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Rota-Baxter Operators of Weight Zero on Cayley-Dickson Algebra with Matrix Images

Alexander S. Panasenکو^{1,2✉}

¹ Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russian Federation

² Novosibirsk State University, Novosibirsk, Russian Federation

✉ a.panasenko@g.nsu.ru

Abstract: Rota-Baxter operators present a natural generalization of integration by parts formula for the integral operator. We consider Rota-Baxter operators of weight zero on split octonion algebra over a field of characteristic not 2. We classify all these operators under a condition of embeddability of their images in second order matrix algebra. With the additional condition of quadratic closure of the field, we obtain 9 operators. In addition, we refine the classification of Rota-Baxter operators on the second order matrix algebra by removing the restriction on the algebraic closure of the field. Classifications were obtained up to multiplication by a scalar, conjugation by automorphisms and antiautomorphisms. In particular, we constructed some automorphisms and antiautomorphisms of octonions.

Keywords: Cayley-Dickson algebra, Rota-Baxter operator, split octonions, automorphism, antiautomorphism

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Научная статья

Операторы Роты – Бакстера нулевого веса на алгебре Кэли – Диксона с матричным образом

А. С. Панасенко^{1,2✉},

¹ Институт математики им. С. Л. Соболева СО РАН, Новосибирск, Российская Федерация

² Новосибирский государственный университет, Новосибирск, Российская Федерация,

✉ a.panasenko@g.nsu.ru

Аннотация: Операторы Роты – Бакстера представляют собой естественное обобщение формулы интегрирования по частям для интегрального оператора. Рассматриваются операторы Роты – Бакстера нулевого веса на алгебре расщепляемых октонионов над полем характеристики, отличной от 2. Все эти операторы классифицируются при условии вложимости их образов в матричную алгебру второго порядка. С дополнительным условием квадратичного замыкания поля получены 9 операторов. Кроме того, уточняется классификация операторов Роты – Бакстера на матричной алгебре второго порядка без ограничения на алгебраическое замыкание поля. Получены классификации с точностью до умножения на скаляр, сопряжения автоморфизмами и антиавтоморфизмами. В частности, построены несколько явных конструкций автоморфизмов и антиавтоморфизмов алгебры октонионов.

Ключевые слова: алгебра Кэли – Диксона, оператор Роты – Бакстера, расщепляемые октонионы, автоморфизм, антиавтоморфизм

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1. Introduction

The Rota-Baxter operator is a formal generalization of the integration by parts formula [3]. In the work [18] Rota-Baxter operators of nonzero weight appeared independently as solutions of the modified Yang-Baxter equation. At present, applications of Rota-Baxter operators to various areas of algebra are known. In [1] and [2] a connection between Rota-Baxter operators and pre-Lie algebras has been researched. In [8] a connection between Rota-Baxter operators and double Lie algebras has been researched.

We are interested in the problem of classifying Rota-Baxter operators on various algebras, especially simple finite-dimensional ones. Descriptions of Rota-Baxter operators of weight zero are known on $M_2(F)$ over an algebraically closed field [4], the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ [14; 16], the simple Jordan superalgebra D_t over an algebraically closed field of characteristic 0 [5], K_3 [4].

Descriptions of Rota-Baxter operators of nonzero weight are known on a simple Jordan algebra of a bilinear form of odd dimension, $M_2(F)$, K_3 (all in [4]), $M_3(\mathbb{C})$ [7;9;12]. Let us notice that classifications of RB-operators of zero weight and RB-operators of non-zero weight typically use significantly different techniques. For example, every decomposition of an algebra into a direct sum of two subalgebras induces a Rota-Baxter operator of non-zero weight.

Composition algebras arose within the framework of a generalized formulation of the Hurwitz problem [20]. They can be of two types: split and division algebras. In the paper [4] it is proved that all Rota-Baxter operators on a division quadratic algebra are trivial. Each composition algebra is quadratic. Over a fixed field F of characteristic not 2 there exist only three split composition algebras: the direct sum of two fields F , the matrix algebra $M_2(F)$, and the split octonions over F . In the article [2] Rota-Baxter operators on the direct sum of two fields were described (later, the description was generalized to the direct sum of a finite number of fields, [11]). In [4], Rota-Baxter operators on the algebra of second-order matrices over an algebraically closed field were described. Thus, the problem of classification of Rota-Baxter operators on composition algebras was reduced to the problem of describing Rota-Baxter operators on split octonions.

Octonions are one of the most famous and well-studied non-associative algebraic systems. Octonions have many mathematical ([19]) and physical ([6]) applications. A brief introduction to octonions can be found in ([20], Chapter 2).

In the recent paper [13], all subalgebras of octonions were described up to automorphism. This description forms the basis for our work. All classifications in this paper are made up to automorphisms and antiautomorphisms of split octonions.

In this paper we complete the classification of Rota-Baxter operators of weight zero on split octonions over a field of characteristic not 2 with a condition that images of these operators can be embedded in $M_2(F)$. The paper is organized as follows. In Section 2 we define several automorphisms and antiautomorphisms on split octonions. In addition, we refine the description of RB-operators on the algebra of second-order matrices from [4], extending the restriction to any field of characteristic not 2. In Section 3 we describe RB-operators of weight zero on split octonions with one-dimensional image. In Section 4 we describe RB-operators of weight zero on split octonions with two-dimensional idempotent image. In Section 5 we collect the auxiliary results of the previous sections into a final result. We formulate two versions of the result: for an arbitrary field of characteristic not 2 and for a quadratically closed field of characteristic not 2.

The results of this work were previously presented in a preprint [15].

2. Preliminaries

In this article we fix a field F with characteristic $\neq 2$.

An antiautomorphism $\varphi : A \rightarrow A$ of an algebra A is called an **involution** if $\varphi^2 = \text{id}|_A$.

Let $\mathbb{O} = M_2(F) + vM_2(F)$ be the split Cayley-Dickson algebra. It has the following multiplication table:

$$a \cdot b = ab, \quad a \cdot vb = v(\bar{a}b), \quad va \cdot b = v(ba), \quad va \cdot vb = b\bar{a},$$

where $x \cdot y$ is a multiplication in $\mathbb{O}$ for any $x, y \in \mathbb{O}$, ab is a multiplication in $M_2(F)$ for any $a, b \in M_2(F)$, $\bar{a}$ is an image of an element a by a symplectic involution in $M_2(F)$, i.e.

$$\overline{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

An involution $x \rightarrow \bar{x}$, $a + vb \rightarrow \bar{a} - vb$ in $\mathbb{O}$ is called **standard involution**.

An algebra $\mathbb{O}$ has the following basis

$$e_{11}, e_{12}, e_{21}, e_{22}, ve_{11}, ve_{12}, ve_{21}, ve_{22}.$$

If $\varphi : M_2(F) \rightarrow M_2(F)$ is (anti)automorphism then it can be extended to (anti)automorphism $\bar{\varphi} : \mathbb{O} \rightarrow \mathbb{O}$ by ([13], Lemma 4.1). The construction is as follows. If $\varphi : M_2(F) \rightarrow M_2(F)$ is an automorphism on $M_2(F)$ then $\bar{\varphi}(a + bv) = \varphi(a) + \varphi(b)v$ for $a, b \in M_2(F)$. If $\varphi : M_2(F) \rightarrow M_2(F)$ is an antiautomorphism on $M_2(F)$ then a map, $\bar{\varphi}(a + bv) = \varphi(a) + \overline{\varphi(b)}v$ for $a, b \in M_2(F)$, where $y = \overline{\varphi(b)}$ is an image of $\varphi(b)$ under an action of symplectic involution and $\overline{yv}$ is an image of yv under an action of standard involution. We will use this fact without further mentions.

If B is a subalgebra in $\mathbb{O}$ then we will call B a **unital** subalgebra if B contains a unit of $\mathbb{O}$.

The proofs of the following examples are straightforward.

Example 1. Let $\varphi : \mathbb{O} \rightarrow \mathbb{O}$ be a linear map such that $\varphi^2 = \text{id}$, $\varphi(ve_{12}) = -ve_{21}$, $\varphi(ve_{11}) = ve_{22}$, $\varphi(e_{11}) = e_{11}$, $\varphi(e_{12}) = e_{21}$, $\varphi(e_{22}) = e_{22}$. Then φ is an involution on the algebra $\mathbb{O}$.

Example 2. Let $\varphi : \mathbb{O} \rightarrow \mathbb{O}$ be a linear map such that $\varphi(e_{ij}) = e_{ij}$ for any $i, j \in \{1, 2\}$, $\varphi(ve_{12}) = ve_{12}$, $\varphi(ve_{22}) = ve_{22}$, $\varphi(ve_{11}) = ve_{11} + \alpha ve_{12}$, $\varphi(ve_{21}) = ve_{21} + \alpha ve_{22}$ for some $\alpha \in F$. Then φ is an automorphism on the algebra $\mathbb{O}$.

Example 3. Let $\varphi : \mathbb{O} \rightarrow \mathbb{O}$ be a linear map such that $\varphi(e_{ij}) = e_{ij}$ for any $i, j \in \{1, 2\}$, $\varphi(ve_{11}) = ve_{11}$, $\varphi(ve_{21}) = ve_{21}$, $\varphi(ve_{12}) = ve_{12} + \alpha ve_{11}$, $\varphi(ve_{22}) = ve_{22} + \alpha ve_{21}$ for some $\alpha \in F$. Then φ is an automorphism on the algebra $\mathbb{O}$.

Example 4. Let $\varphi : \mathbb{O} \rightarrow \mathbb{O}$ be a linear map such that $\varphi(e_{ij}) = e_{ij}$ for any $i, j \in \{1, 2\}$, $\varphi(ve_{12}) = -ve_{11}$, $\varphi(ve_{11}) = ve_{12}$, $\varphi(ve_{21}) = ve_{22}$, $\varphi(ve_{22}) = -ve_{21}$. Then φ is an automorphism on the algebra $\mathbb{O}$.

Example 5. Let $\varphi : \mathbb{O} \rightarrow \mathbb{O}$ be a linear map such that $\varphi(e_{11}) = e_{22}$, $\varphi(e_{12}) = ve_{22}$, $\varphi(e_{21}) = ve_{11}$, $\varphi(e_{22}) = e_{11}$ and $\varphi^2 = \text{id}$. Then φ is an involution on the algebra $\mathbb{O}$.

Example 6. Let $\varphi : \mathbb{O} \rightarrow \mathbb{O}$ be a linear map such that $\varphi(e_{ii}) = e_{ii}$, $\varphi(ve_{ii}) = ve_{ii}$ for any $i \in \{1, 2\}$, $\varphi(e_{12}) = \alpha e_{12}$, $\varphi(ve_{12}) = \alpha ve_{12}$, $\varphi(e_{21}) = \frac{1}{\alpha} e_{21}$, $\varphi(ve_{21}) = \frac{1}{\alpha} ve_{21}$ for some $0 \neq \alpha \in F$. Then φ is an automorphism on the algebra $\mathbb{O}$.

Example 7. Let $\varphi : \mathbb{O} \rightarrow \mathbb{O}$ be a linear map such that $\varphi(e_{ii}) = e_{ii}$, for any $i \in \{1, 2\}$, $\varphi(e_{12}) = \alpha e_{12}$, $\varphi(ve_{11}) = \alpha ve_{11}$, $\varphi(e_{21}) = \frac{1}{\alpha} e_{21}$, $\varphi(ve_{22}) = \frac{1}{\alpha} ve_{22}$, $\varphi(ve_{12}) = ve_{12}$, $\varphi(ve_{21}) = ve_{21}$ for some $0 \neq \alpha \in F$. Then φ is an automorphism on the algebra $\mathbb{O}$.

Example 8. Let $\varphi : \mathbb{O} \rightarrow \mathbb{O}$ be a linear map such that $\varphi(e_{ij}) = e_{ij}$, for any $i, j \in \{1, 2\}$, $\varphi(ve_{11}) = \alpha ve_{11}$, $\varphi(ve_{21}) = \alpha ve_{21}$, $\varphi(ve_{22}) = \frac{1}{\alpha} ve_{22}$, $\varphi(ve_{12}) = \frac{1}{\alpha} ve_{12}$ for some $0 \neq \alpha \in F$. Then φ is an automorphism on the algebra $\mathbb{O}$.

Example 9. Let $\varphi : \mathbb{O} \rightarrow \mathbb{O}$ be a linear map such that $\varphi(e_{ii}) = e_{ii}$, $\varphi(e_{12}) = e_{12} - \alpha ve_{22}$, $\varphi(e_{21}) = e_{21}$, $\varphi(e_{22}) = e_{22}$, $\varphi(ve_{11}) = ve_{11} + \alpha e_{21}$, $\varphi(ve_{12}) = ve_{12}$, $\varphi(ve_{21}) = ve_{21}$, $\varphi(ve_{22}) = ve_{22}$ for some $\alpha \in F$. Then φ is an automorphism on the algebra $\mathbb{O}$.

Example 10. Let $\varphi : \mathbb{O} \rightarrow \mathbb{O}$ be a linear map such that $\varphi(e_{ii}) = e_{ii}$, $\varphi(e_{12}) = e_{12}$, $\varphi(e_{21}) = e_{21} + \alpha ve_{11}$, $\varphi(e_{22}) = e_{22}$, $\varphi(ve_{11}) = ve_{11}$, $\varphi(ve_{12}) = ve_{12}$, $\varphi(ve_{21}) = ve_{21}$, $\varphi(ve_{22}) = ve_{22} - \alpha e_{12}$ for some $\alpha \in F$. Then φ is an automorphism on the algebra $\mathbb{O}$.

Example 11. Let $\varphi : \mathbb{O} \rightarrow \mathbb{O}$ be a linear map such that $\varphi(e_{11}) = e_{11} - \alpha ve_{12}$, $\varphi(e_{12}) = e_{12}$, $\varphi(e_{21}) = e_{21} + \alpha ve_{22}$, $\varphi(e_{22}) = e_{22} + \alpha ve_{12}$, $R(ve_{11}) = ve_{11} - \alpha e_{12}$, $R(ve_{12}) = ve_{12}$, $\varphi(ve_{21}) = ve_{21} - \alpha e_{11} + \alpha e_{22} + \alpha^2 ve_{12}$, $\varphi(ve_{22}) = ve_{22}$ for some $\alpha \in F$. Then φ is an automorphism on the algebra $\mathbb{O}$.

Example 12. Let $\varphi : \mathbb{O} \rightarrow \mathbb{O}$ be a linear map such that $\varphi(e_{11}) = e_{11} + \alpha e_{12}$, $\varphi(e_{12}) = e_{12}$, $\varphi(e_{21}) = e_{21} - \alpha e_{11} + \alpha e_{22} - \alpha^2 e_{12}$, $\varphi(e_{22}) = e_{22} - \alpha e_{12}$, $R(ve_{11}) = ve_{11}$, $R(ve_{12}) = ve_{12}$, $\varphi(ve_{21}) = ve_{21} - \alpha ve_{11}$, $\varphi(ve_{22}) = ve_{22} - \alpha ve_{12}$ for some $\alpha \in F$. Then φ is an automorphism on the algebra $\mathbb{O}$.

When we want to prove that φ is an automorphism, we need to check that $\varphi(xy) = \varphi(x)\varphi(y)$ for all pairs of basis elements x, y . Although the calculations can easily be carried out in a computer algebra system, let us demonstrate this method with Example 2. Since the subspace $B = M_2(F) + Fve_{12} + Fve_{22}$ is a subalgebra and $\varphi|_B = \text{id}$, it suffices to check the

equality $\varphi(xy) = \varphi(x)\varphi(y)$ provided that x or y lies in $\{ve_{11}, ve_{21}\}$. For example, let $x = ve_{k1}$. If $y \in M_2(F)$, then $\varphi(ve_{k1})\varphi(e_{ij}) = (ve_{k1} + \alpha ve_{k2})e_{ij} = \delta_{j,k}(ve_{i1} + \alpha ve_{i2})$. At the same time, $\varphi(ve_{k1}e_{ij}) = \delta_{j,k}\varphi(ve_{i1}) = \delta_{j,k}(ve_{i1} + \alpha ve_{i2})$. If $y \in vM_2(F)$, then $\varphi(ve_{k1})\varphi(ve_{ij}) = (ve_{k1} + \alpha ve_{k2})(ve_{ij} + \delta_{j,1}\alpha ve_{i2})$. Note that for $j = 1$ this expression is zero, so it equals to $\delta_{j,2}(ve_{k1} + \alpha ve_{k2})ve_{i2} = \delta_{j,2}ve_{k1}ve_{i2}$. At the same time, $\varphi(ve_{k1}ve_{ij}) = \delta_{j,2}\varphi(\delta_{k,1}e_{i2} - \delta_{k,2}e_{i1}) = \delta_{j,2}(\delta_{k,1}e_{i2} - \delta_{k,2}e_{i1}) = \delta_{j,2}ve_{k1}ve_{i2}$. Thus, $\varphi(xy) = \varphi(x)\varphi(y)$.

If A is an algebra over F and $R : A \rightarrow A$ is a linear (over F) map, then R is called **Rota-Baxter operator (RB-operator)** of a weight λ if for any $x, y \in A$ we have the following identity

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy).$$

In this article we are interested only in RB-operators of zero weight, $\lambda = 0$.

An image of R is a subalgebra B of an algebra A and $\text{Ker}(R)$ is a $\text{Im}(R)$ -bimodule.

Everywhere below for a linear operator R on an algebra A we will use a notation $G_R(x, y) = R(x)R(y) - R(R(x)y + xR(y))$ for any $x, y \in A$. We will omit the index and write $G(x, y)$ if we have some operator, denoted by R .

In [4] RB-operators on the algebra $M_2(F)$ for algebraically closed F were described. We will need this description for any field with characteristic $\neq 2$.

Proposition 1. *Let $R : M_2(F) \rightarrow M_2(F)$ be a Rota-Baxter operator on $M_2(F)$ of weight zero. Then, up to conjugation by automorphism, anti-automorphism and up to multiplication by a scalar, R acts in one of the following ways:*

- 1) $R(e_{21}) = e_{11}, \quad R(e_{11}) = R(e_{12}) = R(e_{22}) = 0;$
- 2) $R(e_{21}) = e_{12}, \quad R(e_{11}) = R(e_{12}) = R(e_{22}) = 0;$
- 3) $R(e_{21}) = e_{11}, \quad R(e_{22}) = e_{12}, \quad R(e_{11}) = R(e_{12}) = 0;$
- 4) $R(e_{21}) = -e_{11}, \quad R(e_{11}) = e_{12}, \quad R(e_{12}) = R(e_{22}) = 0.$

Proof. In ([10], Lemma 1), it is shown that on a simple unital finite-dimensional non-one-dimensional algebra the kernel of any Rota-Baxter operator of weight 0 has dimension at least two, moreover, the unit does not lie in the image of this operator. Thus, by the theorem on the dimension of the kernel and the image of the linear map, $\dim(\text{Im}(R)) \leq 2$, and $\text{Im}(R)$ does not contain the identity matrix.

The results of paper [13] imply that any one-dimensional non-unital subalgebra in $\mathbb{O}$ has the form Fe_{11} or Fe_{12} up to automorphism and antiautomorphism. Thus, subalgebras in $M_2(F)$ also have the form Fe_{11} or Fe_{12} up to automorphism and antiautomorphism. The results of paper [13] imply that, up to automorphism and antiautomorphism, any two-dimensional non-unital subalgebra in $\mathbb{O}$ either has the form $Fe_{11} + Fe_{12}$ or has zero multiplication. However, $M_2(F)$ does not have two-dimensional subalgebras with zero multiplication. Thus, the only two-dimensional subalgebra in $M_2(F)$ (up to automorphism and antiautomorphism) has the form $Fe_{11} + Fe_{12}$.

Since $\text{Im}(R)$ is a subalgebra, the following cases are possible.

1) $\text{Im}(R) = Fe_{11}$. Let $R(e_{ij}) = \alpha_{ij}e_{11}$ for any i, j . Then $G(e_{11}, e_{11}) = -\alpha_{11}^2 e_{11}$, whence $\alpha_{11} = 0$. Further, $G(e_{22}, e_{22}) = -\alpha_{22}^2 e_{11}$, whence $\alpha_{22} = 0$. Further $G(e_{12}, e_{21}) = -\alpha_{12}\alpha_{21}e_{11}$, whence $\alpha_{12}\alpha_{21} = 0$. Up to an antiautomorphism (transpose), we can assume that $\alpha_{12} = 0$. Up to multiplication by a scalar, we can assume that $R(e_{21}) = e_{11}$.

2) $\text{Im}(R) = Fe_{12}$. Let $R(e_{ij}) = \alpha_{ij}e_{12}$ for any i, j . Then $G(e_{12}, e_{22}) = -\alpha_{12}^2 e_{12}$, whence $\alpha_{12} = 0$. Further, $G(e_{11}, e_{21}) = -\alpha_{11}^2 e_{12}$, whence $\alpha_{11} = 0$. Further, $G(e_{21}, e_{22}) = -\alpha_{22}^2 e_{12}$, whence $\alpha_{22} = 0$.

Up to multiplication by a scalar, we can assume that $R(e_{21}) = e_{12}$.

3) $\text{Im}(R) = Fe_{11} + Fe_{12}$. Let $x = \alpha e_{11} + \beta e_{12} + \gamma e_{21} + \delta e_{22} \in \text{Ker}(R)$. Since $e_{11}, e_{12} \in \text{Im}(R)$ and $\text{Ker}(R)$ is a $\text{Im}(R)$ -bimodule, then $e_{11}x = \alpha e_{11} + \beta e_{12} \in \text{Ker}(R)$, whence $e_{11}xe_{11} = \alpha e_{11} \in \text{Ker}(R)$, so $e_{11}x - e_{11}xe_{11} = \beta e_{12} \in \text{Ker}(R)$. Similarly $\alpha e_{11} + \gamma e_{21} \in \text{Ker}(R)$ and $\gamma e_{21} \in \text{Ker}(R)$, whence $\delta e_{22} \in \text{Ker}(R)$. Thus, if $\alpha e_{11} + \beta e_{12} + \gamma e_{21} + \delta e_{22} \in \text{Ker}(R)$, then $\alpha e_{11}, \beta e_{12}, \gamma e_{21}, \delta e_{22} \in \text{Ker}(R)$.

Suppose that $e_{11} \in \text{Ker}(R)$. Then $e_{12} = e_{11} \cdot e_{12} \in \text{Ker}(R)$ and $\text{Ker}(R) = Fe_{11} + Fe_{12}$. Let $R(e_{22}) = \xi_4 e_{11} + \eta_4 e_{12}$, $R(e_{21}) = \xi_3 e_{11} + \eta_3 e_{12}$. Then $G(e_{22}, e_{22}) = \xi_4^2 e_{11} + \xi_4 \eta_4 e_{12}$, whence $\xi_4 = 0$. Further, we have $G(e_{21}, e_{21}) = -\eta_3 \eta_4 e_{12}$, whence $\eta_3 \eta_4 = 0$. Finally, $G(e_{21}, e_{22}) = (\xi_3 \eta_4 - \eta_4^2) e_{12}$, whence $\eta_4(\eta_4 - \xi_3) = 0$. Since $\dim(\text{Im}(R)) = 2$, then $\eta_4 \neq 0$ and, up to multiplication by a scalar, $R(e_{21}) = e_{11}$, $R(e_{22}) = e_{12}$.

Suppose that $e_{11} \notin \text{Ker}(R)$. Then $e_{21} \notin \text{Ker}(R)$ (otherwise $e_{11} = e_{12} \cdot e_{21} \in \text{Ker}(R)$). Above we prove that if $\alpha e_{11} + \beta e_{12} + \gamma e_{21} + \delta e_{22} \in \text{Ker}(R)$, then $\alpha e_{11}, \gamma e_{21} \in \text{Ker}(R)$. It means that $\alpha = \gamma = 0$ and $\text{Ker}(R) = Fe_{12} + Fe_{22}$. Let $R(e_{11}) = \xi_1 e_{11} + \eta_1 e_{12}$, $R(e_{21}) = \xi_3 e_{11} + \eta_3 e_{12}$. Then $G(e_{11}, e_{11}) = -\xi_1^2 e_{11} - \xi_1 \eta_1 e_{12}$, whence $\xi_1 = 0$. Further, $G(e_{21}, e_{21}) = -\eta_3 \eta_1 e_{12}$, whence $\eta_3 \eta_1 = 0$. Since $\dim(\text{Im}(R)) = 2$, then $\eta_1 \neq 0$ and $\eta_3 = 0$. Finally, $G(e_{11}, e_{21}) = -(\eta_1 + \xi_3) \eta_1 e_{12}$. Since $\eta_1 \neq 0$, then, up to multiplication by a scalar, $R(e_{11}) = e_{12}$, $R(e_{21}) = -e_{11}$. The proposition is proven.

3. RB-Operators with one-dimensional image

In [13] it was proved that there are only two one-dimensional non-unital subalgebras B in $\mathbb{O}$, up to action of automorphism: nilpotent Fe_{12} and idempotent Fe_{11} . Let us describe the Rota-Baxter operators of zero weight on $\mathbb{O}$ with these images.

Lemma 1. *Let R be a Rota-Baxter operator of zero weight on the split Cayley-Dickson algebra $\mathbb{O}$ and $\text{Im}(R) = Fe_{12}$. Then, up to conjugation by automorphism, antiautomorphism and up to multiplication by a scalar, an operator R acts in one of the following ways (an operator R is zero on unspecified basic elements e_{ij} , ve_{ij}):*

$$1) R(e_{21}) = e_{12};$$

$$2) R(ve_{22}) = e_{12}.$$

Proof. Since $\text{Im}(R) \subset M_2(F)$, then $R|_{M_2(F)}$ is the Rota-Baxter operator on the subalgebra $M_2(F)$. According to the Proposition 1, we can assume that $R(e_{11}) = R(e_{12}) = R(e_{22}) = 0$.

Note that $(\text{Im}(R))^2 = 0$, so $e_{12}R(y) = R(x)e_{12} = R(x)R(y) = 0$ for any $x, y \in \mathbb{O}$. Let $R(ve_{12}) = \alpha e_{12}$, then $G(ve_{22}, ve_{12}) = -\alpha^2 e_{12}$, whence $\alpha = 0$ and $R(ve_{12}) = 0$. Let $R(ve_{11}) = \beta e_{12}$, then $G(ve_{11}, ve_{21}) = \beta^2 e_{12}$, whence $\beta = 0$ and $R(ve_{11}) = 0$. Let $R(ve_{21}) = \alpha_1 e_{12}$ and $R(ve_{22}) = \alpha_2 e_{12}$.

If $\alpha_2 \neq 0$, then there exists $\varepsilon \in F$ such that $ve_{21} + \varepsilon ve_{22} \in \text{Ker}(R)$. According to the Example 2, we can assume that $\varepsilon = 0$, that is, $\alpha_1 = 0$. Thus, we can assume that either $\alpha_2 = 0$ or $\alpha_1 = 0$. According to the Example 4, we can assume that $\alpha_1 = 0$. Thus, $R(ve_{22}) = \alpha_2 e_{12}$ and $R(e_{21}) = \alpha_3 e_{12}$.

Let $\alpha_2 = 0$. Then we can assume that $R(e_{21}) = e_{12}$.

Let $\alpha_3 = 0$. Then we can assume that $R(ve_{22}) = e_{12}$.

Let $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$. Up to multiplication by $\frac{1}{\alpha_2}$, we can assume that $R(ve_{22}) = e_{12}$, $R(e_{21}) = \alpha_3 e_{12}$. Conjugation by automorphism from Example 6 with $\alpha = \frac{1}{\alpha_3}$ gives us $R(ve_{22}) = \frac{1}{\alpha_3} e_{12}$, $R(e_{21}) = \frac{1}{\alpha_3} e_{12}$, which after multiplication by α_3 gives $R(ve_{22}) = e_{12}$, $R(e_{21}) = e_{12}$. Example 11 with a scalar $\alpha = 1$ allows us to assume that $R(e_{21}) = 0$ and $R(ve_{22}) = e_{12}$. The lemma is proven.

Lemma 2. *Let R be a Rota-Baxter operator of zero weight on the split Cayley-Dickson algebra $\mathbb{O}$ and $\text{Im}(R) = Fe_{11}$. Then, up to conjugation by automorphism, antiautomorphism and up to multiplication by a scalar, an operator R acts in the following way (an operator R is zero on unspecified basic elements e_{ij} , ve_{ij}):*

$$R(e_{21}) = e_{11}.$$

Proof. Since $\text{Im}(R) \subset M_2(F)$, then R is the Rota-Baxter operator on the subalgebra $M_2(F)$. According to the Proposition 1, we can assume that $R(e_{11}) = R(e_{12}) = R(e_{22}) = 0$. Let us introduce the notation $R(ve_{ij}) = \beta_{ij}e_{11}$. Then $G(ve_{12}, ve_{22}) = -\beta_{12}\beta_{22}e_{11}$, whence $\beta_{12}\beta_{22} = 0$. Further, $G(ve_{21}, ve_{12}) = \beta_{12}\beta_{21}e_{11}$, whence $\beta_{12}\beta_{21} = 0$. Further, $G(ve_{22}, ve_{11}) = \beta_{11}\beta_{22}e_{11}$, whence $\beta_{11}\beta_{22} = 0$. Further, $G(ve_{11}, ve_{21}) = -\beta_{11}\beta_{21}e_{11}$, whence $\beta_{11}\beta_{21} = 0$. Let $R(e_{21}) = \alpha e_{11}$. Then for any $j \in \{1, 2\}$ we have $G(ve_{2j}, e_{21}) = \alpha\beta_{2j}e_{11}$, whence $\alpha\beta_{21} = \alpha\beta_{22} = 0$. Thus, either $R(e_{21}) = R(ve_{11}) = R(ve_{12}) = 0$ or $R(ve_{21}) = R(ve_{22}) = 0$. The involution from Example 1 allows us to assume that $R(ve_{21}) = R(ve_{22}) = 0$.

If $\beta_{12} \neq 0$, then there exists $\varepsilon_1 \in F$ such that $ve_{11} + \varepsilon_1 ve_{12} \in \text{Ker}(R)$. By Example 2 we can assume that $\varepsilon_1 = 0$, so $ve_{11} \in \text{Ker}(R)$. Thus, in any case, either $\beta_{11} = 0$ or $\beta_{12} = 0$. By Example 4 we can assume that $\beta_{12} = 0$. Conjugation by automorphism from Example 6, as above, allows us to assume that either $R(e_{21}) = e_{11}$ and $R(ve_{11}) = e_{11}$, or $R(ve_{11}) = 0$ and $R(e_{21}) = e_{11}$, or $R(e_{21}) = 0$ and $R(ve_{11}) = e_{11}$. The composition of the standard involution and the involution from Example 5 allows us to consider that the second and third cases are equivalent. The Example 9 with a scalar 1 states that the first and the second cases are equivalent. The lemma is proven.

4. RB-Operators with two-dimensional image

In [13] it was proved that there are only two two-dimensional non-unital subalgebras B in $\mathbb{O}$, up to action of automorphism and antiautomorphism: idempotent $Fe_{11} + Fe_{12}$ and nilpotent $Fve_{12} + Fve_{22}$. It is easy to see that $M_2(F)$ does not contain two-dimensional nilpotent subalgebra. Let us describe the Rota-Baxter operators of zero weight on $\mathbb{O}$ with idempotent image.

Lemma 3. *Let R be a Rota-Baxter operator of zero weight on the split Cayley-Dickson algebra $\mathbb{O}$ and $\text{Im}(R) = Fe_{11} + Fe_{12}$. Then, up to conjugation by automorphism, antiautomorphism and up to multiplication by a scalar, an operator R acts in one of the following ways for some $\alpha \in F$ (an operator R is zero on unspecified basic elements e_{ij}, ve_{ij}):*

- 1) $R(e_{21}) = e_{11}, R(e_{22}) = e_{12};$
- 2) $R(e_{21}) = -e_{11}, R(e_{11}) = e_{12};$
- 3) $R(e_{21}) = e_{11}, R(ve_{21}) = e_{12};$
- 4) $R(ve_{11}) = \alpha e_{11}, R(ve_{21}) = e_{12}, \alpha \neq 0;$
- 5) $R(ve_{11}) = e_{12}, R(ve_{21}) = e_{11};$

6) $R(ve_{21}) = \alpha e_{11}, R(ve_{22}) = e_{12}, \alpha \neq 0$.

Proof. According to the Proposition 1, the following cases are possible.

1) $R(e_{21}) = e_{11}, R(e_{22}) = e_{12}, R(e_{11}) = R(e_{12}) = 0$.

Let $R(ve_{11}) = \alpha_1 e_{11} + \beta_1 e_{12}$. Then $G(ve_{11}, e_{21}) = -\beta_1 e_{12}$, whence $\beta_1 = 0$. Further, $G(ve_{11}, e_{22}) = \alpha_1 e_{12}$, whence $\alpha_1 = 0$. Therefore, $R(ve_{11}) = 0$.

Let $R(ve_{12}) = \alpha_2 e_{11} + \beta_2 e_{12}$. Then $G(ve_{12}, e_{21}) = -\beta_2 e_{12}$, whence $\beta_2 = 0$. Further, $G(ve_{12}, e_{22}) = \alpha_2 e_{12}$, whence $\alpha_2 = 0$. Therefore, $R(ve_{12}) = 0$.

Let $R(ve_{21}) = \alpha_3 e_{11} + \beta_3 e_{12}$. Then $G(ve_{21}, e_{21}) = \alpha_3 e_{11}$, whence $\alpha_3 = 0$. Further, $G(e_{21}, ve_{21}) = -\beta_3 e_{12}$, whence $\beta_3 = 0$. Therefore, $R(ve_{21}) = 0$.

Let $R(ve_{22}) = \alpha_4 e_{11} + \beta_4 e_{12}$. Then $G(ve_{22}, e_{21}) = \alpha_4 e_{11}$, whence $\alpha_4 = 0$. Further, $G(e_{21}, ve_{22}) = -\beta_4 e_{12}$, whence $\beta_4 = 0$. Therefore, $R(ve_{22}) = 0$. We have obtained case (1) from the statement of the lemma.

2) $R(e_{21}) = -e_{11}, R(e_{11}) = e_{12}, R(e_{12}) = R(e_{22}) = 0$.

Let $R(ve_{11}) = \alpha_1 e_{11} + \beta_1 e_{12}$. Then $G(e_{11}, ve_{11}) = -\alpha_1 e_{12}$, whence $\alpha_1 = 0$. Further, $G(e_{21}, ve_{11}) = -\beta_1 e_{12}$, whence $\beta_1 = 0$. Therefore, $R(ve_{11}) = 0$.

Let $R(ve_{22}) = \alpha_4 e_{11} + \beta_4 e_{12}$. Then $G(e_{21}, ve_{22}) = \alpha_4 e_{11}$, whence $\alpha_4 = 0$. Further, $G(ve_{22}, e_{21}) = -\beta_4 e_{12}$, whence $\beta_4 = 0$. Therefore, $R(ve_{22}) = 0$.

Let $R(ve_{12}) = \alpha_2 e_{11} + \beta_2 e_{12}$. Then $G(e_{11}, ve_{12}) = -\alpha_2 e_{12}$, whence $\alpha_2 = 0$. Further, $G(e_{11}, ve_{22}) = \beta_2 e_{12}$, whence $\beta_2 = 0$. Therefore, $R(ve_{12}) = 0$.

Let $R(ve_{21}) = \alpha_3 e_{11} + \beta_3 e_{12}$. Then $G(e_{11}, ve_{21}) = -\alpha_3 e_{12}$, whence $\alpha_3 = 0$. Further, $G(ve_{21}, e_{21}) = -\beta_3 e_{12}$, whence $\beta_3 = 0$. Therefore, $R(ve_{21}) = 0$. We have obtained case (2) from the statement of the lemma.

In the remaining three cases, the dimension of $R(M_2(F))$ does not exceed one. Then the dimension of $R(vM_2(F))$ is not less than one. Let $0 \neq x = \alpha ve_{11} + \beta ve_{12} + \gamma ve_{21} + \delta ve_{22} \in \text{Ker}(R)$. Since $e_{11}, e_{12} \in \text{Im}(R)$ and $\text{Ker}(R)$ is an $\text{Im}(R)$ -bimodule, then $e_{11}x = \gamma ve_{21} + \delta ve_{22} \in \text{Ker}(R)$, whence $\alpha ve_{11} + \beta ve_{12} \in \text{Ker}(R)$. But then $e_{12}(e_{11}x) = -\gamma ve_{11} - \delta ve_{12} \in \text{Ker}(R)$. Let $V_1 = \text{Ker}(R) \cap (Fve_{11} + Fve_{12})$, $V_2 = \text{Ker}(R) \cap (Fve_{21} + Fve_{22})$. Thus, $\text{Ker}(R) \cap vM_2(F) = V_1 \oplus V_2$, and $\dim V_1 \geq \dim V_2$. Since $2 \leq \dim(\text{Ker}(R) \cap vM_2(F)) \leq 3$ (by the theorem on the dimension of the kernel and image for $R|_{vM_2(F)}$), then either $\dim V_1 = 2$ and $\dim V_2 = 1$, or $\dim V_1 = 2$ and $\dim V_2 = 0$, or $\dim V_1 = \dim V_2 = 1$. In the first case, $ve_{11}, ve_{12} \in \text{Ker}(R)$ and we can assume (by Examples 2–4) that $ve_{21} \in \text{Ker}(R)$. In the second case, $ve_{11}, ve_{12} \in \text{Ker}(R)$. In the third case, we can assume (by Examples 2–4) that $ve_{11}, ve_{21} \in \text{Ker}(R)$.

3) $R(e_{21}) = e_{11}, R(e_{11}) = R(e_{12}) = R(e_{22}) = 0$.

Let $R(ve_{11}) = \alpha_1 e_{11} + \beta_1 e_{12}$. Then $G(ve_{11}, e_{21}) = -\beta_1 e_{12}$, whence $\beta_1 = 0$. Let $R(ve_{12}) = \alpha_2 e_{11} + \beta_2 e_{12}$. Then $G(ve_{12}, e_{21}) = -\beta_2 e_{12}$, whence $\beta_2 = 0$. Let $R(ve_{21}) = \alpha_3 e_{11} + \beta_3 e_{12}$. Then $G(ve_{21}, e_{21}) = \alpha_3 e_{11}$, whence $\alpha_3 = 0$. Let $R(ve_{22}) = \alpha_4 e_{11} + \beta_4 e_{12}$. Then $G(ve_{22}, e_{21}) = \alpha_4 e_{11}$, whence $\alpha_4 = 0$. Further, $G(ve_{22}, ve_{21}) = (\beta_4 \alpha_1 - \beta_3 \alpha_2) e_{11}$, whence $\beta_3 \alpha_2 = \beta_4 \alpha_1$.

3a) Let $\dim V_1 = 2$ and $\dim V_2 = 1$. Then, by above and by Example 4, we can assume that

$$R(e_{11}) = R(e_{12}) = R(e_{22}) = R(ve_{11}) = R(ve_{12}) = R(ve_{22}) = 0, \\ R(e_{21}) = e_{11}, \quad R(ve_{21}) = \beta_4 e_{12}.$$

Consider the automorphism φ from Example 6 for $\alpha = \frac{1}{\beta_4}$. Then

$$\varphi^{-1}R\varphi(e_{21}) = \frac{1}{\beta_4} e_{11}, \varphi^{-1}R\varphi(ve_{21}) = \frac{1}{\beta_4} e_{12}.$$

After multiplication by the scalar β_4 we can assume that

$$R(e_{21}) = e_{11}, R(ve_{21}) = e_{12}.$$

We have obtained case (3) in the statement of the lemma.

3b) Let $\dim V_1 = 2$ and $\dim V_2 = 0$. If we consider the restriction of the mapping R to V_2 , we will find that the image of this mapping has a dimension at most one, which implies (according to the the theorem on the dimension of the kernel and the image) that the kernel must have a dimension at least one. Therefore, this case is impossible.

3c) Let $\dim V_1 = \dim V_2 = 1$. Then, by above, we can assume that

$$R(e_{11}) = R(e_{12}) = R(e_{22}) = R(ve_{11}) = R(ve_{22}) = 0, \\ R(e_{21}) = e_{11}, \quad R(ve_{11}) = \alpha_2 e_{11}, \quad R(ve_{21}) = \beta_4 e_{12}.$$

Consider the automorphism φ from Example 6 for $\alpha = \frac{1}{\beta_4}$. Then

$$\varphi^{-1}R\varphi(e_{21}) = \frac{1}{\beta_4} e_{11}, \varphi^{-1}R\varphi(ve_{11}) = \alpha_2 e_{11}, \varphi^{-1}R\varphi(ve_{21}) = \frac{1}{\beta_4} e_{12}.$$

After multiplication by the scalar β_4 we can assume that (here $\varepsilon = \alpha_2 \beta_4$)

$$R(e_{21}) = e_{11}, \quad R(ve_{11}) = \varepsilon e_{11}, \quad R(ve_{21}) = e_{12}.$$

The Example 10 with a scalar ε allows us to assume that $R(e_{21}) = 0$. We have obtained the case (4) in the statement of the lemma.

4) $R(e_{21}) = e_{12}$, $R(e_{11}) = R(e_{22}) = R(e_{12}) = 0$. Let $R(ve_{11}) = \alpha_1 e_{11} + \beta_1 e_{12}$. Then $G(ve_{11}, e_{21}) = \alpha_1 e_{12}$, whence $\alpha_1 = 0$. Let $R(ve_{12}) = \alpha_2 e_{11} + \beta_2 e_{12}$. Then $G(ve_{12}, e_{21}) = \alpha_2 e_{12}$, whence $\alpha_2 = 0$. Let $R(ve_{21}) = \alpha_3 e_{11} + \beta_3 e_{12}$. Then $G(ve_{21}, e_{21}) = (\alpha_3 - \beta_1) e_{12}$, whence $\alpha_3 = \beta_1$. Let $R(ve_{22}) = \alpha_4 e_{11} + \beta_4 e_{12}$. Then $G(ve_{22}, e_{21}) = (\alpha_4 - \beta_2) e_{12}$, whence $\alpha_4 = \beta_2$.

4a) Let $\dim V_1 = 2$. By above we have $\beta_1 = \beta_2 = 0$, hence $\alpha_3 = \alpha_4 = 0$. Then $\text{Im}(R) = Fe_{12}$, a contradiction. Therefore, this case is impossible.

4b) Let $\dim V_1 = \dim V_2 = 1$. By above and Example 4, we can assume that $ve_{12}, ve_{22} \in \text{Ker}(R)$. Therefore, $\beta_2 = \alpha_4 = \beta_4 = 0$. We have

$$R(e_{11}) = R(e_{12}) = R(e_{22}) = R(ve_{12}) = R(ve_{22}) = 0, \\ R(e_{21}) = e_{12}, \quad R(ve_{11}) = \beta_1 e_{12}, \quad R(ve_{21}) = \beta_1 e_{11} + \beta_3 e_{12}.$$

Consider the automorphism φ from Example 6 for $\alpha = \beta_1$. Then

$$\varphi^{-1}R\varphi(e_{21}) = \beta_1^2 e_{12}, \varphi^{-1}R\varphi(ve_{11}) = \beta_1^2 e_{12}, R(ve_{21}) = \beta_1^2 e_{11} + \beta_3 \beta_1^2 e_{12}.$$

After multiplication by $\frac{1}{\beta_1^2}$ we can assume that

$$R(e_{21}) = e_{12}, \quad R(ve_{11}) = e_{12}, \quad R(ve_{21}) = e_{11} + \beta_3 e_{12}.$$

Example 10 allows us to assume that $R(e_{21}) = 0, R(ve_{11}) = e_{12}, R(ve_{21}) = e_{11} + \beta_3 e_{12}$. Example 12 with a scalar $\alpha = -\frac{\beta_3}{2}$ allows us to assume that $R(e_{21}) = 0, R(ve_{11}) = e_{12}, R(ve_{21}) = e_{11}$. We have obtained case (5) in the statement of the lemma.

5) $R(M_2(F)) = 0$. By above we can assume that either $\text{Ker}(R) \cap (vM_2(F)) = Fve_{11} + Fve_{12}$, or $\text{Ker}(R) \cap (vM_2(F)) = Fve_{11} + Fve_{21}$. Let $R(ve_{12}) = \alpha_2 e_{11} + \beta_2 e_{12}, R(ve_{21}) = \alpha_3 e_{11} + \beta_3 e_{12}, R(ve_{22}) = \alpha_4 e_{11} + \beta_4 e_{12}$.

5a) $R(ve_{21}) = 0$, that is $\alpha_3 = \beta_3 = 0$. Then

$$G(ve_{12}, ve_{22}) = (\beta_2 - \alpha_4)\alpha_2 e_{11} + (\beta_2 - \alpha_4)\beta_2 e_{12}.$$

Since $R(ve_{12}) = \alpha_2 e_{11} + \beta_2 e_{12} \neq 0$, then $\alpha_4 = \beta_2$.

If $\beta_2 = 0$, then we can assume (up to Example 4) that $R(ve_{11}) = \alpha_2 e_{11}, R(ve_{21}) = e_{12}$. We have obtained case (4) from the statement of the lemma.

If $\beta_2 \neq 0$, then conjugation by automorphism from Example 7 with $\alpha = \frac{1}{\beta_2}$ gives us (with $\varepsilon = \frac{\gamma}{\beta_2^2}$)

$$R(ve_{12}) = \alpha_2 e_{11} + e_{12}, \quad R(ve_{22}) = e_{11} + \varepsilon e_{12}.$$

The conjugation by automorphism from Example 6 for $\alpha = \frac{1}{\varepsilon}$ and Example 4 allow us to assume that R has the form $R(ve_{11}) = \alpha_2 \varepsilon e_{11} + e_{12}, R(ve_{21}) = e_{11} + e_{12}$. After conjugation with an automorphism from Example 12 with a scalar $-\frac{1}{\alpha_2}$ and a multiplication by a scalar, we obtain case (4) in the statement of the lemma (if $\alpha_2 \neq 1$, otherwise we obtain $R(ve_{21}) = 0$, it is a contradiction).

It is easy to see that the operator R with these conditions is a Rota-Baxter operator. It remains to note that in order for the condition $\text{Im}(R) = Fe_{11} + Fe_{12}$ to be satisfied, it is necessary and sufficient that $\alpha_2 \varepsilon \neq 1$.

5b) $R(ve_{12}) = 0$, that is $\alpha_2 = \beta_2 = 0$. Then $R(ve_{21}) = \alpha_3 e_{11} + \beta_3 e_{12}, R(ve_{22}) = \alpha_4 e_{11} + \beta_4 e_{12}$. It is easy to see that the operator R with these conditions is a Rota-Baxter operator. Note that in order for the condition $\text{Im}(R) = Fe_{11} + Fe_{12}$ to be satisfied, it is necessary and sufficient that $\alpha_3 \beta_4 \neq \alpha_4 \beta_3$. Next, let $\beta_3 = 0$. Then we can assume that

$$R(ve_{21}) = e_{11}, \quad R(ve_{22}) = \alpha_4 e_{11} + \beta_4 e_{12}. \quad (4.1)$$

By the Example 7 we can assume that $R(ve_{21}) = e_{11}, R(ve_{22}) = e_{11} + \frac{\beta_4}{\alpha_4} e_{12}$. Then, by the Example 8, we can assume that $R(ve_{21}) = \gamma e_{11}, R(ve_{22}) =$

$e_{11} + e_{12}$, where $\gamma = \frac{\beta_4}{\alpha_4}$. According to the Example 3 we can assume that $R(ve_{21}) = \gamma e_{11}$, $R(ve_{22} + \gamma^{-1}ve_{21}) = e_{11} + e_{12}$, whence $R(ve_{21}) = \gamma e_{11}$, $R(ve_{22}) = e_{12}$. We obtain case (6) in the statement of the lemma.

Let $\beta_3 \neq 0$. Then we can assume that $R(ve_{21}) = \alpha_3 e_{11} + e_{12}$. Further, by the Example 3 with a scalar β_4 we can assume that

$$R(ve_{21}) = \alpha_3 e_{11} + e_{12}, \quad R(ve_{22}) = \gamma e_{12},$$

where $\gamma = \alpha_4 - \beta_4 \alpha_3$. Then we can assume that

$$R(ve_{21}) = \alpha'_3 e_{11} + \beta'_3 e_{12}, \quad R(ve_{22}) = e_{11}.$$

According to the Example 4 we can assume that

$$R(ve_{21}) = e_{11}, \quad R(ve_{22}) = -\alpha'_3 e_{11} - \beta'_3 e_{12}.$$

This is exactly the already discussed case $\beta_3 = 0$, formula (4.1). The lemma is proven.

Corollary 1. *Let R be a Rota-Baxter operator of zero weight on the split Cayley-Dickson algebra $\mathbb{O}$ and $\text{Im}(R) = Fe_{11} + Fe_{12}$. If a field F is quadratically closed then, up to conjugation by automorphism, antiautomorphism and up to multiplication by a scalar, an operator R acts in one of the following ways for some $\alpha \in F$ (an operator R is zero on unspecified basic elements e_{ij} , ve_{ij}):*

- 1) $R(e_{21}) = e_{11}$, $R(e_{22}) = e_{12}$;
- 2) $R(e_{21}) = -e_{11}$, $R(e_{11}) = e_{12}$;
- 3) $R(e_{21}) = e_{11}$, $R(ve_{21}) = e_{12}$;
- 4) $R(ve_{11}) = e_{11}$, $R(ve_{21}) = e_{12}$;
- 5) $R(ve_{11}) = e_{12}$, $R(ve_{21}) = e_{11}$;
- 6) $R(ve_{21}) = e_{11}$, $R(ve_{22}) = e_{12}$;

Proof. Let us consider the resulting cases on R in Lemma 3. Cases 1–3, 5–7 remained the same.

4) Example 7 with a scalar $\sqrt{\alpha}$ allows us to assume that $R(ve_{11}) = \sqrt{\alpha}e_{11}$, $R(ve_{21}) = \sqrt{\alpha}e_{12}$. Up to multiplication by a scalar, we obtain the required operator.

6) Example 7 with a scalar $\sqrt{\alpha}$ allows us to assume that $R(ve_{21}) = \alpha e_{11}$, $R(ve_{22}) = \alpha e_{12}$. Up to multiplication by a scalar, we obtain the required operator. The corollary is proven.

5. Main Theorem

We are ready to formulate the main result.

Theorem 1. *Let R be the Rota-Baxter operator of zero weight on the split Cayley-Dickson algebra $\mathbb{O}$ and $\text{Im}(R)$ can be embedded in $M_2(F)$. Then, up to the action of automorphism, antiautomorphism and multiplication by a scalar, R acts on $\mathbb{O}$ in one of the following ways for some $\alpha \in F$ (an operator R is zero on unspecified basic elements e_{ij} , ve_{ij}):*

- 1) $R(e_{21}) = e_{12}$;
- 2) $R(ve_{22}) = e_{12}$;
- 3) $R(e_{21}) = e_{11}$;
- 4) $R(e_{21}) = e_{11}$, $R(e_{22}) = e_{12}$;
- 5) $R(e_{21}) = -e_{11}$, $R(e_{11}) = e_{12}$;
- 6) $R(e_{21}) = e_{11}$, $R(ve_{21}) = e_{12}$;
- 7) $R(ve_{11}) = \alpha e_{11}$, $R(ve_{21}) = e_{12}$, $\alpha \neq 0$;
- 8) $R(ve_{11}) = e_{12}$, $R(ve_{21}) = e_{11}$;
- 9) $R(ve_{21}) = \alpha e_{11}$, $R(ve_{22}) = e_{12}$, $\alpha \neq 0$.

Proof. In [13] it was proved that there are only three non-zero non-unital matrix subalgebras B in $\mathbb{O}$, up to action of automorphism. They are precisely the subalgebras from the statements of Lemmas 1–3. The theorem is proven.

Corollary 2. *Let R be the Rota-Baxter operator of zero weight on the split Cayley-Dickson algebra $\mathbb{O}$ and $\text{Im}(R)$ can be embedded in $M_2(F)$. If a field F is quadratically closed, then, up to the action of automorphism, antiautomorphism and multiplication by a scalar, R acts on $\mathbb{O}$ in one of the following ways (unspecified basic elements e_{ij} , ve_{ij} lie in $\text{Ker}(R)$):*

- 1) $R(e_{21}) = e_{12}$;
- 2) $R(ve_{22}) = e_{12}$;
- 3) $R(e_{21}) = e_{11}$;
- 4) $R(e_{21}) = e_{11}$, $R(e_{22}) = e_{12}$;
- 5) $R(e_{21}) = -e_{11}$, $R(e_{11}) = e_{12}$;
- 6) $R(e_{21}) = e_{11}$, $R(ve_{21}) = e_{12}$;

$$7) R(ve_{11}) = e_{11}, R(ve_{21}) = e_{12};$$

$$8) R(ve_{11}) = e_{12}, R(ve_{21}) = e_{11};$$

$$9) R(ve_{21}) = e_{11}, R(ve_{22}) = e_{12}.$$

Remark 1. In the Corollary 2 operator (5) is the unique one, where $R^2 \neq 0$, but $R^3 = 0$. Operators (1)–(4), (6)–(9) are the ones, where $R^2 = 0$.

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Об авторах

Панасенко Александр Сергеевич, канд. физ.-мат. наук, Институт математики им. С. Л. Соболева СО РАН, Новосибирск, 630090, Российская Федерация, a.panasenko@g.nsu.ru, <https://orcid.org/0000-0003-1637-3779>

About the authors

Alexander S. Panasenko, Cand. Sci. (Phys.-Math.), Sobolev Institute of Mathematics SB RAS, Novosibirsk, 630090, Russian Federation, a.panasenko@g.nsu.ru, <https://orcid.org/0000-0003-1637-3779>

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