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Algebras of Binary Isolating Formulas for Homomorphic Product Theories

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Abstract.

Algebras of distributions of binary isolating and semi-isolating formulas are objects that are derived for a given theory, and they specify the relations between binary formulas of the theory. These algebras are useful for classifying theories and determining which algebras correspond to which theories. In the paper, we discuss algebras of binary formulas for strong products and provide Cayley tables for these algebras. On the basis of constructed tables we formulate a theorem describing all algebras of distributions of binary formulas for the theories of strong multiplications of regular polygons on an edge. In addition, we shows that these algebras can be absorbed by simplex algebras, which simplify the study of that theory and connect it with other algebraic structures. This concept is a useful tool for understanding the relationships between binary formulas of a theory.

Keywords: algebra of binary isolating formulas, homomorphic product, model theory, Cayley tables

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Научная статья

Алгебры бинарных изолирующих формул для теорий гомоморфных произведений

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Аннотация. Показаны алгебры распределений бинарных изолирующих и полуизолирующих формул — объекты, которые являются производными для данной теории, они указывают отношения между бинарными формулами теории. Эти алгебры полезны для классификации теорий и определения того, какие алгебры соответствуют каким теориям. Рассматриваются алгебры бинарных формул для гомоморфных произведений и приводятся таблицы Кэли для этих алгебр. На основе построенных таблиц формулируются теоремы, описывающие все алгебры распределений бинарных формул для теорий гомоморфных умножений правильных многоугольников на ребро и симплекса на правильные многоугольники. Эта концепция является полезным инструментом для понимания отношений между бинарными формулами данной теории.

Ключевые слова: алгебра бинарных изолирующих формул, гомоморфное произведение, теория моделей, таблицы Кэли

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1. Introduction

This paper continues the investigation of distribution algebras associated with binary isolation formulae [1–3; 6; 10; 12], which have been extensively studied for various theories, including unary ones, Cartesian products of graphs, Archimedean solids, strong and zig-zag products, and Mychelski graphs, as well as for polygonometric theories and their generalizations for semi-isolation formulae [4; 5; 11]. These distribution algebras serve as derived structures for these theories, reflecting binary relationships between types and their realizations. A comprehensive analysis of the general properties of such algebras can be found in the work by [9], which characterizes a specific class of these structures.

The article presents multiplication tables for a variety of examples that consider the complex interaction between these algebraic structures and the underlying theories. Thanks to the study of algebras related to binary isolating formulas, it becomes possible to classify theories of this class through the prism of derived algebraic constructions. This paper proposes such a classification for homomorphic product graphs, where the products of graphs are limited to sets of realizations belonging to a fixed type. These implementations serve as valuable tools in the process of reconstructing, to some extent, the binary structure inherent in this 1-type.

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2. Algebras of binary isolating formulas for homomorphic product theories

Homomorphic products of graphs are operations that allow for the construction of new graphs from existing ones by establishing relationships between the vertices of one graph and the sets of labels of another graph. This process aids in the study of various structural properties of graphs and their applications in information theory, coding theory, and optimization. These products illustrate how the properties of one graph can be transferred to another, making them a useful tool for exploring more complex network structures.

In particular, homomorphic products find application in graph coloring problems and related optimization tasks. In this area, the concept of label sets is often used to construct mappings that preserve or transmit certain properties, such as the chromatic number or the stability of the graph. This enables an in-depth exploration of graph properties on a more intuitive level.

Homomorphic products of graphs can be paralleled with model theory approaches through the use of labels to denote properties that define how one structure can be mapped to another. Label sets serve to model specific properties of nodes and edges, simplifying the understanding of how one representation of a structure can be transformed into another through homomorphisms.

Labels can roughly be considered as elements that trace which properties have been preserved when transitioning to a homomorphic product. This brings the concepts used in model theory for describing structure closer to graph modeling. Here, the labels of nodes and edges play a key role, providing a connection between theoretical-model aspects and practical analysis of graphs.

Definition 1. Homomorphic product of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with homomorphism $f : V_1 \to V_2$ is a graph $H = (V_H, E_H)$ where: set of vertices $V_H = \{(v, u) \mid v \in V_1, u \in V_2\}$; set of edges $E_H = E_1^H \cup E_2^H$, where

$$\begin{split} \vec{E}_1^H &= \{((v_1, u_1), (v_2, u_2)) \mid (v_1, v_2) \in E_1, u_1, u_2 \in V_2, f(v_1) = f(v_2)\}, \\ E_2^H &= \{((v, u_1), (v, u_2)) \mid v \in V_1, u_1, u_2 \in V_2, f(v) = u_1 u_2\}. \end{split}$$

Subsequently, we delve into the realm of algebras derived through the process of homomorphic multiplication of edges within graphs constructed from polygons.

The algebras for the homomorphic product of graphs of edge to edge $H \times H$, with a set of labels $\rho_{\nu(p)} = \{0, 1\}$ they are set by the following multiplication rules:

$$0 \cdot 0 = \{0\},\$$

$$0 \cdot 1 = \{1\}, \\ 1 \cdot 1 = \{0, 1\}, \\ 1 \cdot 0 = \{1\}.$$

Algebra for the homomorphic product of an edge graph by a triangle $H \times T$ with labels $\rho_{\nu(p)} = \{0, 1, 2, 3\}$, it will be set by the following products of labels:

$$\begin{array}{ll} 0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\ 1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0, 2\}, \\ 1 \cdot 2 = \{0, 1\}, & 1 \cdot 3 = \{0, 2\}, \\ 2 \cdot 0 = \{2\}, & 2 \cdot 1 = \{1, 2\}, \\ 2 \cdot 2 = \{0, 1\}, & 2 \cdot 3 = \{1, 3\}, \\ 3 \cdot 0 = \{3\}, & 3 \cdot 1 = \{0, 2\}, \\ 3 \cdot 2 = \{1, 3\}, & 3 \cdot 3 = \{0, 2\}. \end{array}$$

The product of the graph of an edge and a square $H \times Q$ corresponds to two identical algebras of isolating formulas consisting of $\rho_{\nu(p)} = \{0, 1, 2\}$ labels, the multiplication of which is given below:

$$\begin{array}{ll} 0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\ 1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0, 2\}, \\ 1 \cdot 2 = \{1, 3\}, & 2 \cdot 0 = \{2\}, \\ 2 \cdot 1 = \{1, 3\}, & 2 \cdot 2 = \{0, 2\}. \end{array}$$

The cardinality of the label set in an algebra is directly proportional to the diameter of the corresponding graph.

Remark 1. If the diameters of two graphs obtained from the homomorphic product of n-graphs on an edge are the same, then the algebras of binary isolating formulas for these products will be isomorphic.

Consider the following n-gon, in particular, a pentagon, and multiply it by the graph of the edge $H \times P$. As a result, we get an algebra with labels $\rho_{\nu(p)} = \{0, 1, 2, 3, 4, 5\}$, given their works:

$$\begin{array}{lll} 0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\ 1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0, 2\}, \\ 1 \cdot 2 = \{1, 3\}, & 1 \cdot 3 = \{0, 2, 4\}, \\ 1 \cdot 4 = \{1, 3, 5\}, & 1 \cdot 5 = \{0, 2, 4\}, \\ 2 \cdot 0 = \{2\}, & 2 \cdot 1 = \{1, 3\}, \\ 2 \cdot 2 = \{0, 2, 4\}, & 2 \cdot 3 = \{1, 3, 5\}, \\ 2 \cdot 4 = \{0, 2, 4\}, & 2 \cdot 5 = \{1, 3, 5\}, \\ 3 \cdot 0 = \{3\}, & 3 \cdot 1 = \{0, 2, 4\}, \\ 3 \cdot 2 = \{1, 3, 5\}, & 3 \cdot 3 = \{0, 2, 4\}, \end{array}$$

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When we produce an edge graph and a hexagon $H \times Q$, we get two identical algebras of isolating formulas consisting of labels $\rho_{\nu(p)} = \{0, 1, 2\}$, the multiplication of which is described below:

$$\begin{array}{ll} 0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\ 1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0, 2\}, \\ 1 \cdot 2 = \{1, 3\}, & 1 \cdot 3 = \{0, 2\}, \\ 2 \cdot 0 = \{2\}, & 2 \cdot 1 = \{1, 3\}, \\ 2 \cdot 2 = \{0, 2\}, & 2 \cdot 3 = \{1, 3\}, \\ 3 \cdot 0 = \{3\}, & 3 \cdot 1 = \{0, 2\}, \\ 3 \cdot 2 = \{1, 3\}, & 3 \cdot 3 = \{0, 2\}. \end{array}$$

When we multiply the graph of an edge by the graph of a heptagon $H \times He$, the algebra of isolating formulas consisting of labels $\rho_{\nu(p)} = \{0, 1, 2, 3, 4, 5, 6, 7\}$, the multiplication of which is described below:

$0 \cdot 1 = \{1\},\$	$0\cdot 6 = \{6\},$
$1 \cdot 1 = \{0, 2\},$	$1 \cdot 6 = \{1, 3, 5, 7\},\$
$1 \cdot 3 = \{0, 2, 4\},$	$2 \cdot 6 = \{0, 2, 4, 6\},\$
$1 \cdot 5 = \{0, 2, 4, 6\},\$	$3 \cdot 6 = \{1, 3, 5, 7\},\$
$2 \cdot 1 = \{1, 3\},$	$4 \cdot 6 = \{0, 2, 4, 6\},\$
$2 \cdot 3 = \{1, 3, 5\},\$	$5 \cdot 6 = \{1, 3, 5, 7\},\$
$2 \cdot 5 = \{1, 3, 5, 7\},\$	$6 \cdot 6 = \{0, 2, 4, 6\},\$
$3 \cdot 1 = \{0, 2, 4\},$	$7 \cdot 6 = \{1, 3, 5, 7\},\$
$3 \cdot 3 = \{0, 2, 4, 6\},\$	$0\cdot 7 = \{7\},$
$3 \cdot 5 = \{0, 2, 4, 6\},\$	$1 \cdot 7 = \{0, 1, 4, 6\},\$
$4 \cdot 1 = \{1, 3, 5\},$	$2 \cdot 7 = \{1, 3, 5, 7\},\$
$4 \cdot 3 = \{1, 3, 5, 7\},\$	$3 \cdot 7 = \{0, 2, 4, 6\},\$
$4 \cdot 5 = \{1, 3, 5, 7\},\$	$4 \cdot 7 = \{1, 3, 5, 7\},\$
$5 \cdot 1 = \{0, 2, 4, 6\},\$	$5 \cdot 7 = \{0, 2, 4, 6\},\$
$5 \cdot 3 = \{0, 2, 4, 6\},\$	$6 \cdot 7 = \{1, 3, 5, 7\},\$
$5 \cdot 5 = \{0, 2, 4, 6\},\$	$7 \cdot 7 = \{0, 2, 4, 6\}.$
	$0 \cdot 1 = \{1\}, \\1 \cdot 1 = \{0, 2\}, \\1 \cdot 3 = \{0, 2, 4\}, \\1 \cdot 5 = \{0, 2, 4, 6\}, \\2 \cdot 1 = \{1, 3\}, \\2 \cdot 3 = \{1, 3, 5\}, \\2 \cdot 5 = \{1, 3, 5, 7\}, \\3 \cdot 1 = \{0, 2, 4\}, \\3 \cdot 3 = \{0, 2, 4, 6\}, \\3 \cdot 5 = \{0, 2, 4, 6\}, \\4 \cdot 1 = \{1, 3, 5, 7\}, \\4 \cdot 3 = \{1, 3, 5, 7\}, \\4 \cdot 5 = \{1, 3, 5, 7\}, \\5 \cdot 1 = \{0, 2, 4, 6\}, \\5 \cdot 3 = \{0, 2, 4, 6\}, \\5 \cdot 5 = \{0, 2, 4, 6\}, \end{cases}$

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The analysis of the examples discussed above reveals a consistent pattern and a clear correlation between the algebras and the diameter of the graph, allowing us to characterize algebras with n labels in a systematic manner.

For the homomorphic product of graphs $G \times H$, the algebra is denoted by $\mathfrak{H}\mathfrak{p}_{\mathfrak{o}}$, which is defined as follows. Multiple labels $\{1, 2, 3, \ldots, n\}$, where n is an even number equal to the diameter of the graph obtained by multiplying the graphs G and H. The algebra $\mathfrak{H}\mathfrak{p}_{\mathfrak{e}}$ is given by the following table:

For the homomorphic product of graphs $G \times H$, the algebra is denoted by $\mathfrak{H}\mathfrak{p}_{\mathfrak{o}}$, which is defined as follows. Multiple labels $\{1, 2, 3, \ldots, n\}$, where n is an odd number equal to the diameter of the graph obtained by multiplying the graphs G and H. The algebra $\mathfrak{H}\mathfrak{p}_{\mathfrak{o}}$ is given by the following table:

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Theorem 1. If T is the theory of the homomorphic product of an edge on a graph, and \mathfrak{B} is the algebra of binary isolating formulas T, then the algebra \mathfrak{B} is defined by one of the following two options: $\mathfrak{Hp}_{\mathfrak{o}}$ or $\mathfrak{Hp}_{\mathfrak{o}}$.

Let's take a closer look at the process of multiplying a simplex into polygonal graphs. In early studies devoted to the study of product algebras of graphs, it was found that the algebra of simplices is able to absorb other algebras. Moreover, when producing graphs by a simplex, the resulting graph will always contain a simplex, which leads to the absorption of its algebra by the algebra of simplices.

Algebra for the homomorphic product of a triangle graph by a triangle $T \times T$ with labels $\rho_{\nu(p)} = \{0, 1, 2\}$, the resulting algebra of binary isolating formulas is equivalent to the algebra of simplices [2], it will be set by the following products of labels:

$$0 \cdot 0 = \{0\}, \qquad 0 \cdot 1 = \{1\}, \\1 \cdot 0 = \{1\}, \qquad 1 \cdot 1 = \{0, 1, 2\}, \\1 \cdot 2 = \{0, 1, 2\}, \qquad 2 \cdot 0 = \{2\}, \\2 \cdot 1 = \{0, 1, 2\}, \qquad 2 \cdot 2 = \{0, 1, 2\}$$

The following algebras for homomorphic product theories of graphs will coincide, since the diameters of the resulting graphs are the same, these are the product algebras of the simplex into a quadrilateral, pentagon, hexagon and heptagon. This algebra has $\rho_{\nu(p)} = \{0, 1, 2, 3\}$ labels, and the following multiplication rules:

$$\begin{array}{ll} 0 \cdot 0 = \{0\}, & 0 \cdot 1 = \{1\}, \\ 1 \cdot 0 = \{1\}, & 1 \cdot 1 = \{0, 2\}, \\ 1 \cdot 2 = \{0, 1\}, & 1 \cdot 3 = \{0, 2\}, \\ 2 \cdot 0 = \{2\}, & 2 \cdot 1 = \{1, 2\}, \\ 2 \cdot 2 = \{0, 1\}, & 2 \cdot 3 = \{1, 3\}, \\ 3 \cdot 0 = \{3\}, & 3 \cdot 1 = \{0, 2\}, \\ 3 \cdot 2 = \{1, 3\}, & 3 \cdot 3 = \{0, 2\}. \end{array}$$

The algebras for the theories of the homomorphic product of simplex graphs on an octagon and a ninagon will be isomorphic, the algebra has $\rho_{\nu(p)} = \{0, 1, 2, 3, 4\}$ labels, and the following multiplication rules:

$$4 \cdot 0 = \{4\} \qquad 4 \cdot 1 = \{1,3\} \qquad 4 \cdot 2 = \{0,2,4\}$$
$$4 \cdot 3 = \{1,3,5\} \qquad 4 \cdot 4 = \{0,2,4\}$$

The diameters of the graphs when the simplex is produced by the graphs of regular polygons after reaching a diameter equal to three, will be identical for both results of the product. So, for example, the product of a simplex into an octagon and a ninagon will give us a graph with a diameter equal to four, which, in turn, indicates the isomorphism of the corresponding algebras. Similarly, the product of a simplex by a decagon and an elevensided one will give us a graph with a diameter equal to five, and so on.

If we continue multiplying the simplex by regular polygons, then the algebra of the product of the graph of the simplex by an n-gon will be isomorphic to the algebra $\mathfrak{Hp}_{\mathfrak{e}}$ or $\mathfrak{Hp}_{\mathfrak{o}}$, depending on the parity of the diameter of the graph from 1.

3. Conclusion

In the presented papers [1–3;6], algebras corresponding to the theories of various graph products were studied. These algebras are derived structures with respect to the original theories. The question of the possibility of restoring the original structures based on these algebras was considered. It has been found that different products of graphs can generate the same algebras of binary formulas. This means that the question of restoring the original theories based on the algebras of binary formulas receives a partial negative answer: different products of graphs can generate identical algebras of binary formulas. This can be seen in the presented algebras in the article, that the algebras for the products of different graphs coincide.

It is noted that for a homomorphic product of a simplex on graphs of regular *n*-gons at n > 3, absorption by the algebra of simplices is not observed, as was the case with tensor, root and zigzag products. Upon closer examination, such algebras are described by general rules for homomorphic products of graphs.

According to Theorem 1, when graphs are homomorphically multiplied by regular n-gons, they can be characterized by one of two algebras $\mathfrak{Hp}_{\mathfrak{o}}$ or $\mathfrak{Hp}_{\mathfrak{e}}$, and these algebras are isomorphic to algebras corresponding to tensor theories, zigzags and root products of graphs.

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