its formulae are stable. Models of a stable theory are called *stable*, too. If a formula/theory/structure is not stable, it is called *unstable* or having the *order property*.

Following [12] it is said that a formula $\varphi(\overline{x}, \overline{y})$ has the *strict order property* if there are parameters $\overline{a}_i \in N$, $i \in \omega$, such that the sets $\varphi(\overline{a}_i, \mathcal{N})$, $i \in \omega$, form a strictly descending chain with $\varphi(\overline{a}_i, \mathcal{N}) \supsetneqq \varphi(\overline{a}_{i+1}, \mathcal{N})$, $i \in \omega$.

Again following [12] it is said that an unstable formula $\varphi(\overline{x},\overline{y})$ has the *independence property* if in every/some model $\mathcal{N}$ of T there is, for each $n\in\omega$, a family of tuples $\overline{a}_i$, $i\in n$, such that for each of the 2^n subsets X of n there is a tuple $\overline{b}\in N$ for which

$$\mathcal{N}\models\varphi(\overline{a}_i,\overline{b})\Leftrightarrow i\in X.$$

It is well known that a formula $\varphi(\overline{x},\overline{y})$ is unstable iff it has the strict order property or the independence property.

Let $P=P_{\rm st}\subseteq B(\mathcal{M})$ be the property of stability of a structure.

Remark 12. By the definition, similarly to the strongly minimal case, the property P is closed under intersections of structures, via intersections of signatures, and under marking of subsets of M by unary predicates.

Thus P forms a lower semilattice represented by the equality (6.1). At the same time P admits unions of chains of regular expansions by new unary predicates preserving the regularity of structures.

Remind [5] that Boolean combinations of stable formulae stay stable. So the violations of stability for fusions of stable structures can be obtained via quantifiers only. The following example illustrates this violation showing that indeed fusions of stable structures can fail the property of stability.

Example 6. Let $R_1=\{\langle a_i,c_{ik}\rangle\mid i,k\in\omega\}$, $R_2=\{\langle d_{lj},b_j\rangle\mid l,j\in\omega\}$ be binary relations with pairwise distinct elements a_i,b_i,c_{ik},d_{lj}. We identify some elements c_{ik} and d_{lj} as follows: $c_{ik}=d_{lj}\Leftrightarrow i\leq k=l\leq j$. The obtained structure $\mathcal{N}=\langle M;R_1,R_2\rangle$, with the universe consisting of the elements a_i,b_i,c_{ik},d_{lj} having the identifications above only, is unstable since the formula $\varphi(x,y)=\exists z(R_1(x,z)\wedge R_2(z,y))$ witnesses the strict order property by the sequences a_i,b_j, $i,j\in\omega$. At the same time the restrictions $\mathcal{N}_i=\mathcal{N}|_{R_i}$, $i=1,2$, are stable consisting of countably many pairwise isomorphic countable acyclic connected components of diameter 2 and countably many isolated elements. Here $\mathcal{N}=\mathcal{N}_1\cup\mathcal{N}_2$.

Besides this example again confirms that ω-categorical structures $\mathcal{N}_1,\mathcal{N}_2$ can have a fusion $\mathcal{N}_1\cup\mathcal{N}_2$ which is not ω-categorical.

Example 7. Example 6 can be easily modified to the random bipartite graph with the relation R defined by the same formula $\varphi(x,y)$ as follows. We enumerate the family Z of all pairs of finite disjoint sets $X,Y\subset\omega$: $\nu\colon\omega\to Z$. Now for each $i\in\omega$ and $\nu(i)=\langle X,Y\rangle$, we identify the elements c_{ik} and d_{lj} iff $k=l=j$ and $j\in X$. Thus after this identification we obtain the structure $\mathcal{N}=\langle M;R_1,R_2\rangle$ with $\varphi(a_i,\mathcal{N})=\{b_j\mid j\in X\}$. It implies that each finite subset of $\{b_j\mid j\in\omega\}$ is defined by a copy of the formula $\varphi(x,y)$, producing the random bipartite graph having the independence property.

The restrictions $\mathcal{N}_i = \mathcal{N}|_{R_i}$, $i = 1, 2$, are again stable consisting of countably many pairwise isomorphic countable acyclic connected components of diameter 2 and countably many isolated elements. Here again $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$.

Theorem 9. *The family $P_{\rm st} \subseteq B(\mathcal{M})$ of stable structures is represented as the union of lower cones of all their elements. This family is closed under permutations and not closed under unions, and these unions can produce both the strict order property and the independence property.*

10. Conclusion

We considered hierarchies of properties on a Boolean algebra of regular expansions and restrictions based on the universe of an arbitrary regular structure. Some general properties of these hierarchies are studied with respect to elementary theories of structures, lower and upper cones, lattices, permutations. A general approach is applied for several natural classes of structures and their theories including the ω-categoricity, Ehrenfeuchtness, strong minimality, ω_1-categoricity, stability. Properties of these classes in a Boolean algebra are described. It would be interesting to spread this approach describing properties of further natural classes in these Boolean algebras.

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Поступила в редакцию / Received 07.02.2025
Поступила после рецензирования / Revised 27.03.2025
Принята к публикации / Accepted 07.04.2025