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Algebras of Binary Formulas for Weakly Circularly Minimal Theories: Monotonic-to-left Case

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Abstract. This article concerns the notion of weak circular minimality being a variant of o-minimality for circularly ordered structures. Algebras of binary isolating formulas are studied for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank greater than 1 with a trivial definable closure having a non-trivial monotonic-to-left function acting on the universe of a structure. On the basis of the study, the authors present a description of these algebras. It is shown that for this case there exist only non-commutative algebras. A strict m-deterministicity of such algebras for some natural number m is also established.

Keywords: algebra of binary formulas, $\aleph_0$-categorical theory, weak circular minimality, circularly ordered structure, convexity rank

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Научная статья

Алгебры бинарных формул для слабо циклически минимальных теорий: монотонный влево случай

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Аннотация. Рассматривается понятие слабой циклической минимальности, являющееся вариантом o-минимальности для циклически упорядоченных структур. Исследуются алгебры бинарных изолирующих формул для $\aleph_0$-категоричных 1-транзитивных непримитивных слабо циклически минимальных теорий ранга выпуклости большего 1 с тривиальным определимым замыканием, имеющих нетривиальную монотонную влево функцию, действующую на основном множестве структуры. Представлено описание этих алгебр. Показано, что для данного случая существуют только некоммутативные алгебры. Также устанавливается строгая m-детерминированность таких алгебр для некоторого натурального числа m.

Ключевые слова: алгебра бинарных формул, $\aleph_0$-категоричная теория, слабая циклическая минимальность, циклически упорядоченная структура, ранг выпуклости

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1. Preliminaries

Let L be a countable first-order language. Throughout we consider L-structures and assume that L contains a ternary relational symbol K, interpreted as a circular order in these structures (unless otherwise stated).

The *circular order* is described by a ternary relation K satisfying the following conditions:

(co1) $\forall x\forall y\forall z(K(x,y,z)\to K(y,z,x))$;
(co2) $\forall x\forall y\forall z(K(x,y,z)\land K(y,x,z)\Leftrightarrow x=y\lor y=z\lor z=x)$;
(co3) $\forall x\forall y\forall z(K(x,y,z)\to \forall t[K(x,y,t)\lor K(t,y,z)])$;
(co4) $\forall x\forall y\forall z(K(x,y,z)\lor K(y,x,z))$.

The following observation relates linear and circular orders.

Proposition 1. [4] *If $\langle \mathcal{M},\le\rangle$ is a linear ordering and K is the ternary relation derived from $\le$ by the rule*

$$K(x,y,z) :\Leftrightarrow (x\le y\le z)\lor(z\le x\le y)\lor(y\le z\le x)$$

then K is a circular order relation on $\mathcal{M}$.

The notion of *weak circular minimality* was studied initially in [14]. Let $A \subseteq M$, where $\mathcal{M}$ is a circularly ordered structure. The set A is called *convex* if for any $a, b \in A$ the following property is satisfied: for any $c \in M$ with $K(a, c, b)$, $c \in A$ holds, or for any $c \in M$ with $K(b, c, a)$, $c \in A$ holds. A *weakly circularly minimal structure* is a circularly ordered structure $\mathcal{M} = \langle M, K, \ldots \rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in M. The study of weakly circularly minimal structures was continued in the papers [15]– [21].

Let $\mathcal{M}$ be an $\aleph_0$-categorical weakly circularly minimal structure, $G := \mathrm{Aut}(\mathcal{M})$. Following the standard group theory terminology, the group G is called *k-transitive* if for any pairwise distinct $a_1, a_2, \ldots, a_k \in M$ and pairwise distinct $b_1, b_2, \ldots, b_k \in M$ there exists $g \in G$ such that $g(a_1) = b_1, g(a_2) = b_2, \ldots, g(a_k) = b_k$. A *congruence* on $\mathcal{M}$ is an arbitrary G-invariant equivalence relation on $\mathcal{M}$. The group G is called *primitive* if G is 1-transitive and there are no non-trivial proper congruences on $\mathcal{M}$.

Definition 1. (1) $K_0(x, y, z) := K(x, y, z) \land y \neq x \land y \neq z \land x \neq z$.

(2) $K'(u_1, \ldots, u_n)$ denotes a formula saying that all subtuples of the tuple $\langle u_1, \ldots, u_n \rangle$ having the length 3 (in ascending order) satisfy K, i.e.

$$K'(u_1, \ldots, u_n) := \land \land_{1 \leq i < j < m \leq n} K(u_i, u_j, u_m) \land K(u_{n-1}, u_n, u_1)$$

$$\land K(u_n, u_1, u_2).$$

Similar notations are used for K_0.

(3) Let A, B, C be disjoint convex subsets of a circularly ordered structure $\mathcal{M}$. We write $K(A, B, C)$ if for any $a, b, c \in M$ with $a \in A$, $b \in B$, $c \in C$ we have $K(a, b, c)$.

Further we need the notion of the definable completion of a circularly ordered structure, introduced in [14]. Its linear analog was introduced in [24]. A *cut* $C(x)$ in a circularly ordered structure $\mathcal{M}$ is maximal consistent set of formulas of the form $K(a, x, b)$, where $a, b \in M$. A cut is said to be *algebraic* if there exists $c \in M$ that realizes it. Otherwise, such a cut is said to be *non-algebraic*. Let $C(x)$ be a non-algebraic cut. If there is some $a \in M$ such that either for all $b \in M$ the formula $K(a, x, b) \in C(x)$, or for all $b \in M$ the formula $K(b, x, a) \in C(x)$, then $C(x)$ is said to be *rational*. Otherwise, such a cut is said to be *irrational*. A *definable cut* in M is a cut $C(x)$ with the following property: there exist $a, b \in M$ such that $K(a, x, b) \in C(x)$ and the set $\{c \in M \mid K(a, c, b) \text{ and } K(a, x, c) \in C(x)\}$ is definable. The *definable completion* $\overline{M}$ of a structure $\mathcal{M}$ consists of M together with all definable cuts in $\mathcal{M}$ that are irrational (essentially $\overline{M}$ consists of endpoints of definable subsets of the structure $\mathcal{M}$).

Definition 2. [14] Let $F(x,y)$ be an L-formula such that $F(M,b)$ is convex infinite co-infinite for each $b \in M$. Let $F^l(y)$ be the formula saying y is a left endpoint of $F(M,y)$:

$$\exists z_1 \exists z_2 [K_0(z_1, y, z_2) \land \forall t_1 (K(z_1, t_1, y) \land t_1 \neq y \to \neg F(t_1, y)) \land$$

$$\forall t_2 (K(y, t_2, z_2) \land t_2 \neq y \to F(t_2, y))].$$

We say that $F(x,y)$ is *convex-to-right* if

$$M \models \forall y \forall x [F(x,y) \to F^l(y) \land \forall z (K(y,z,x) \to F(z,y))].$$

Consider $F(M,a)$ for arbitrary $a \in M$. In general, $F(M,a)$ has no the right endpoint in M. For example, if $dcl(a) = \{a\}$ holds for some $a \in M$ then for any convex-to-right formula $F(x,y)$ and any $a \in M$ the formula $F(M,a)$ has no the right endpoint in M. We write $f(y) :=$ rend $F(M,y)$, assuming that $f(y)$ is the right endpoint of the set $F(M,y)$ that lies in general in the definable completion $\overline{M}$ of $\mathcal{M}$. Then f is a function mapping $\mathcal{M}$ in $\overline{M}$.

Definition 3. Let $E(x,y)$ be an $\emptyset$–definable equivalence relation partitioning M into infinite convex classes. Suppose that y lies in $\overline{M}$ (non-obligatory in M). Then

$$E^*(x,y) := \exists y_1 \exists y_2 [y_1 \neq y_2 \land \forall t (K(y_1, t, y_2) \to E(t,x)) \land K_0(y_1, y, y_2)].$$

Let $\mathcal{M}$, $\mathcal{N}$ be circularly ordered structures. The 2-*reduct* of $\mathcal{M}$ is a circularly ordered structure with the same universe of $\mathcal{M}$ and consisting of predicates for each $\emptyset$-definable relation on $\mathcal{M}$ of arity ≤ 2 as well as of the ternary predicate K for the circular order, but does not have other predicates of arities more than two. We say that the structure $\mathcal{M}$ is *isomorphic* to $\mathcal{N}$ *up to binarity* or *binarily isomorphic to* $\mathcal{N}$ if the 2-reduct of $\mathcal{M}$ is isomorphic to the 2-reduct of $\mathcal{N}$.

Let f be a unary function from $\mathcal{M}$ to $\overline{M}$. We say that f is *monotonic-to-right (left) on* M if it preserves (reverses) the relation K_0, i.e. for any $a, b, c \in M$ such that $K_0(a,b,c)$, we have $K_0(f(a), f(b), f(c))$ $(K_0(f(c), f(b), f(a)))$.

Let $F'(x,y) := \exists t [F(t,y) \land F(x,t)]$, where $F(x,y)$ is a convex-to-right formula. Denote by $f^2(y)$ the right endpoint of $F'(M,y)$ in $\overline{M}$.

Lemma 1. [15] *Let $\mathcal{M}$ be an $\aleph_0$–categorical 1-transitive weakly circularly minimal structure, $F(x,y)$ be a convex-to-right formula so that $f(y) :=$ rend $F(M,y)$ is monotonic-to-left on M. Then $f^2(a) = a$ for all $a \in M$.*

The following definition can be used in a circular ordered structure as well.

Definition 4. [22], [23] Let T be a weakly o-minimal theory, $\mathcal{M}$ be a sufficiently saturated model of T, $A \subseteq M$. *The rank of convexity of the set* A ($RC(A)$) is defined as follows:

1) $RC(A) = 0$ if A is finite and non-empty.

2) $RC(A) \geq 1$ if A is infinite.

3) $RC(A) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:

– For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;

– For every $i \in \omega$, $RC(E(M, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A.

4) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α, we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A)) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, i.e. $RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$.

The following theorem characterizes up to binarity $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structures $\mathcal{M}$ of convexity rank greater than 1 having both a trivial definable closure and a convex-to-right formula $R(x, y)$ such that $r(y) :=$ rend $R(M, y)$ is monotonic-to-left on M:

Theorem 1. [15] *Let $\mathcal{M}$ be an $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank greater than 1 such that $dcl(a) = \{a\}$ for some $a \in M$. Suppose that there exists a convex-to-right formula $R(x, y)$ such that $r(y) :=$ rend $R(M, y)$ is monotonic-to-left on M. Then $\mathcal{M}$ is isomorphic up to binarity to*

$$\mathcal{M}'_{s,2,2} := \langle M, K^3, E_1^2, E_2^2, \ldots, E_s^2, E_{s+1}^2, R^2 \rangle,$$

where $\mathcal{M}$ is a circularly ordered structure, M is densely ordered, $s \geq 1$; E_{s+1} is an equivalence relation partitioning M into two infinite convex classes without endpoints; E_i for every $1 \leq i \leq s$ is an equivalence relation partitioning every E_{i+1}-class into infinitely many infinite convex E_i-subclasses without endpoints so that the induced order on E_i-subclasses is dense without endpoints; $R(M, a)$ has no right endpoint in M and $r^2(a) = a$ for all $a \in M$, where $r^2(y) := r(r(y))$.

Algebras of binary formulas are a tool for describing relationships between elements of the sets of realizations of an one-type at the binary level with respect to the superposition of binary definable sets. A *binary isolating formula* is a formula of the form $\varphi(x, y)$ such that for some parameter a the formula $\varphi(a, y)$ isolates a complete type in $S(\{a\})$. The concepts and notations related to these algebras can be found in the papers [25; 26]. In recent years, algebras of binary formulas have been studied intensively and have been continued in the works [1], [3], [7]– [13].

In [9] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories with a primitive automorphism group. In [10] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [11]– [12] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [13] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a trivial definable closure. Here we describe algebras of binary isolating formulas for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and a trivial definable closure.

2. Results

Definition 5. [26] Let $p \in S_1(\emptyset)$ be non-algebraic. The algebra $\mathcal{P}_{\nu(p)}$ is said to be *deterministic* if $u_1 \cdot u_2$ is a singleton for any labels $u_1, u_2 \in \rho_{\nu(p)}$.

Generalizing the last definition, we say that the algebra $\mathcal{P}_{\nu(p)}$ is *m-deterministic* if the product $u_1 \cdot u_2$ consists of at most m elements for any labels $u_1, u_2 \in \rho_{\nu(p)}$. We also say that an m-deterministic algebra $\mathcal{P}_{\nu(p)}$ is *strictly m-deterministic* if it is not $(m-1)$-deterministic.

Example 1. Consider the structure $\mathcal{M}'_{1,2,2} := \langle M, K^3, E_1^2, E_2^2, R^2 \rangle$ from Theorem 1 with the condition that the function $r(y) :=$ rend $R(M, y)$ is monotonic-to-left on M. Here $E_2(x, y)$ is an equivalence relation partitioning M into two infinite convex classes.

We assert that $Th(\mathcal{M}'_{1,2,2})$ has nine binary isolating formulas:

$$\theta_0(x,y) := x = y, \quad \theta_1(x,y) := K_0(x,y,r(x)) \wedge E_1(x,y),$$

$$\theta_2(x,y) := K_0(x,y,r(x)) \wedge E_2(x,y) \wedge \neg E_1(x,y),$$

$$\theta_3(x,y) := K_0(x,y,r(x)) \wedge \neg E_2(x,y) \wedge \neg E_1^*(y,r(x)),$$

$$\theta_4(x,y) := K_0(x,y,r(x)) \wedge \neg E_2(x,y) \wedge E_1^*(y,r(x)),$$

$$\theta_5(x,y) := K_0(r(x),y,x) \wedge \neg E_2(x,y) \wedge E_1^*(y,r(x)),$$

$$\theta_6(x,y) := K_0(r(x),y,x) \wedge \neg E_2(x,y) \wedge \neg E_1^*(y,r(x)),$$

$$\theta_7(x,y) := K_0(r(x),y,x) \wedge E_2(x,y) \wedge \neg E_1(x,y),$$

$$\theta_8(x,y) := K_0(r(x),y,x) \wedge E_1(x,y),$$

and the following holds for any $a \in M$:

$$K_0'(\theta_0(a,M),\theta_1(a,M),\theta_2(a,M),\theta_3(a,M),\ldots,\theta_7(a,M),\theta_8(a,M)).$$

Define labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y), \text{ where } 0 \le k \le 8.$$

It easy to check that for the algebra $\mathfrak{P}_{\mathcal{M}'_{1,2,2}}$ the following equalities hold:
$0 \cdot k = k \cdot 0 = \{k\}$ for each $0 \le k \le 8$,
$1 \cdot k = \{k\}$ for each $1 \le k \le 4$, $1 \cdot 5 = \{4,5\}$, $1 \cdot 6 = \{6\}$, $1 \cdot 7 = \{7\}$, and $1 \cdot 8 = \{0,1,8\}$,
$2 \cdot 1 = \{2\}$, $2 \cdot 2 = \{2\}$, $2 \cdot 3 = \{3\}$, $2 \cdot 4 = \{3\}$, $2 \cdot 5 = \{3\}$,
$2 \cdot 6 = \{3,4,5,6\}$, $2 \cdot 7 = \{0,1,2,7,8\}$, and $2 \cdot 8 = \{2\}$,
$3 \cdot 1 = \{3\}$, $3 \cdot 2 = \{3,4,5,6\}$, $3 \cdot 3 = \{0,1,2,7,8\}$,
$3 \cdot 4 = \{2\}$, $3 \cdot 5 = \{2\}$, $3 \cdot 6 = \{2\}$, $3 \cdot 7 = \{3\}$, $3 \cdot 8 = \{3\}$,
$4 \cdot 1 = \{4,5\}$, $4 \cdot 2 = \{6\}$, $4 \cdot 3 = \{7\}$, $4 \cdot 4 = \{0,1,8\}$,
$4 \cdot 5 = \{1\}$, $4 \cdot 6 = \{2\}$, $4 \cdot 7 = \{3\}$, $4 \cdot 8 = \{4\}$,
$5 \cdot 1 = \{5\}$, $5 \cdot 2 = \{6\}$, $5 \cdot 3 = \{7\}$, $5 \cdot 4 = \{8\}$, $5 \cdot 5 = \{0,1,8\}$,
$5 \cdot 6 = \{2\}$, $5 \cdot 7 = \{3\}$, $5 \cdot 8 = \{4,5\}$,
$6 \cdot 1 = \{6\}$, $6 \cdot 2 = \{6\}$, $6 \cdot 3 = \{7\}$, $6 \cdot 4 = \{7\}$, $6 \cdot 5 = \{7\}$,
$6 \cdot 6 = \{0,1,2,7,8\}$, $6 \cdot 7 = \{3,4,5,6\}$, $6 \cdot 8 = \{6\}$,
$7 \cdot 1 = \{7\}$, $7 \cdot 2 = \{0,1,2,7,8\}$, $7 \cdot 3 = \{3,4,5,6\}$, $7 \cdot 4 = \{6\}$, $7 \cdot 5 = \{6\}$,
$7 \cdot 6 = \{6\}$, $7 \cdot 7 = \{7\}$, $7 \cdot 8 = \{7\}$,
$8 \cdot 1 = \{0,1,8\}$, $8 \cdot 2 = \{2\}$, $8 \cdot 3 = \{3\}$, $8 \cdot 4 = \{4,5\}$, $8 \cdot 5 = \{5\}$,
$8 \cdot 6 = \{6\}$, $8 \cdot 7 = \{7\}$, $8 \cdot 8 = \{8\}$.
According to these equalities, the algebra $\mathfrak{P}_{\mathcal{M}'_{1,2,2}}$ is strictly 5-deterministic and not commutative.

The following theorem describes the algebra $\mathfrak{P}_{\mathcal{M}'_{s,2,2}}$ for every $s \ge 1$ by giving the Cayley table for this algebra.

Theorem 2. *The algebra $\mathfrak{P}_{\mathcal{M}'_{s,2,2}}$ of binary isolating formulas with monotonic-to-left function r has $4s+5$ labels, is strictly $(2s+3)$-deterministic and not commutative for every $s \ge 1$.*

Proof. We assert that the algebra $\mathfrak{P}_{\mathcal{M}'_{s,2,2}}$ has $4s+5$ binary isolating formulas: $\theta_0(x,y) := x = y$, $\quad \theta_1(x,y) := K_0(x,y,r(x)) \wedge E_1(x,y)$,

$$\theta_{l_1}(x,y) := K_0(x,y,r(x)) \wedge E_{l_1}(x,y) \wedge \neg E_{l_1-1}(x,y), \text{ where } 2 \le l_1 \le s+1$$

$$\theta_{s+2}(x,y) := K_0(x,y,r(x)) \wedge \neg E_{s+1}(x,y) \wedge \neg E^*_s(y,r(x)),$$

$$\theta_{l_2}(x,y) := K_0(x,y,r(x)) \wedge \neg E_{s+1}(x,y) \wedge E^*_{2s+3-l_2}(y,r(x)) \wedge$$
$$\neg E^*_{2s+2-l_2}(y,r(x)), \text{ where } s+3 \le l_2 \le 2s+1,$$

$$\theta_{2s+2}(x,y) := K_0(x,y,r(x)) \wedge \neg E_{s+1}(x,y) \wedge E_1^*(y,r(x)),$$

$$\theta_{2s+3}(x,y) := K_0(r(x),y,x) \wedge \neg E_{s+1}(x,y) \wedge E_1^*(y,r(x)),$$

$$\theta_{l_3}(x,y) := K_0(r(x),y,x) \wedge \neg E_{s+1}(x,y) \wedge E^*_{l_3-(2s+2)}(y,r(x))\wedge$$

$$\neg E^*_{l_3-(2s+3)}(y,r(x)), \text{ where } 2s+4 \le l_3 \le 3s+2,$$

$$\theta_{3s+3}(x,y) := K_0(r(x),y,x) \wedge \neg E_{s+1}(x,y) \wedge \neg E_s^*(y,r(x)),$$

$$\theta_{l_4}(x,y) := K_0(r(x),y,x) \wedge E_{4s+5-l_4}(x,y) \wedge \neg E_{4s+4-l_4}(x,y),$$

$$\text{where } 3s+4 \le l_4 \le 4s+3,$$

$$\theta_{4s+4}(x,y) := K_0(r(x),y,x) \wedge E_1(x,y).$$

Thus, we have $2+s+1+(s-1)+2+(s-1)+1+s+1 = 4s+5$ binary isolating formulas. Moreover, we have defined the formulas so that for any $a \in M$ the following holds:

$$K_0'(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \ldots, \theta_{4s+3}(a,M), \theta_{4s+4}(a,M)).$$

Prove now that the algebra $\mathfrak{P}_{\mathcal{M}'_{s,2,2}}$ is strictly $(2s+3)$-deterministic and not commutative for every $s \ge 1$. Firstly, obviously that $0 \cdot k = k \cdot 0 = \{k\}$ for any $0 \le k \le 4s+4$. Suppose further that $k_1 \ne 0$ and $k_2 \ne 0$.

Consider the following formula: $\exists t[\theta_{k_1}(x,t) \wedge \theta_{k_2}(t,y)]$.

Case 1: $k_1 = 1$. Then we have: $E_1(x,t)$ and $K_0(x,t,r(x))$.

If $k_2 = 1$ then $E_1(t,y)$ and $K_0(t,y,r(t))$. Whence we obtain: $E_1(x,y)$ and $K_0(x,y,r(x))$, i.e. $1 \cdot 1 = \{1\}$.

Suppose now that $2 \le k_2 \le s+1$. Then we have: $K_0(t,y,r(t))$, $E_l(t,y)$ and $\neg E_{l-1}(t,y)$ for some $2 \le l \le s+1$. Consequently, we obtain the following: $K_0(x,y,r(x))$, $E_l(x,y)$ and $\neg E_{l-1}(x,y)$, i.e. $k_1 \cdot k_2 = \{k_2\}$.

Consider the product $k_2 \cdot 1$. We have the following: $K_0(x,t,r(x))$, $E_l(x,t)$ and $\neg E_{l-1}(x,t)$ for some $2 \le l \le s+1$; $K_0(t,y,r(t))$ and $E_1(t,y)$. Whence we obtain: $K_0(x,y,r(x))$, $E_l(x,y)$ and $\neg E_{l-1}(x,y)$, i.e. $k_2 \cdot 1 = \{k_2\}$.

Suppose that $k_2 = s+2$. Then we have: $K_0(t,y,r(t))$, $\neg E_{s+1}(t,y)$ and $\neg E_s^*(y,r(t))$. Consequently, we obtain: $K_0(x,y,r(x))$, $\neg E_{s+1}(x,y)$ and $\neg E_s^*(y,r(x))$, i.e. $k_1 \cdot k_2 = \{k_2\}$.

Consider the product $(s+2) \cdot 1$. We have the following: $K_0(x,t,r(x))$, $\neg E_{s+1}(x,t)$ and $\neg E_s^*(t,r(x))$; $K_0(t,y,r(t))$ and $E_1(t,y)$. Whence we obtain: $K_0(x,y,r(x))$, $\neg E_{s+1}(x,y)$ and $\neg E_s^*(y,r(x))$, i.e. $(s+2) \cdot 1 = \{s+2\}$.

Suppose now that $s+3 \le k_2 \le 2s+1$. Then we have: $K_0(t,y,r(t))$, $\neg E_{s+1}(t,y)$, $E_l^*(y,r(t))$ and $\neg E_{l-1}^*(y,r(t))$ for some $2 \le l \le s$. Whence we obtain: $K_0(x,y,r(x))$, $\neg E_{s+1}(x,y)$, $E_l^*(y,r(x))$ and $\neg E_{l-1}^*(y,r(x))$, i.e. $k_1 \cdot k_2 = \{k_2\}$. We can show similarly that $k_2 \cdot k_1 = \{k_2\}$.

Suppose that $k_2 = 2s+2$. Then we have: $K_0(t,y,r(t))$, $\neg E_{s+1}(t,y)$ and $E_1^*(y,r(t))$. Whence we obtain: $K_0(x,y,r(x))$, $\neg E_{s+1}(x,y)$ and $E_1^*(y,r(x))$, i.e. $k_1 \cdot k_2 = \{k_2\}$.

Consider the product $(2s+2)\cdot 1$. We have the following: $K_0(x,t,r(x))$, $\neg E_{s+1}(x,t)$ and $E_1^*(t,r(x))$; $K_0(t,y,r(t))$ and $E_1(t,y)$. Consequently, we obtain: $\neg E_{s+1}(x,y)$ and $E_1^*(y,r(x))$, but it can be $K_0(x,y,r(x))$ or $K_0(r(x)$, y, $x)$, i.e. $(2s+2)\cdot 1=\{2s+2,2s+3\}$.

Thus, the algebra $\mathfrak{P}_{M'_{s,2,2}}$ is not commutative for every $s\geq 1$.

Suppose that $k_2=2s+3$. Then we have: $K_0(r(t),y,t)$, $\neg E_{s+1}(t,y)$ and $E_1^*(y,r(t))$. Whence we obtain: $\neg E_{s+1}(x,y)$ and $E_1^*(y,r(x))$; moreover, by monotonicity-to-left of the function r we have $K_0(x,t,r(t),r(x))$, i.e. it can be either $K_0(x,y,r(x))$ or $K_0(r(x),y,x)$; consequently, $k_1\cdot k_2=\{2s+2,2s+3\}$. By considering the product $(2s+3)\cdot 1$, we can see that $(2s+3)\cdot 1=\{2s+3\}$.

Suppose now that $2s+4\leq k_2\leq 3s+2$. Then we have: $K_0(r(t),y,t)$, $\neg E_{s+1}(t,y)$, $E_l^*(y,r(t))$ and $\neg E_{l-1}^*(y,r(t))$ for some $2\leq l\leq s$. Whence we obtain: $K_0(r(x),y,x)$, $\neg E_{s+1}(x,y)$, $E_l^*(y,r(x))$ and $\neg E_{l-1}^*(y,r(x))$, i.e. $k_1\cdot k_2=\{k_2\}$. We can show similarly that $k_2\cdot k_1=\{k_2\}$.

Suppose that $k_2=3s+3$. Then we have: $K_0(r(t),y,t)$, $\neg E_{s+1}(t,y)$ and $\neg E_s^*(y,r(t))$. Then we also obtain that $k_1\cdot k_2=\{k_2\}=k_2\cdot k_1$.

Suppose now that $3s+4\leq k_2\leq 4s+3$. Then we have: $K_0(r(t),y,t)$, $E_l(t,y)$ and $\neg E_{l-1}(t,y)$ for some $2\leq l\leq s+1$. Whence we obtain: $K_0(r(x),y,x)$, $E_l(x,y)$ and $\neg E_{l-1}(x,y)$, i.e. $k_1\cdot k_2=\{k_2\}$. We can show similarly that $k_2\cdot k_1=\{k_2\}$.

Suppose that $k_2=4s+4$. Then we have: $K_0(r(t),y,t)$ and $E_1(t,y)$. Whence we obtain: $E_1(x,y)$, wherein y can be anywhere in this class, i.e. $k_1\cdot k_2=\{0,1,4s+4\}$. We can show similarly that $(4s+4)\cdot 1=\{0,1,4s+4\}$.

Case 2: $2\leq k_1\leq s+1$. Then we have: $K_0(x,t,r(x))$, $E_{l_1}(x,t)$ and $\neg E_{l_1-1}(x,t)$ for some $2\leq l_1\leq s+1$.

Suppose that $2\leq k_2\leq s+1$. Then we have: $K_0(t,y,r(t))$, $E_{l_2}(t,y)$ and $\neg E_{l_2-1}(t,y)$ for some $2\leq l_2\leq s+1$. Whence we obtain: if $l_1\leq l_2$ then $K_0(x,y,r(x))$, $E_{l_2}(x,y)$ and $\neg E_{l_2-1}(x,y)$, i.e. $k_1\cdot k_2=\{k_2\}$; if $l_1>l_2$ then $K_0(x,y,r(x))$, $E_{l_1}(x,y)$ and $\neg E_{l_1-1}(x,y)$, i.e. $k_1\cdot k_2=\{k_1\}$.

Suppose that $k_2=s+2$. Then we have: $K_0(t,y,r(t))$, $\neg E_{s+1}(t,y)$ and $\neg E_s^*(y,r(t))$. Consequently, we obtain: $K_0(x,y,r(x))$, $\neg E_{s+1}(x,y)$ and $\neg E_s^*(y,r(x))$, i.e. $k_1\cdot k_2=\{k_2\}$.

Consider the product $(s+2)\cdot k_1$. We have the following: $K_0(x,t,r(x))$, $\neg E_{s+1}(x,t)$ and $\neg E_s^*(t,r(x))$; $K_0(t,y,r(t))$, $E_l(t,y)$ and $\neg E_{l-1}(t,y)$ for some $2\leq l\leq s+1$. Whence we obtain: $\neg E_{s+1}(x,y)$. Since $\neg E_{s+1}(x,t)$, we have: $K_0(x,r(t),t,r(x))$. Consequently, it can be either $K_0(x,y,r(x))$ or $K_0(r(x),y,x)$. Then $(s+2)\cdot k_1=\{s+2,s+3,\ldots,3s+3\}$.

Suppose now that $s+3\leq k_2\leq 2s+1$. Then we have: $K_0(t,y,r(t))$, $\neg E_{s+1}(t,y)$, $E_{l_2}^*(y,r(t))$ and $\neg E_{l_2-1}^*(y,r(t))$ for some $2\leq l_2\leq s$. Whence we obtain: $K_0(x,y,r(x))$ and $\neg E_{s+1}(x,y)$. If $l_1=s+1$ then $k_1\cdot k_2=\{s+2\}$. If $l_1\geq l_2$ and $l_1\neq s+1$ then $E_{l_1}^*(y,r(x))$ and $\neg E_{l_1-1}^*(y,r(x))$, whence $k_1\cdot k_2=\{2s+3-l_1\}$. If $l_1<l_2$ then $E_{l_2}^*(y,r(x))$ and $\neg E_{l_2-1}^*(y,r(x))$,

whence $k_1 \cdot k_2 = \{k_2\}$. By considering the product $k_2 \cdot k_1$, we obtain: $K_0(x,t,r(x))$, $\neg E_{s+1}(x,t)$, $E^*_{l_1}(t,r(x))$ and $\neg E^*_{l_1-1}(t,r(x))$ for some $2 \leq l_1 \leq s$; $K_0(t,y,r(t))$, $E_{l_2}(t,y)$ and $\neg E_{l_2-1}(t,y)$ for some $2 \leq l_2 \leq s+1$. Whence we obtain: $\neg E_{s+1}(x,y)$. If $l_1 \geq l_2$ then $k_2 \cdot k_1 = \{k_2, k_2+1, \dots, k_2 + 2l_1 - 1\}$. If $l_1 < l_2$ then $k_2 \cdot k_1 = \{k_2\}$.

Suppose that $k_2 = 2s+2$. Then we have: $K_0(t,y,r(t))$, $\neg E_{s+1}(t,y)$ and $E^*_1(y,r(t))$. Whence we obtain: $K_0(x,y,r(x))$, $\neg E_{s+1}(x,y)$, $E^*_{l_1}(y,r(x))$ and $\neg E^*_{l_1-1}(y,r(x))$ for some $2 \leq l_1 \leq s+1$. Consequently, $k_1 \cdot k_2 = \{2s+3-l_1\}$. Consider the product $(2s+2)\cdot k_1$. We have the following: $K_0(x,t,r(x))$, $\neg E_{s+1}(x,t)$ and $E^*_1(t,r(x))$; $K_0(t,y,r(t))$, $E^*_l(t,y)$ and $\neg E^*_{l-1}(t,y)$ for some $2 \leq l \leq s+1$. Whence we obtain: $K_0(r(x),y,x)$, $\neg E_{s+1}(x,y)$, $E^*_l(y,r(x))$ and $\neg E^*_{l-1}(y,r(x))$, i.e. $(2s+2)\cdot k_1 = \{2s+2+l\}$. We can establish similarly that if $k_2 = 2s+3$ then $k_1 \cdot k_2 = \{2s+3-l_1\}$ and $2s+3)\cdot k_1 = \{2s+2+l\}$.

Suppose now that $2s+4 \leq k_2 \leq 3s+2$. Then we have: $K_0(r(t),y,t)$, $\neg E_{s+1}(t,y)$, $E^*_{l_2}(y,r(t))$ and $\neg E^*_{l_2-1}(y,r(t))$ for some $2 \leq l_2 \leq s$. Whence we obtain: $\neg E_{s+1}(x,y)$. If $l_1 \geq l_2$ then it is possible $K_0(x,y,r(x))$ or $K_0(r(x),y,x)$. If $l_1 > l_2$ then $k_1 \cdot k_2 = \{2s+3-l_1\}$. If $l_1 = l_2$ then $k_1 \cdot k_2 = \{k_1, k_1+1, \dots, k_1+2l_1-1\}$. If $l_1 < l_2$ then we have $K_0(r(x),y,x)$ and $k_1 \cdot k_2 = \{2s+2+l_2\}$. By considering the product $k_2 \cdot k_1$ we obtain: $K_0(r(x),t,x)$, $\neg E_{s+1}(x,t)$, $E^*_{l_1}(t,r(x))$ and $\neg E^*_{l_1-1}(t,r(x))$ for some $2 \leq l_1 \leq s$; $K_0(t,y,r(t))$, $E_{l_2}(t,y)$ and $\neg E_{l_2-1}(t,y)$ for some $2 \leq l_2 \leq s+1$. Whence we obtain: $\neg E_{s+1}(x,y)$ and $K_0(r(x),y,x)$. If $l_1 \geq l_2$ then $k_2 \cdot k_1 = \{k_2\}$. If $l_1 < l_2$ then $k_2 \cdot k_1 = \{k_1\}$.

Suppose that $k_2 = 3s+3$. Then we have: $K_0(r(t),y,t)$, $\neg E_{s+1}(t,y)$ and $\neg E^*_s(y,r(t))$. Consequently, we obtain: $K_0(r(x),y,x)$, $\neg E_{s+1}(x,y)$ and $\neg E^*_s(y,r(x))$. Consequently, $k_1 \cdot k_2 = \{k_2\}$. By considering the product $(3s+3)\cdot k_1$ we obtain similarly that $k_2 \cdot k_1 = \{k_2\}$.

Suppose now that $3s+4 \leq k_2 \leq 4s+3$. Then we have: $K_0(r(t),y,t)$, $E_{l_2}(t,y)$ and $\neg E_{l_2-1}(t,y)$ for some $2 \leq l_2 \leq s+1$. If $l_1 > l_2$ then $K_0(x, y, r(x))$, $E_{l_1}(x,y)$ and $\neg E_{l_1-1}(x,y)$, and $k_1 \cdot k_2 = \{k_1\}$. If $l_1 = l_2$ then it is possible $K_0(x,y,r(x))$ or $K_0(r(x),y,x)$, and $k_1 \cdot k_2 = \{0, 1, \dots, l_1, 4s+5-l_1, \dots, 4s+4\}$. If $l_1 < l_2$ then $K_0(r(x),y,x)$, $E_{l2}(x,y)$ and $\neg E_{l_2-1}(x,y)$, and $k_1 \cdot k_2 = \{k_2\}$. By considering the product $k_2 \cdot k_1$ we obtain: $K_0(r(x),t,x)$, $E_{l_1}(x,t)$, $\neg E_{l_1-1}(x,t)$ for some $2 \leq l_1 \leq s+1$; $K_0(t,y,r(t))$, $E_{l_2}(t,y)$ and $\neg E_{l_2-1}(t,y)$ for some $2 \leq l_2 \leq s+1$. Then if $l_1 > l_2$, we have $K_0(r(x),y,x)$ and $k_2 \cdot k_1 = \{k_2\}$. If $l_1 = l_2$ then it is possible $K_0(x,y,r(x))$ or $K_0(r(x),y,x)$, whence we obtain: $k_2 \cdot k_1 = \{0, 1, \dots, l_1, 4s+5-l_1, \dots, 4s+4\}$. If $l_1 < l_2$ then $K_0(x,y,r(x))$ and $k_2 \cdot k_1 = \{k_1\}$.

Suppose that $k_2 = 4s+4$. Then we have: $K_0(r(t),y,t)$ and $E_1(t,y)$. Whence we obtain: $K_0(x,y,r(x))$, $E_{l_1}(x,y)$ and $\neg E_{l_1-1}(x,y)$, i.e. $k_1 \cdot k_2 = \{k_1\}$. We can show similarly that $(4s+4)\cdot k_1 = \{k_1\}$.

Case 3: $k_1 = s+2$. We have the following: $K_0(x,t,r(x))$, $\neg E_{s+1}(x,t)$ and $\neg E^*_s(t,r(x))$.

If $k_2 = s+2$ then we obtain: $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$ and $\neg E^*_s(y, r(t))$. Then we have: $E_{s+1}(x, y)$. Since $\neg E_{s+1}(x, t)$, we have $K_0(x, r(t), t, r(x))$. Consequently, it is possible $K_0(x, y, r(x))$ or $K_0(r(x), y, x)$. Whence we obtain: $k_1 \cdot k_2 = (0, 1, \ldots, s+1, 3s+4, \ldots, 4s+4\}$, i.e. the product $(s+2)\cdot(s+2)$ gives $2s+3$ labels of the algebra.

Suppose now that $s+3 \le k_2 \le 2s+1$. Then we have: $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$, $E^*_{l_2}(y, r(t))$ and $\neg E^*_{l_2-1}(y, r(t))$ for some $2 \le l_2 \le s$. Whence we obtain: $E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{s+1\}$. By considering the product $k_2 \cdot k_1$ we obtain: $K_0(x, t, r(x))$, $\neg E_{s+1}(x, t)$, $E^*_l(t, r(x))$ and $\neg E^*_{l-1}(t, r(x))$ for some $2 \le l \le s$; $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$ and $\neg E^*_s(y, r(t))$. Whence we have: $E_{s+1}(x, y)$ and $k_2 \cdot k_1 = \{3s+4\}$.

Suppose that $k_2 = 2s+2$. Then we have: $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$ and $E^*_1(y, r(t))$. Whence we obtain: $E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{s+1\}$. By considering the product $(2s+2)\cdot(s+2)$ we obtain: $K_0(x, t, r(x))$, $\neg E_{s+1}(x, t)$ and $E^*_1(t, r(x))$; $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$ and $\neg E^*_s(y, r(t))$. Whence we have: $E_{s+1}(x, y)$ and $(2s+2)\cdot(s+2) = \{3s+4\}$. We can establish similarly that if $k_2 = 2s+3$ then $k_1 \cdot k_2 = \{s+1\}$ and $(2s+3)\cdot(s+2) = \{3s+4\}$.

Suppose now that $2s+4 \le k_2 \le 3s+2$. Then we have the following: $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$, $E^*_l(y, r(t))$ and $\neg E^*_{l-1}(y, r(t))$ for some $2 \le l \le s$. Whence we obtain: $E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{s+1\}$. By considering the product $k_2 \cdot k_1$ we show similarly that $k_2 \cdot k_1 = \{3s+4\}$.

Suppose that $k_2 = 3s+3$. Then we have: $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$ and $\neg E^*_s(y, r(t))$. Whence we obtain: $E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{s+1\}$. By considering the product $(3s+3)\cdot(s+2)$ we show similarly that $k_2 \cdot k_1 = \{3s+4\}$.

Suppose now that $3s+4 \le k_2 \le 4s+3$. Then we have: $K_0(r(t), y, t)$, $E_l(t, y)$ and $\neg E_{l-1}(t, y)$ for some $2 \le l \le s+1$. We establish in this case that $\neg E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{s+2\}$. By considering the product $k_2 \cdot k_1$ we show similarly that $\neg E_{s+1}(x, y)$ and $k_2 \cdot k_1 = \{s+2\}$.

Suppose that $k_2 = 4s+4$. Then we have: $K_0(r(t), y, t)$ and $E_1(t, y)$. Whence we obtain: $\neg E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{s+2\}$. By considering the product $(4s+4)\cdot(s+2)$ we show similarly that $\neg E_{s+1}(x, y)$ and $k_2 \cdot k_1 = \{s+2\}$.

Case 4: $s+3 \le k_1 \le 2s+1$. We have the following: $K_0(x, t, r(x))$, $\neg E_{s+1}(x, t)$, $E^*_{l_1}(t, r(x))$ and $\neg E^*_{l_1-1}(t, r(x))$ for some $2 \le l_1 \le s$.

If $s+3 \le k_2 \le 2s+1$ then $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$, $E^*_{l_2}(y, r(t))$ and $\neg E^*_{l_2-1}(y, r(t))$ for some $2 \le l_2 \le s$. Then we obtain: $E_{s+1}(x, y)$. If $l_1 > l_2$ then $k_1 \cdot k_2 = \{l_1\}$. If $l_1 = l_2$ then $k_1 \cdot k_2 = \{0, 1, \ldots, l_1, 4s+5-l_1, \ldots, 4s+4\}$. If $l_1 < l_2$ then $k_1 \cdot k_2 = \{4s+5-l_2\}$.

Suppose that $k_2 = 2s+2$. Then we have: $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$ and $E^*_1(y, r(t))$. Whence we obtain: $E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{l_1\}$.

Consider the product $(2s+2)\cdot k_1$. We have the following: $K_0(x, t, r(x))$, $\neg E_{s+1}(x, t)$ and $E^*_1(t, r(x))$; $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$, $E^*_l(y, r(t))$ and $\neg E^*_{l-1}(y, r(t))$ for some $2 \le l \le s$. Whence we obtain: $E_{s+1}(x, y)$ and

$k_1 \cdot k_2 = \{4s+5-l\}$. We establish similarly that if $k_2 = 2s+3$ then $k_1 \cdot k_2 = \{l_1\}$ and $(2s+3)\cdot(s+2) = \{4s+5-l\}$.

Suppose now that $2s+4 \le k_2 \le 3s+2$. Then we have: $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$, $E^*_{l_2}(y, r(t))$ and $\neg E^*_{l_2-1}(y, r(t))$ for some $2 \le l_2 \le s$. Whence we obtain: $E_{s+1}(x, y)$. If $l_1 \ge l_2$ then $k_1 \cdot k_2 = \{l_1\}$. If $l_1 < l_2$ then $k_2 \cdot k_1 = \{l_2\}$.

Consider the product $k_2 \cdot k_1$. We have the following: $K_0(r(x), t, x)$, $\neg E_{s+1}(x, t)$, $E^*_{l_1}(t, r(x))$ and $\neg E^*_{l_1-1}(t, r(x))$ for some $2 \le l_1 \le s$; $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$, $E^*_{l_2}(y, r(t))$ and $\neg E^*_{l_2-1}(y, r(t))$ for some $2 \le l_2 \le s$. Whence we obtain: $E_{s+1}(x, y)$. If $l_1 \ge l_2$ then $k_2 \cdot k_1 = \{4s+5-l_1\}$. If $l_1 < l_2$ then $k_2 \cdot k_1 = \{4s+5-l_2\}$.

Suppose that $k_2 = 3s+3$. Then we have: $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$ and $\neg E^*_s(y, r(t))$. Whence we obtain: $E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{s+1\}$.

By considering the product $k_2 \cdot k_1$ we prove that $k_2 \cdot k_1 = \{3s+4\}$.

Suppose now that $3s+4 \le k_2 \le 4s+3$. Then we have: $K_0(r(t), y, t)$, $E_{l_2}(t, y)$ and $\neg E_{l_2-1}(t, y)$ for some $2 \le l_2 \le s+1$. We establish that in this case $\neg E_{s+1}(x, y)$. If $l_1 \ge l_2$ then $k_1 \cdot k_2 = \{2s+3-l_1\}$. If $l_1 < l_2$ then $k_1 \cdot k_2 = \{2s+3-l_2\}$. Consider the product $k_2 \cdot k_1$. We have the following: $K_0(r(x), t, x)$, $E_{l_1}(x, t)$ and $\neg E_{l_1-1}(x, t)$ for some $2 \le l_1 \le s+1$; $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$, $E^*_{l_2}(y, r(t))$ and $\neg E^*_{l_2-1}(y, r(t))$ for some $2 \le l_2 \le s$. Whence we obtain: $\neg E_{s+1}(x, y)$. If $l_1 > l_2$ then $k_2 \cdot k_1 = \{2s+2+l_1\}$. If $l_1 = l_2$ then $k_2 \cdot k_1 = \{2s+3-l_1, \ldots, 2s+2+l_1\}$. If $l_1 < l_2$ then $k_2 \cdot k_1 = \{2s+3-l_2\}$.

Suppose that $k_2 = 4s+4$. Then we have: $K_0(r(t), y, t)$ and $E_1(t, y)$. Whence we obtain: $\neg E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{2s+3-l_1\}$. Consider the product $(4s+4)\cdot k_1$. We have the following: $K_0(r(x), t, x)$ and $E_1(x, t)$; $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$, $E^*_l(y, r(t))$ and $\neg E^*_{l-1}(y, r(t))$ for some $2 \le l \le s$. Then we obtain $\neg E_{s+1}(x, y)$ and $k_2 \cdot k_1 = \{2s+3-l\}$.

Case 5: $k_1 = 2s+2$. We have the following: $K_0(x, t, r(x))$, $\neg E_{s+1}(x, t)$ and $E^*_1(t, r(x))$.

If $k_2 = 2s+2$ then we have $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$ and $E^*_1(y, r(t))$. Whence we obtain: $E_{s+1}(x, y)$. Since $E^*_1(t, r(x))$, we have $E^*_1(x, r(t))$, and consequently $E_1(x, y)$. Then $k_1 \cdot k_2 = \{0, 1, 4s+4\}$.

Suppose that $k_2 = 2s+3$. Then we have: $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$ and $E^*_1(y, r(t))$. Whence we obtain: $E_1(x, y)$ and $k_1 \cdot k_2 = \{1\}$. Consider the product $(2s+3)\cdot k_1$. We have the following: $K_0(r(x), t, x)$, $\neg E_{s+1}(x, t)$ and $E^*_1(t, r(x))$; $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$ and $E^*_1(y, r(t))$. Consequently, we obtain: $E_1(x, y)$ and $k_2 \cdot k_1 = \{4s+4\}$.

Suppose now that $2s+4 \le k_2 \le 3s+2$. Then we have: $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$, $E^*_l(y, r(t))$ and $\neg E^*_{l-1}(y, r(t))$ for some $2 \le l \le s$. Whence we obtain: $E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{l\}$. Consider the product $k_2 \cdot k_1$. We have the following: $K_0(r(x), t, x)$, $\neg E_{s+1}(x, t)$, $E^*_l(t, r(x))$ and $\neg E^*_{l-1}(t, r(x))$ for some $2 \le l \le s$; $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$ and $E^*_1(y, r(t))$. Whence we obtain: $E_{s+1}(x, y)$ and $k_2 \cdot k_1 = \{4s+5-l\}$.

Suppose that $k_2 = 3s+3$. Then we have: $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$ and $\neg E^*_s(y, r(t))$. Whence we obtain: $E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{s+1\}$. By considering the product $k_2 \cdot k_1$ we establish that $k_2 \cdot k_1 = \{3s+4\}$.

Suppose now that $3s+4 \leq k_2 \leq 4s+3$. We have the following: $K_0(r(t), y, t)$, $E_l(t, y)$ and $\neg E_{l-1}(t, y)$ for some $2 \leq l \leq s+1$. We establish that in this case $\neg E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{2s+3-l\}$. Consider the product $k_2 \cdot k_1$. We have the following: $K_0(r(x), t, x)$, $E_l(x, t)$ and $\neg E_{l-1}(x, t)$ for some $2 \leq l \leq s+1$; $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$, $E^*_1(y, r(t))$. Whence we obtain: $\neg E_{s+1}(x, y)$ and $k_2 \cdot k_1 = \{2s+2+l\}$.

Suppose that $k_2 = 4s+4$. Then we have: $K_0(r(t), y, t)$ and $E_1(t, y)$. Whence we obtain: $\neg E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{2s+2\}$. Consider the product $(4s+4) \cdot k_1$. We have the following: $K_0(r(x), t, x)$ and $E_1(x, t)$; $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$, $E^*_1(y, r(t))$. Consequently, we obtain $\neg E_{s+1}(x, y)$ and $k_2 \cdot k_1 = \{2s+2, 2s+3\}$.

Case 6: $k_1 = 2s+3$. We have the following: $K_0(r(x), t, x)$, $\neg E_{s+1}(x, t)$ and $E^*_1(t, r(x))$.

If $k_2 = 2s+3$ then we have: $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$ and $E^*_1(y, r(t))$. Whence we obtain: $E_1(x, y)$ and $k_1 \cdot k_2 = \{0, 1, 4s+4\}$.

Suppose now that $2s+4 \leq k_2 \leq 3s+2$. Then we have: $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$, $E^*_l(y, r(t))$ and $\neg E^*_{l-1}(y, r(t))$ for some $2 \leq l \leq s$. Whence we obtain: $E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{l\}$. By considering the product $k_2 \cdot k_1$ we establish similarly that $k_2 \cdot k_1 = \{4s+5-l\}$.

Suppose that $k_2 = 3s+3$. Then we have: $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$ and $\neg E^*_s(y, r(t))$. Whence we obtain: $E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{s+1\}$. By considering the product $k_2 \cdot k_1$ we establish that $k_2 \cdot k_1 = \{3s+4\}$.

Suppose now that $3s+4 \leq k_2 \leq 4s+3$. Then we have: $K_0(r(t), y, t)$, $E_l(t, y)$ and $\neg E_{l-1}(t, y)$ for some $2 \leq l \leq s+1$. We establish in this case that $\neg E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{2s+3-l\}$. Consider the product $k_2 \cdot k_1$. We have the following: $K_0(r(x), t, x)$, $E_l(x, t)$ and $\neg E_{l-1}(x, t)$ for some $2 \leq l \leq s+1$; $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$, $E^*_1(y, r(t))$. Whence we obtain: $\neg E_{s+1}(x, y)$ and $k_2 \cdot k_1 = \{2s+2+l\}$.

Suppose that $k_2 = 4s+4$. Then we have: $K_0(r(t), y, t)$ and $E_1(t, y)$. Whence we obtain: $\neg E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{2s+2, 2s+3\}$. Consider the product $(4s+4) \cdot k_1$. We have the following: $K_0(r(x), t, x)$ and $E_1(x, t)$; $K_0(t, y, r(t))$, $\neg E_{s+1}(t, y)$, $E^*_1(y, r(t))$. Consequently, we obtain $\neg E_{s+1}(x, y)$ and $k_2 \cdot k_1 = \{2s+3\}$.

Case 7: $2s+4 \leq k_1 \leq 3s+2$. We have the following: $K_0(r(x), t, x)$, $\neg E_{s+1}(x, t)$, $E^*_{l_1}(t, r(x))$ and $\neg E^*_{l_1-1}(t, r(x))$ for some $2 \leq l_1 \leq s$.

If $2s+4 \leq k_2 \leq 3s+2$ then $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$, $E^*_{l_2}(y, r(t))$ and $\neg E^*_{l_2-1}(y, r(t))$ hold for some $2 \leq l_2 \leq s$. Whence we obtain: $E_{s+1}(x, y)$. If $l_1 > l_2$ then $k_1 \cdot k_2 = \{4s+5-l_1\}$. If $l_1 = l_2$ then $k_1 \cdot k_2 = \{0, 1, \ldots, l_1, 4s+5-l_1, \ldots, 4s+4\}$. If $l_1 < l_2$ then $k_1 \cdot k_2 = \{l_2\}$.

Suppose that $k_2 = 3s+3$. Then we have: $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$ and $\neg E^*_s(y, r(t))$. Whence we obtain: $E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{s+1\}$. By considering the product $k_2 \cdot k_1$ we establish that $k_2 \cdot k_1 = \{3s+4\}$.

Suppose now that $3s+4 \leq k_2 \leq 4s+3$. Then we have: $K_0(r(t), y, t)$, $E_{l_2}(t, y)$ and $\neg E_{l_2-1}(t, y)$ for some $2 \leq l_2 \leq s+1$. We establish in this case that $\neg E_{s+1}(x, y)$. If $l_1 > l_2$ then $k_1 \cdot k_2 = \{2s+2+l_1\}$. If $l_1 = l_2$ then $k_1 \cdot k_2 = \{2s+3-l_1, \ldots, 2s+2+l_1\}$. If $l_1 < l_2$ then $k_1 \cdot k_2 = \{2s+3-l_2\}$. Consider the product $k_2 \cdot k_1$. We have the following: $K_0(r(x), t, x)$, $E_{l_1}(x, t)$ and $\neg E_{l_1-1}(x, t)$ for some $2 \leq l_1 \leq s+1$; $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$, $E^*_{l_2}(y, r(t))$ and $\neg E^*_{l_2-1}(y, r(t))$ for some $2 \leq l_2 \leq s$. Whence we obtain: $\neg E_{s+1}(x, y)$. If $l_1 > l_2$ then $k_1 \cdot k_2 = \{4s+5-l_1\}$. If $l_1 = l_2$ then $k_1 \cdot k_2 = \{0, 1, \ldots, l_1, 4s+5-l_1, \ldots, 4s+4\}$. If $l_1 < l_2$ then $k_1 \cdot k_2 = \{l_2\}$.

Suppose that $k_2 = 4s+4$. Then we have: $K_0(r(t), y, t)$ and $E_1(t, y)$. Whence we obtain: $\neg E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{2s+2+l_1\}$. Consider the product $(4s+4) \cdot k_1$. We have the following: $K_0(r(x), t, x)$ and $E_1(x, t)$; $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$, $E^*_l(y, r(t))$ and $\neg E^*_{l-1}(y, r(t))$ for some $2 \leq l \leq s$. Then we obtain $\neg E_{s+1}(x,\ y)$ and $k_2 \cdot k_1 = \{2s+2+l\}$.

Case 8: $k_1 = 3s+3$. We have the following: $K_0(r(x), t, x)$, $\neg E_{s+1}(x, t)$ and $\neg E^*_s(t, r(x))$.

If $k_2 = 3s+3$ then we have: $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$ and $\neg E^*_s(y, r(t))$. Whence we obtain: $E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{0, 1, \ldots, s+1, 3s+4, \ldots, 4s+4\}$.

Suppose now that $3s+4 \leq k_2 \leq 4s+3$. Then we have: $K_0(r(t), y, t)$, $E_l(t, y)$ and $\neg E_{l-1}(t, y)$ for some $2 \leq l \leq s+1$. We establish in this case that $\neg E_{s+1}(x, y)$. If $l = s+1$ then $k_1 \cdot k_2 = \{s+2, \ldots, 3s+3\}$. If $l < s+1$ then $k_1 \cdot k_2 = \{3s+3\}$. Consider the product $k_2 \cdot k_1$. We have the following: $K_0(r(x), t, x)$, $E_l(x, t)$ and $\neg E_{l-1}(x, t)$ for some $2 \leq l \leq s+1$; $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$ and $E^*_s(y, r(t))$. Whence we obtain: $\neg E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{3s+3\}$.

Suppose that $k_2 = 4s+4$. Then we have: $K_0(r(t), y, t)$ and $E_1(t, y)$. Whence we obtain: $\neg E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{3s+3\}$.

Consider the product $(4s+4) \cdot k_1$. We have the following: $K_0(r(x), t, x)$ and $E_1(x, t)$; $K_0(r(t), y, t)$, $\neg E_{s+1}(t, y)$ and $\neg E^*_s(y, r(t))$. Then we obtain $\neg E_{s+1}(x, y)$ and $k_2 \cdot k_1 = \{3s+3\}$.

Case 9: $3s+4 \leq k_1 \leq 4s+3$. We have the following: $K_0(r(x), t, x)$, $E_{l_1}(x, t)$ and $\neg E_{l_1-1}(x, t)$ for some $2 \leq l_1 \leq s+1$.

If $3s+4 \leq k_2 \leq 4s+3$ then we have: $K_0(r(t), y, t)$, $E_{l_2}(t, y)$ and $\neg E_{l_2-1}(t, y)$ for some $2 \leq l_2 \leq s+1$. Whence we obtain: $E_{s+1}(x, y)$. If $l_1 \geq l_2$ then $k_1 \cdot k_2 = \{4s+5-l_1\}$. If $l_1 < l_2$ then $k_1 \cdot k_2 = \{4s+5-l_2\}$.

Suppose that $k_2 = 4s+4$. Then we have: $K_0(r(t), y, t)$ and $E_1(t, y)$. Whence we obtain: $E_{s+1}(x, y)$ and $k_1 \cdot k_2 = \{4s+5-l_1\}$.

Consider the product $(4s+4) \cdot k_1$. We have the following: $K_0(r(x), t, x)$ and $E_1(x, t)$; $K_0(r(t), y, t)$, $E_l(t, y)$ and $\neg E_{l-1}(t, y)$ for some $2 \leq l \leq s+1$. Then we obtain $E_{s+1}(x, y)$ and $k_2 \cdot k_1 = \{4s+5-l\}$.

Case 10: $k_1 = 4s+4$. We have the following: $K_0(r(x), t, x)$ and $E_1(x, t)$.

If $k_2 = 4s+4$ then we have: $K_0(r(t), y, t)$ and $E_1(t, y)$. Whence we obtain: $E_1(x, y)$ and $k_1 \cdot k_2 = \{4s+4\}$.

Thus, the product $k_1 \cdot k_2$ has the greatest number of labels, i.e. $2s+3$, in four cases:

(1) $k_1 = s+1$ and $k_2 = 3s+4$;
(2) $k_1 = k_2 = s+2$;
(3) $k_1 = k_2 = 3s+3$;
(4) $k_1 = 3s+4$ and $k_2 = s+1$.

Consequently, the algebra $\mathfrak{P}_{\mathcal{M}'_{s,2,2}}$ is strictly $(2s+3)$-deterministic. □

3. Conclusion

We investigated algebras of binary isolating formulas for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank greater than 1 with a trivial definable closure having a non-trivial monotonic-to-left function acting on the universe of a structure. We also proved their non-commutativity and established their strict m-deterministicity for some natural m. It would now be interesting to describe the corresponding algebras for theories having a non-trivial monotonic-to-right function.

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