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Non-Orthogonality of 1-types in Theories with a Linear Order

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Abstract:
Non-orthogonality of complete types is an important concept for such classes of first-order theories as o-minimal, weakly-o-minimal and quite o-minimal theories. This concept is used in studying countable spectrum of such theories, since orthogonality affects omission and realization of types. Further study of the Vaught's conjecture for small ordered theories requires the use of the relation between incomplete types, in particular, convex closures of 1-types. In this paper, two notions of non-orthogonality of convex incomplete types are introduced. Connections between different kinds of non-orthogonality are shown. Theorems on preservation of properties of types under non-orthogonality are proven.

Keywords: linear order, convex closure, orthogonality (weak and almost), definable type, quasirational type

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Научная статья

Неортогональность 1-типов в теориях с линейным порядком

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Аннотация: Неортогональность полных типов является важным понятием для таких классов теорий первого порядка, как о-минимальные, слабо-о-минимальные и вполне о-минимальные теории. Это понятие используется при изучении счётного спектра таких теорий, поскольку ортогональность влияет на опускание и реализацию типов. Дальнейшее изучение гипотезы Воота для малых упорядоченных теорий требует использования связи между неполными типами, в частности выпуклыми замыканиями 1-типов. Вводятся два понятия неортогональности выпуклых неполных типов. Показаны связи между различными видами неортогональности. Доказаны теоремы о сохранении свойств типов при неортогональности.

Ключевые слова: линейный порядок, выпуклое замыкание, ортогональность (слабая и почти), определимый тип, квазирациональный тип

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1. Introduction

According to S. Shelah, two complete types $p(\bar{x})$ and $q(\bar{y})$ are *weakly orthogonal*, if $p(\bar{x}) \cup q(\bar{y})$ is a complete type [20]. The following is an equivalent definition of weak orthogonality of 1-types.

Let $A \subseteq N$, $p, q \in S_1(A)$, and $\mathfrak{N}$ be an $|A|^+$-saturated structure of a language $\mathcal{L}$. We use the standard notations: $p(\mathfrak{N}) = \{\alpha \mid \mathfrak{N} \models p(\alpha)\}$ and $\varphi(\mathfrak{N}, \alpha) = \{\beta \mid \mathfrak{N} \models \varphi(\beta, \alpha)\}$. We say that p is *weakly orthogonal* to q and write $p \perp^w q$ for this, if for every A-definable formula $\varphi(x, y)$ and every $\alpha \in p(\mathfrak{N})$ either $\varphi(\mathfrak{N}, \alpha) \cap q(\mathfrak{N}) = \emptyset$ or $q(\mathfrak{N}) \subseteq \varphi(\mathfrak{N}, \alpha)$.

Equivalently, the definition of non-weak-orthogonality can be given. We say that p is *not weakly orthogonal* to q and write $p \not\perp^w q$ for this, if there exist an A-definable formula $\varphi(x, y)$, $\alpha \in p(\mathfrak{N})$, and realizations $\beta_1, \beta_2 \in q(\mathfrak{N})$ such that $\beta_1 \in \varphi(\mathfrak{N}, \alpha)$ and $\beta_2 \notin \varphi(\mathfrak{N}, \alpha)$.

In [2], B. Baizhanov studied properties of non orthogonality of 1-types for the class of weakly o-minimal theories.

From now on we consider only linearly ordered structures and their elementary theories.

Definition 1. *[3] The type p is not almost orthogonal to the type q, $p \not\perp^a q$, if there exists an A-definable formula $\varphi(x, y)$, such that for some $\alpha \in p(\mathfrak{N})$ and $\gamma_1, \gamma_2 \in q(\mathfrak{N})$, $\gamma_1 < \varphi(\mathfrak{N}, \alpha) < \gamma_2$ and $\varphi(\mathfrak{N}, \alpha) \cap q(\mathfrak{N}) \neq \emptyset$.*

The relations of non-weak and non-almost orthogonality in weakly o-minimal theories are equivalence relations on $S_1(A)$ [2].

Many authors refer to the question of the number of countable pairwise non-isomorphic models [8; 9; 17; 18; 20; 23]. The Vaught's conjecture is an important direction of study in model theory [7; 10; 11; 13; 21; 22; 24]. The number of countable models of theories with an $\emptyset$-definable relation of a linear order has been studied in [14; 15]. Also, the Vaught's conjecture has been confirmed for several classes of theories with linear order [12; 16; 19]. The relations of non-orthogonality play an important role in counting the number of countable non-isomorphic models [1; 6; 12; 14]. The relation of non almost orthogonality allows us to realize types, while the relation of non weak orthogonality allows us to omit them.

In o-minimal, weakly-o-minimal, and quite o-minimal theories, the set of all realizations of an arbitrary complete 1-type is convex. In general, this is not true. There can be several 1-types with the same convex closure (Definition 4). Later in the article, we generalize the notions of non-orthogonality for weakly o-minimal theories and introduce definitions of weak convex-orthogonality (Definition 6) and almost convex-orthogonality (Definition 7) of the convex closures of complete 1-types in theories with a linear order. In Remark 1, we describe the relationships between weak, almost orthogonalities, weak and almost convex-orthogonalities of types. We show that the relation of non weak convex-orthogonality is symmetric (Theorem 3), preserves quasirationality (Corollary 1) and definability (Theorem 2) of convex types.

2. The main part

For subsets A and B of a linearly ordered structure $\mathfrak{M} = \langle M; =, <, \ldots \rangle$ we use the following notations:

$$A^+ := \{\gamma \in M \mid \mathfrak{M} \models a < \gamma, \text{ for all } a \in A\};$$
$$A^- := \{\gamma \in M \mid \mathfrak{M} \models \gamma < a, \text{ for all } a \in A\}.$$

We write $A < B$ if $\mathfrak{M} \models a < b$ for all $a \in A$, $b \in B$. We write $a < B$ ($A < b$) if $\{a\} < B$ ($A < \{b\}$). If sets A and B are C-definable ($C \subseteq M$), then A^+, A^- and $A < B$ are C-definable as well.

Definition 2. *A subset A of a totally ordered structure $\mathfrak{M}$ is said to be convex if $a < c < b$ implies $c \in A$ for all $a, b \in A$ and all $c \in M$.*

Definition 3. *A formula $\varphi(x, \bar{y}, \bar{a})$ is a convex formula, if for every $\bar{b} \in N$ the set $\varphi(\mathfrak{N}, \bar{b}, \bar{a})$ is convex in every model $\mathfrak{N} = \langle N; =, <, \dots \rangle$ of $Th(\mathfrak{M})$ containing $\bar{a}$.*

Definition 4. *1) The convex closure of a formula $\varphi(x, \bar{a})$ is the following:*

$$\varphi^c(x, \bar{a}) := \exists y_1 \exists y_2 \big(\varphi(y_1, \bar{a}) \wedge \varphi(y_2, \bar{a}) \wedge (y_1 \leq x \leq y_2)\big).$$

2) The convex closure of a type $p(x) \in S_1(A)$ is the following type:

$$p^c(x) := \{\varphi^c(x, \bar{a}) \mid \varphi(x, \bar{a}) \in p\}.$$

Similarly we denote $tp^c(\alpha/A) := \{\varphi^c(x, \bar{a}) \mid \varphi(x, \bar{a}) \in tp(\alpha/A)\}$. We define $S_{p^c}(A) := \{q \in S_1(A) \mid q^c = p^c\}$.

The convex closure defines the smallest convex set which contains the set of all realizations of a given formula or a type. From Definition 4 it follows that the type p^c is, in some way, complete: for every convex A-formula φ one and only one of φ, φ^+, φ^- belongs to p^c up to equivalence of formulas. Up to equivalence, $p^c = \{\varphi(x, \bar{a}) \in p \mid \varphi(x, \bar{a})$ is convex in every model of $T\}$. Note that in general the type p^c is not necessarily complete. From Definition 4 it follows that if $p^c(\mathfrak{N}) \cap q^c(\mathfrak{N}) \neq \emptyset$, then $p^c(\mathfrak{N}) = q^c(\mathfrak{N})$. Indeed, let $\alpha \in p^c(\mathfrak{N}) \cap q^c(\mathfrak{N})$, and let $\varphi \in q^c$ be a convex formula. Then $\mathfrak{N} \models \varphi(\alpha)$, and therefore $\varphi \in p^c$. Since φ is arbitrary, $p^c \subseteq q^c$. Analogically, $q^c \subseteq p^c$. Then $p^c = q^c$ and $p^c(\mathfrak{N}) = q^c(\mathfrak{N})$. So, the relation $p^c = q^c$ is an equivalence relation on the set $S_1(A)$.

If T is a weakly o-minimal theory and p is a 1-type over a subset of a model $\mathfrak{M} \models T$, it is easy to see, that the set $p(\mathfrak{M})$ is either convex or empty, and $p(\mathfrak{M}) = p^c(\mathfrak{M})$.

Let $A \subset N$, $p, q \in S_1(A)$, and $\mathfrak{N}$ be an $|A|^+$-saturated model of a theory with a linear order. Definitions 6 and 7 are direct generalizations of non-orthogonality of complete types in weakly o-minimal theories.

Definition 5. *We say that an A-formula $\varphi(x, y)$ monotonically increases (decreases) on $B \subseteq N$, if for all $b_1, b_2 \in B$, with $b_1 < b_2$, the following holds:*

$$\varphi(\mathfrak{N}, b_2)^+ \subseteq \varphi(\mathfrak{N}, b_1)^+ \ (\varphi(\mathfrak{N}, b_1)^+ \subseteq \varphi(\mathfrak{N}, b_2)^+),$$

and for some distinct $b_3, b_4 \in B$, $\varphi(\mathfrak{N}, b_3) \neq \varphi(\mathfrak{N}, b_4)$.

Definition 6. *We say that p^c is not weakly convex-orthogonal to q^c and write $p^c \not\perp^{cw} q^c$ for this, if there exists a convex monotonic on $p^c(\mathfrak{N})$ A-definable formula $\varphi(x, y)$, such that for each $\alpha \in p^c(\mathfrak{N})$, there exist $\beta_1, \beta_2 \in q^c(\mathfrak{N})$ with $\beta_1 \in \varphi(\mathfrak{N}, \alpha)$, $\beta_2 \notin \varphi(\mathfrak{N}, \alpha)$ and $\beta_1 < \beta_2$.*

Definition 7. *We say that p^c is not almost convex-orthogonal to q^c and write $p^c \not\perp^{ca} q^c$ for this, if there exists an A-definable convex monotonic on $p^c(\mathfrak{N})$ formula $\varphi(x,y)$, such that for all $\alpha \in p^c(\mathfrak{N})$ there are $\gamma_1, \gamma_2 \in q^c(\mathfrak{N})$ such that $\emptyset \neq \varphi(\mathfrak{N},\alpha)$ and $\gamma_1 < \varphi(\mathfrak{N},\alpha) < \gamma_2$.*

We say that a formula $\psi(x)$ *divides* a (partial) type q, if $\psi(\mathfrak{N}) \cap q(\mathfrak{N}) \neq \emptyset$ and $\neg\psi(\mathfrak{N}) \cap q(\mathfrak{N}) \neq \emptyset$. So, for two types $p, q \in S_1(A)$, p^c is not weakly convex-orthogonal to q^c if and only if there exists an A-definable formula $\varphi(x,y)$ such that the right border of $\varphi(x,\alpha)$ divides q^c for all $\alpha \in p^c(\mathfrak{N})$, and the formula φ is monotonic but not constant on the set $q^c(\mathfrak{N})$.

The following remark is obvious.

Remark 1. Let $A \subset N$, $\mathfrak{N}$ be an $|A|^+$-saturated model of a theory with a linear order, $p, q \in S_1(A)$ be non-algebraic types. Then

1) $p \not\perp^w q \Leftrightarrow q \not\perp^w p$;

2) $p \not\perp^a q \Rightarrow p \not\perp^w q$;

3) $p^c \not\perp^{ca} q^c \Rightarrow p^c \not\perp^{cw} q^c$;

4) $p^c \not\perp^{cw} q^c \Rightarrow p_0 \not\perp^w q_0$ for each $p_0 \in S_1(A)$ with $p_0^c = p^c$ and each $q_0 \in S_1(A)$ with $q_0^c = q^c$;

5) $p^c \not\perp^{ca} q^c \Rightarrow p_0 \not\perp^a q_0$ for each $p_0 \in S_1(A)$ with $p_0^c = p^c$ and some $q_0 \in S_1(A)$ with $q_0^c = q^c$.

Definition 8. *Let $p \in S_1(A)$, $A \subset N$, and $\mathfrak{N}$ be an $|A|^+$-saturated model of a theory with a linear order. We say that p^c is quasirational to the right (left) if there exists an A-formula $U(x)$, such that for every $\alpha \in p^c(\mathfrak{N})$*

$$(p^c(\mathfrak{N}))^- \cup p^c(\mathfrak{N}) = U(\mathfrak{N})$$
$$((p^c(\mathfrak{N}))^+ \cup p^c(\mathfrak{N}) = U(\mathfrak{N})).$$

We say that p^c is quasirational if it is either quasirational to the left, or quasirational to the right.

Note that p^c is quasirational to the right if and only if there exists an A-formula $R(x)$ such that $p^c(\mathfrak{N})^+ = R(\mathfrak{N})$, similarly, p^c is quasirational to the left if and only if there exists an A-formula $L(x)$ such that $p^c(\mathfrak{N})^- = L(\mathfrak{N})$.

Note that if $p = p^c$, and p^c is quasirational to the right and to the left, then p is isolated.

Theorem 1. *Let $A \subset N$, $\mathfrak{N}$ be an $|A|^+$-saturated model of a theory with a linear order, p and $q \in S_1(A)$ be non-algebraic types such that $p^c \not\perp^{cw} q^c$. Then if p^c is quasirational, then q^c is quasirational.*

Proof. Let $\varphi(x,y)$ be a convex A-formula as in Definition 6. Without loss of generality let p^c be quasirational to the right, let $U(x)$ be as in Definition 8, and let $\varphi(x,y)$ be monotonically increasing on $p^c(\mathfrak{N})$. Then there exists a convex A-formula $U_1(x)$ such that $\sup U_1(\mathfrak{N}) = \sup U(\mathfrak{N})$ and $\varphi(x,y)$ is increasing on $U_1(\mathfrak{N})$.

If any of the types from $S_1(A)$, whose convex closure equals q^c, is principal, then the convex closure of its isolating formula guarantees quasirationality of q^c. Therefore, let q^c be non-principal. Replace $\varphi(\mathfrak{N},\alpha)$ with $\varphi(\mathfrak{N},\alpha)\cup\varphi(\mathfrak{N},\alpha)^-$. This way, $\varphi(\mathfrak{N},\alpha)$ will contain an initial segment of $\mathfrak{N}$.

Consider the following convex A-formula:

$$R(t) := \exists z\Big[U_1(z)\wedge\varphi(t,z)\Big].$$

For each $\alpha\in p^c(\mathfrak{N})$ the formula $\varphi(x,\alpha)$ divides $q^c(\mathfrak{N})$. First we show that for each $\beta\in q^c(\mathfrak{N})$ there is $\alpha\in p^c(\mathfrak{N})$ such that $\beta\in\varphi(\mathfrak{N},\alpha)$. Let $\alpha\in p^c(\mathfrak{N})$ and $\gamma\in q(\mathfrak{N})$ be such that $\mathfrak{N}\models\varphi(\gamma,\alpha)$. Then for each $P(y)\in p^c$, $\mathfrak{N}\models\exists y(P(y)\wedge\varphi(\gamma,y))$. Therefore $\big(\exists y(P(y)\wedge\varphi(x,y))\big)\in q(x)$. Then since $P(y)\in p^c$ is arbitrary, the desired property holds for each $\beta\in q(\mathfrak{N})$. To show that it holds for an arbitrary $\beta\in q^c(\mathfrak{N})$, take $\beta'\in q(\mathfrak{N})$ such that $\beta'>\beta$. Then there is $\alpha'\in p^c(\mathfrak{N})$ with $\mathfrak{N}\models\varphi(\beta',\alpha')$. By monotonicity of φ and since $\beta'>\beta$, $\mathfrak{N}\models\varphi(\beta,\alpha')$, and so, α' is the desired element. Moreover there exists $\beta''\in q^c(\mathfrak{N})$ such that $\beta''>\varphi(\mathfrak{N},\alpha')$.

Then since U_1 defines the right border of $p^c(\mathfrak{N})$ and the formula φ monotonically increases on $U_1(\mathfrak{N})$, $R(\mathfrak{N})$ defines exactly the right border of $q^c(\mathfrak{N})$. This means q^c is quasirational to the right. Analogically, if φ monotonically decreases, q^c is quasirational to the left. □

Definition 9. *[20] Let $A\subseteq N$, $\Delta\subseteq\mathcal{L}$, $p\in S_\Delta(A)$. We say that p is definable, if for every formula $\varphi(x,\bar y)\in\Delta$ there exists an A-definable formula $d_\varphi(\bar y)$ such that for every $\bar b\in A$*

$$\varphi(x,\bar b)\in p\Leftrightarrow\mathfrak{N}\models d_\varphi(\bar b).$$

In the case of the convex closure p^c, Δ is the set of all convex A-formulas.

In weakly o-minimal theories, if $p,q\in S_1(A)$ are such that $p\not\perp^w q$, then p is definable if and only if q is definable [2]. In general this is not true because of the following example.

Example 1. Let $\mathfrak{M}=\langle M;=,<,P^1,S^1,\varphi^2\rangle$, where $M=(0,1)\cup(1,+\infty)\subset\mathbb{R}$, P and S are unary predicates such that $P(\mathfrak{M})=(0,1)$ and $S(\mathfrak{M})=(1,+\infty)$, and $<$ comes from the natural ordering of $\mathbb{R}$. For all $a,b\in M$ let $\mathfrak{M}\models\varphi(b,a)$ imply $\mathfrak{M}\models(P(b)\wedge S(a))$. And let for all $a_1,a_2\in S(\mathfrak{M})$ the sets $\varphi(\mathfrak{M},a_1)$ and $\varphi(\mathfrak{M},a_2)$ be infinite disjoint mutually dense subsets of $P(\mathfrak{M})$. More precisely, $\varphi(\mathfrak{M},a_1)$ and $\varphi(\mathfrak{M},a_2)$ are both dense in (0,1) and disjoint.

Let $A:=\mathbb{Q}\cap S(\mathfrak{M})$ and $a\in S(\mathfrak{M})$. We put $q_a:=tp(a/A)$. Let $\gamma_a\in\varphi(\mathfrak{M},a)$. Let $p_a:=tp(\gamma_a/A)$. Notice that p_a is principal if $a\in A$. Also, $p_{\sqrt2}\not\perp^w q_{\sqrt2}$ such that for each $\delta\in p_{\sqrt2}(\mathfrak{M})$, $\varphi(\delta,\mathfrak{M})=\{\sqrt2\}$ and $\delta\in\varphi(\mathfrak{M},\sqrt2)$. The type $p_{\sqrt2}$ is non-principal and undefinable. To prove this, let $\theta(x,z_1,z_2):=\exists y(\varphi(x,y)\wedge z_1<y<z_2)$. For all $a_1,a_2\in A$, $\theta(x,a_1,a_2)\in$

$p_{\sqrt{2}}$ if and only if $\mathbb{R} \models a_1 < \sqrt{2} < a_2$. The element $\sqrt{2}$ defines in A an irrational cut. If there is a formula guaranteeing definability of the type $p_{\sqrt{2}}$, it means the irrational cut is definable, which is impossible in this structure. Also $p_{\sqrt{2}} \not\perp^w p_a$ for all $a \in S(\mathfrak{M})$ and since the relation of non weak orthogonality is symmetric, $p_a \not\perp^w p_{\sqrt{2}}$. It shows that non weak orthogonality does not preserve definability and isolation of types. Indeed, for $a \in A$, p_a is isolated and consequently is definable, and $p_{\sqrt{2}}$ is non definable and non-isolated. At the same time $p_{\sqrt{2}}$ and p_a are non weakly orthogonal. Later, we show that non weak convex-orthogonality preserves these properties.

An example of a non-principal type that is not weakly orthogonal to a principal type was also given in [5], and an example of a definable type that is not weakly orthogonal to an undefinable type was given in [4].

Theorem 2. *Let $A \subset N$, $\mathfrak{N}$ be an $|A|^+$-saturated model of a theory with a linear order, p and $q \in S_1(A)$ be non-algebraic types such that $p^c \not\perp^{cw} q^c$. Then if p^c is definable, then q^c is definable.*

Proof. Let $\varphi(x, y)$ be a convex A-formula as in Definition 6. Let p^c be definable, $\alpha \in p(\mathfrak{N})$, and let $\beta_1, \beta_2 \in q^c(\mathfrak{N})$ be such that $\beta_1 \in \varphi(\mathfrak{N}, \alpha) < \beta_2$.

Let φ be increasing on some definable convex set $U(\mathfrak{N})$, where $U \in p^c$. Let $\psi(x, \bar{z})$ be an arbitrary convex $\emptyset$-definable formula. For every $\bar{b} \in A$ the following hold:

$$\begin{aligned} \psi(x,\bar{b}) \in q^c \quad &\Leftrightarrow \quad (\beta_1, \beta_2) \subset q^c(\mathfrak{N}) \subset \psi(\mathfrak{N}, \bar{b}) \\ &\Leftrightarrow \quad \mathfrak{N} \models \forall x(\beta_1 < x < \beta_2 \to \psi(x, \bar{b})) \\ &\Leftrightarrow \quad \mathfrak{N} \models \exists x_1 \exists x_2 \Big(\varphi(x_1, \alpha) \wedge \neg\varphi(x_2, \alpha) \wedge x_1 < x_2 \wedge \\ &\qquad\qquad U(x_1) \wedge U(x_2) \wedge \forall x\big(x_1 < x < x_2 \to \psi(x, \bar{b})\big)\Big). \end{aligned}$$

We denote the last formula by $H(\alpha, \bar{b})$. Then, since $p = tp(\alpha/A)$ is definable, there exist a formula d and a tuple $\bar{c} \in A$ such that

$$\mathfrak{N} \models H(\alpha, \bar{b}) \Leftrightarrow H(y, \bar{b}) \in p \Leftrightarrow \mathfrak{N} \models d(\bar{b}, \bar{c}).$$

Therefore $\psi(x, \bar{b}) \in q^c$ if and only if $\mathfrak{N} \models d(\bar{b}, \bar{c})$. □

Theorem 3. *Let $A \subset N$, $\mathfrak{N}$ be $|A|^+$-saturated, $p, q \in S_1(A)$ be non-principal types. Then $p^c \not\perp^{cw} q^c \Leftrightarrow q^c \not\perp^{cw} p^c$.*

Proof. Let $p^c \not\perp^{cw} q^c$, and let $\varphi(x, y)$ be a convex A-definable monotonic formula such that $\varphi(\mathfrak{N}, \alpha) \cap q^c(\mathfrak{N}) \neq \emptyset$ and $\neg\varphi(\mathfrak{N}, \alpha) \cap q^c(\mathfrak{N}) \neq \emptyset$ for each $\alpha \in p^c(\mathfrak{N})$. Without loss of generality we suppose that φ monotonically increases on $p^c(\mathfrak{N})$. Rename φ to be $\varphi \vee \varphi^-$.

We claim that the formula $\psi(x, y) := \varphi(y, x)$ is the desired A-formula which guarantees that $q^c \not\perp^{cw} p^c$.

Let $\alpha_1, \alpha_2, \alpha_3 \in p^c(\mathfrak{N})$ and $\beta \in q^c(\mathfrak{N})$ be such that $\alpha_1 < \alpha_2 < \alpha_3$ and $\alpha_1, \alpha_3 \in \varphi(\beta, \mathfrak{N})$. From the last statement, $\beta \in \varphi(\mathfrak{N}, \alpha_1)$ and $\beta \in \varphi(\mathfrak{N}, \alpha_3)$. By monotonicity of the formula φ on $p^c(\mathfrak{N})$, $\beta \in \varphi(\mathfrak{N}, \alpha_2)$. And then, $\alpha_2 \in \varphi(\beta, \mathfrak{N})$. This implies that the formula $\psi(x, y) = \varphi(y, x)$ is convex.

To show that $\psi(x, y)$ divides $p^c(\mathfrak{N})$ for realizations of q^c, towards a contradiction, we suppose that there exists an element $\beta \in q^c(\mathfrak{N})$ (which we fix) such that $p^c(\mathfrak{N}) \subseteq \varphi(\beta, \mathfrak{N})$. First, we claim that this property holds for all elements from some convex A-definable set. Namely, we claim there is a convex A-formula $P(y) \in p^c(y)$ such that $P(\mathfrak{N}) \subseteq \varphi(\beta, \mathfrak{N})$. If not, each $P(y) \in p^c(y)$ has an element $\alpha \in P(\mathfrak{N})$ with $\beta \notin \varphi(\mathfrak{N}, \alpha)$. But then the set $p^c(y) \cup \{\beta \notin \varphi(\mathfrak{N}, y)\}$ is locally consistent, which is impossible.

Then the formula $\forall y(P(y) \to \varphi(x, y))$ belongs to $tp(\beta/A)$. Let $\gamma \in q^c(\mathfrak{N})$, $\gamma > \beta$. Since $tp^c(\gamma/A) = tp^c(\beta/A)$, we can take $\beta' \in q^c(\mathfrak{N})$ such that $tp(\beta'/A) = tp(\beta/A)$ and $\beta' > \gamma > \beta$. Then $(\forall y(P(y) \to \varphi(x, y))) \in tp(\beta'/A)$. By monotonicity, when $\beta, \beta' \in \varphi(\mathfrak{N}, \alpha)$, then $\gamma \in \varphi(\mathfrak{N}, \alpha)$ as well. Since γ is arbitrary, $\varphi(\mathfrak{N}, \alpha) \supseteq q^c(\mathfrak{N})$. This is a contradiction since $\varphi(\mathfrak{N}, \alpha)$ should divide $q^c(\mathfrak{N})$.

Similarly, we can show that there is no $\beta \in q^c(\mathfrak{N})$ such that $\beta \notin \varphi(\mathfrak{N}, \alpha)$ for all $\alpha \in p^c(\mathfrak{N})$.

To show monotonicity of ψ, we first confirm that there is a formula $R(y) \in p^c(y)$ such that ψ monotonically increases on $R(\mathfrak{N})$. Suppose the contrary: for each $R(y) \in p^c(y)$ there are $\alpha_1, \alpha_2 \in P(\mathfrak{N})$, $\alpha_1 < \alpha_2$, with $\varphi(\mathfrak{N}, \alpha_1)^+ \subsetneq \varphi(\mathfrak{N}, \alpha_2)^+$. But then the set

$$p^c(y_1) \cup p^c(y_2) \cup \{\forall x(\varphi(x, y_1)^+ \to \varphi(x, y_2)^+)\}$$

is locally consistent, which contradicts to monotonicity of φ on $p^c(\mathfrak{N})$.

Rename $\varphi(x, y)$ to be $\varphi(x, y) \wedge R(y)$. This formula still guarantees that $p^c \not\perp^{cw} q^c$. Therefore since φ is increasing then ψ should also be increasing.

For each $\beta \in q^c(\mathfrak{N})$ the formula $\psi(x, \beta)$ divides $p^c(\mathfrak{N})$, therefore it can not be constant on $q^c(\mathfrak{N})$. This finishes the proof of monotonicity of $\psi(x, y)$ on $q^c(\mathfrak{N})$.

By this, the formula $\psi(x, y) = \varphi(y, x)$ satisfies all the necessary conditions for the theorem to be proved. □

Corollary 1. *Let $A \subset N$, $\mathfrak{N}$ be $|A|^+$-saturated, and $p, q \in S_1(A)$ be non-principal types such that $p^c \not\perp^{cw} q^c$. Then p^c is quasirational if and only if q^c is quasirational.*

Proof. Follows from Theorems 1 and 3. □

3. Conclusion

The aim of this paper is introducing notions that allow us to study the countable spectrum of theories with a linear order. Notions of not convex weak and not convex almost orthogonality of the convex closures of types have been given. Their connection with each other, with already known relations of weak and almost orthogonality of complete types, as well as with such concepts as quasirationality and definability of types has been shown.

Non-orthogonality of types plays an important role in counting the number of non-isomorphic models, in particular for o-minimal and quite o-minimal theories. For weakly o-minimal theories a classification of 1-types and the notions of non-orthogonality have been previously introduced, but they have not been considered in the context of non-complete 1-types. The relevance of the results of the article is in the future application of the introduced types of non-orthogonality to investigation of the class of ordered stable theories.

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