Серия «Математика»
2025. Т. 52. С. 88—104

Онлайн-доступ к журналу:
http://mathizv.isu.ru

И З В Е С Т И Я
Иркутского государственного университета

Research article

УДК 517.9
MSC 35G61, 35J57, 35J62, 35K70, 74N20
DOI https://doi.org/10.26516/1997-7670.2025.52.88

Weak Solution to KWC Systems of Pseudo-parabolic Type

Daiki Mizuno[1]✉

[1] Chiba University, Chiba, Japan
✉ d-mizuno@chiba-u.jp

Abstract. In this paper, a class of systems of pseudo-parabolic PDEs is considered. These systems $(\mathrm{S})_\varepsilon$ are derived as a pseudo-parabolic dissipation system of Kobayashi–Warren–Carter energy, proposed by [Kobayashi et al., Physica D, 140, 141–150 (2000)], to describe planar grain boundary motion. In this context, ε is a value to control the relaxation of singular diffusivity. These systems have been studied in [Antil et al., SIAM J. Math. Anal., 56(5), 6422–6455], and solvability, uniqueness and strong regularity of the solution have been reported under the setting that the initial data is sufficiently smooth. Meanwhile, in this paper, we impose weaker regularity on the initial data, and work on the weak formulation of the systems. In this light, we set our goal of this paper to prove two theorems, concerned with the existence and the uniqueness of weak solution to $(\mathrm{S})_\varepsilon$, and the continuous dependence with respect to the index ε, initial data and forcings.

Keywords: planar grain boundary motion, pseudo-parabolic KWC system, energy-dissipation, singular diffusion, time-discretization

Acknowledgements: This work was supported by JST SPRING, Grant Number JP-MJSP2109.

For citation: Mizuno D. Weak Solution to KWC Systems of Pseudo-parabolic Type. *The Bulletin of Irkutsk State University. Series Mathematics*, 2025, vol. 52, pp. 88–104. https://doi.org/10.26516/1997-7670.2025.52.88

Научная статья

Слабое решение систем KWC псевдопараболического типа

Д. Мизуно[1]✉

[1] Университет Тиба, Тиба, Япония
✉ d-mizuno@chiba-u.jp

Аннотация. Рассматривается класс систем псевдопараболических уравнений в частных производных. Эти системы $(\mathrm{S})_\varepsilon$ являются версией псевдопараболической диссипативной системы энергии Кобаяси – Уоррен – Картер, предложенной [Kobayashi et al., Physica D, 140, 141–150 (2000)] для описания движения плоских границ зерен. В этом контексте ε – это значение для контроля релаксации сингулярной диффузии. Эти системы были изучены в [Antil et al., SIAM J. Math. Anal., 56(5), 6422–6455], где была доказана разрешимость, единственность и сильная регулярность решения при условии, что начальные данные достаточно гладкие. Накладываются более слабые условия регулярности на начальные данные и происходит работа над слабой формулировкой систем. Доказываются две теоремы, касающиеся существования и единственности слабого решения для $(\mathrm{S})_\varepsilon$, а также непрерывной зависимости от индекса ε, начальных данных и воздействий.

Ключевые слова: плоское движение границ, псевдопараболическая система KWC, рассеяние энергии, сингулярная диффузия, дискретизация по времени

Благодарности: Работа выполнена при поддержке JST SPRING (грант № JPMJSP2109).

Ссылка для цитирования: Mizuno D. Weak Solution to KWC Systems of Pseudo-parabolic Type // Известия Иркутского государственного университета. Серия Математика. 2025. Т. 52. С. 88–104.
https://doi.org/10.26516/1997-7670.2025.52.88

1. Introduction

Let $N \in \{1,2,3,4\}$ be a fixed spatial dimension, and let $\Omega \subset \mathbb{R}^N$ be a bounded domain. When $N \geq 2$, Γ denotes a smooth boundary of Ω, and n_Γ denotes the outer unit normal on Γ. Also, let $T > 0$ be a fixed time constant, and set $Q := (0,T) \times \Omega$, $\Sigma := (0,T) \times \Gamma$, $H := L^2(\Omega)$, and $V := H^1(\Omega)$.

In this paper, we consider a class of pseudo-parabolic systems, denoted by $(\mathrm{S})_\varepsilon$ for $\varepsilon \in [0,\infty)$:

$$(\mathrm{S})_\varepsilon \quad \begin{cases} \partial_t \eta - \mathit{\Delta}(\eta + \mu^2 \partial_t \eta) + g(\eta) + \alpha'(\eta)\sqrt{\varepsilon^2 + |\nabla\theta|^2} = u, \text{ in } Q, \\ \nabla(\eta + \mu^2 \partial_t \eta) \cdot n_\Gamma = 0, \text{ on } \Sigma, \\ \eta(0) = \eta_0, \text{ in } \Omega, \end{cases}$$

$$\begin{cases} \alpha_0(\eta)\partial_t\theta - \mathrm{div}\Big(\alpha(\eta)\dfrac{\nabla\theta}{\sqrt{\varepsilon^2 + |\nabla\theta|^2}} + \nu^2 \nabla\partial_t\theta\Big) = v, \text{ in } Q, \\ \Big(\alpha(\eta)\dfrac{\nabla\theta}{\sqrt{\varepsilon^2 + |\nabla\theta|^2}} + \nu^2 \nabla\partial_t\theta\Big) \cdot n_\Gamma = 0, \text{ on } \Sigma, \\ \theta(0) = \theta_0, \text{ in } \Omega. \end{cases}$$

The system $(\mathrm{S})_\varepsilon$ is pseudo-parabolic version of KWC-model of planer grain boundary motion, which is proposed by [7;8]. In this system, η and θ represent *orientation order* and *orientation angle* in a polycrystal, respectively. $\varepsilon \geq 0$ is a value to control the relaxation of singular diffusivity. g is a perturbation for the orientation order η, with a non-negative primitive G. α and α_0 are positive-valued functions, called mobilities of grain boundary motion. u, v are forcing terms. Finally, a pair of functions $[\eta_0, \theta_0]$ is the initial data of $[\eta, \theta]$.

In this paper, the components of the system $(\mathrm{S})_\varepsilon$ are considered under the following assumptions.

(A0) $\mu > 0$, $\nu > 0$ are fixed positive constants.

(A1) $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a locally Lipschitz continuous function with a non-negative primitive $G \in C^1(\mathbb{R})$. In addition, g satisfies the condition; $\underline{\lim}_{\xi \downarrow -\infty} g(\xi) = -\infty$ and $\overline{\lim}_{\xi \uparrow \infty} g(\xi) = \infty$.

(A2) $\alpha : \mathbb{R} \longrightarrow [0, \infty)$ is a C^2-class convex function, such that $\alpha'(0) = 0$ and $\alpha'' \geq 0$ on $\mathbb{R}$. $\alpha_0 : \mathbb{R} \longrightarrow (0, \infty)$ is a locally Lipschitz continuous function. Moreover, we suppose $\delta_* := \inf \alpha_0(\mathbb{R}) > 0$.

(A3) $u \in L^\infty(Q)$, and $v \in L^2(0, T; H)$.

(A4) The initial data $[\eta_0, \theta_0]$ belong to $[V \cap L^\infty(\Omega)] \times V$.

The system $(\mathrm{S})_\varepsilon$ is derived from the following energy-dissipation flow:

$$-\begin{bmatrix} I - \mu^2 \mathit{\Delta}_N & \\ & \alpha_0(\eta(t)) I - \nu^2 \mathit{\Delta}_N \end{bmatrix} \begin{bmatrix} \partial_t \eta(t) \\ \partial_t \theta(t) \end{bmatrix} = \nabla_{[\eta,\theta]} \mathcal{F}_\varepsilon(\eta(t), \theta(t)) + \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \text{ in } [H]^2, \quad \text{a.e. } t > 0, \tag{1.1}$$

where $\mathit{\Delta}_N$ denotes the Laplace operator subject to the homogeneous Neumann boundary condition, and $\mathcal{F}_\varepsilon$ is free-energy of grain boundary motion, called *KWC-energy*, and defined as follows:

$$\mathcal{F}_\varepsilon(\eta, \theta) := \frac{1}{2} \int_\Omega |\nabla \eta|^2 \, dx + \int_\Omega G(\eta) \, dx + \int_\Omega \alpha(\eta) \sqrt{\varepsilon^2 + |D\theta|^2} \in [0, \infty].$$

In this context, the last term is given by:

$$\int_\Omega \alpha(\eta) \sqrt{\varepsilon^2 + |D\theta|^2} := \inf \left\{ \underline{\lim}_{n \to \infty} \int_\Omega \alpha(\eta) \sqrt{\varepsilon^2 + |\nabla \varphi|^2} \, dx \;\middle|\; \begin{array}{l} \{\varphi_n\}_{n=1}^\infty \subset W^{1,1}(\Omega) \\ \text{such that } \varphi_n \to \theta \text{ in} \\ L^1(\Omega) \text{ as } n \to \infty \end{array} \right\},$$
$$\text{for } [\eta, \theta] \in H^1(\Omega) \times BV(\Omega), \text{ and } \varepsilon \in [0, \infty).$$

We note that if $\theta \in H^1(\Omega)$, then the integral $\int_\Omega \alpha(\eta) \sqrt{\varepsilon^2 + |D\theta|^2}$ coincides with $\int_\Omega \alpha(\eta) \sqrt{\varepsilon^2 + |\nabla \theta|^2} \, dx$. Also, a function $\boldsymbol{w} \in \mathbb{R}^N \mapsto \sqrt{\varepsilon^2 + |\boldsymbol{w}|^2} \in [0, \infty)$ is a smooth approximation of the Euclidean norm $|\cdot|$, which is often used in numerical analysis and advanced mathematical topics, such as optimal control problems.

The system (1.1) can be regarded as a generalized version of KWC system, and the case $\mu = \nu = \varepsilon = 0$ corresponds to the original. So far, many researchers have worked on mathematical verification of KWC-type systems, and amount of mathematical results have been reported such as solvability results and large-time behavior on variational settings (cf. [9–13]). But uniqueness question is still a difficult problem, because of the nonlinear flux $\alpha(\eta)\frac{\nabla\theta}{\sqrt{\varepsilon^2+|\nabla\theta|^2}}$, and the unknown-dependent mobility α and α_0. In fact, we need suitable regularization for the energy, and simplification for the mobility to obtain uniqueness result (cf. [2; 6; 17]).

Recently, [3] reported the solvability, uniqueness results under a smoothness condition for initial data to a KWC-type system with pseudo-parabolic regularization, which is often found in fluid dynamics as Foigt regularization, and provides better properties such as well-posedness and strong regularity (cf. [4; 5; 14; 15]). The novelty of [3] lies in the fact that it does not require resetting the free-energy and the mobilities. In this light, this article is focused on the extensions of the existence and uniqueness results of [3] to the weak formulation in Theorem 1.

Theorem 1. *Under the assumptions (A0)–(A4), the system* $(S)_\varepsilon$ *admits a unique solution* $[\eta, \theta]$ *in the following sense:*

(S0) $\eta \in W^{1,2}(0,T;V) \cap L^\infty(Q)$, *and* $\theta \in W^{1,2}(0,T;V)$. *In particular, if* $\theta_0 \in L^\infty(\Omega)$ *and* $v \equiv 0$, *then* $\theta \in L^\infty(Q)$.

(S1) η *solves the following variational identity:*

$$\int_\Omega \big(\partial_t\eta(t) + g(\eta(t)) + \alpha'(\eta(t))\sqrt{\varepsilon^2 + |\nabla\theta(t)|^2}\big)\varphi\,dx + \int_\Omega \nabla(\eta + \mu^2\partial_t\eta)(t)\cdot\nabla\varphi\,dx = \int_\Omega u(t)\varphi\,dx,$$

for any $\varphi \in V$, *and a.e.* $t \in (0,T)$, *subject to* $\eta(0) = \eta_0$.

(S2) θ *solves the following variational inequality:*

$$\int_\Omega (\alpha_0(\eta)\partial_t\theta)(t)(\theta(t)-\psi)\,dx + \int_\Omega \alpha(\eta(t))\sqrt{\varepsilon^2+|\nabla\theta(t)|^2}\,dx + \nu^2\int_\Omega \nabla\partial_t\theta(t)\cdot\nabla(\theta(t)-\psi)\,dx \leq \int_\Omega \alpha(\eta(t))\sqrt{\varepsilon^2+|\nabla\psi|^2}\,dx + \int_\Omega v(t)(\theta(t)-\psi)\,dx,$$

for any $\psi \in V$, *and a.e.* $t \in (0,T)$, *subject to* $\theta(0) = \theta_0$.

(S3) $[\eta, \theta]$ *fulfills the following energy-inequality:*

$$\begin{aligned} C_0 \int_s^t \left(|\partial_t \eta(r)|_V^2 + |\partial_t \theta(r)|_V^2 \right) dr + \mathcal{F}_\varepsilon(\eta(t), \theta(t)) \\ \leq \mathcal{F}_\varepsilon(\eta(s), \theta(s)) + \frac{1}{2} \int_s^t \left(|u(r)|_H^2 + \frac{1}{\delta_*} |v(r)|_H^2 \right) dr, \\ \text{for any } 0 \leq s \leq t \leq T, \ \text{where } C_0 = \min\left\{ \tfrac{1}{4}, \mu^2, \tfrac{\delta_*}{2}, \nu^2 \right\}. \end{aligned} \tag{1.2}$$

Subsequently, the arguments used in Theorem 1 will derive a result for continuous dependence of solution, which provides a mathematical fundamental in optimal control problems.

Theorem 2. *Let* $\{\varepsilon_n\}_{n=1}^\infty \subset [0, \infty)$, $\{[\eta_{0,n}, \theta_{0,n}]\}_{n=1}^\infty \subset [V \cap L^\infty(\Omega)] \times V$ *and* $\{[u_n, v_n]\}_{n=1}^\infty \subset L^\infty(Q) \times L^2(0, T; H)$ *be sequences satisfying the following conditions:*

$$\begin{cases} \bullet \ \sup_{n \in \mathbb{N}} |\eta_{0,n}|_{L^\infty(\Omega)} < \infty, \ \text{and} \ \sup_{n \in \mathbb{N}} |u_n|_{L^\infty(Q)} < \infty, \\ \bullet \ \begin{array}{l} \varepsilon_n \to \varepsilon \text{ in } \mathbb{R}, \ \eta_{0,n} \to \eta_0, \ \theta_{0,n} \to \theta_0 \text{ in } V, \ \text{and} \\ u_n \to u, \ v_n \to v \text{ weakly in } [L^2(0, T; H)]^2, \ \text{as } n \to \infty. \end{array} \end{cases} \tag{1.3}$$

Let $[\eta, \theta]$ *be the unique solution to (S)*$_\varepsilon$*, corresponding to the initial data* $[\eta_0, \theta_0]$ *and the forcings* $[u, v]$*, and let* $[\eta_n, \theta_n]$ *be the unique solution to (S)*$_{\varepsilon_n}$*, for the initial data* $[\eta_{0,n}, \theta_{0,n}]$ *and the forcings* $[u_n, v_n]$*, for any* $n = 1, 2, 3, \ldots$. *Then, we can obtain the following convergences as* $n \to \infty$*:*

$$\begin{cases} \eta_n \to \eta \text{ in } C([0, T]; H), \ L^2(0, T; V), \ \text{weakly in } W^{1,2}(0, T; V) \\ \qquad \text{and weakly-}* \text{ in } L^\infty(Q), \\ \theta_n \to \theta \text{ in } C([0, T]; H), L^2(0, T; V), \ \text{and weakly in } W^{1,2}(0, T; V), \\ \eta_n(t) \to \eta(t), \ \theta_n(t) \to \theta(t) \text{ in } V, \quad \text{for all } t \in [0, T]. \end{cases} \tag{1.4}$$

The outline of this paper is as follows. Notations and a key-lemma for the proof are given in Section 2. On account of these preparations, the proof of the theorems are given in Section 3.

2. Preliminaries

We first prescribe notations and known results used in this paper.

Specific notations. We define $r \vee s := \max\{r, s\}$ and $r \wedge s := \min\{r, s\}$ for all $r, s \in [-\infty, \infty]$. We denote by $[\cdot]^+$, $[\cdot]^-$ positive part and negative part, respectively. We denote by $\lfloor \cdot \rfloor$, $\lceil \cdot \rceil$ floor function and ceiling function, respectively. For an abstract Hilbert space X, we denote by $(\cdot,\cdot)_X$ the inner product of X.

Next, to simplify notations, we set $\{\gamma_\varepsilon\}_{\varepsilon\in[0,\infty)}$ by:

$$\gamma_\varepsilon : y \in \mathbb{R}^N \mapsto \gamma_\varepsilon(y) := \sqrt{\varepsilon^2 + |y|^2} \in [0,\infty).$$

As is easily seen that γ_ε is convex and non-expansive, and for a non-negative $\alpha^\circ \in H$, the following functional is convex and continuous on $[H]^N$:

$$\boldsymbol{w} \in [H]^N \mapsto \int_\Omega \alpha^\circ \gamma_\varepsilon(\boldsymbol{w})\,dx \in [0,\infty).$$

Also, for any $\varepsilon_0 \in [0,\infty)$, γ_ε converges to γ_{ε_0} uniformly on $\mathbb{R}^N$ as $\varepsilon \to \varepsilon_0$.

Finally, we use the following fact($*$) in the proof (cf. [1, Proposition 1.80]): if $a, b \in \mathbb{R}$ and $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty \subset \mathbb{R}$ satisfy:

$$\varliminf_{n\to\infty} a_n \geq a, \ \varliminf_{n\to\infty} b_n \geq b, \text{ and } \varlimsup_{n\to\infty} (a_n + b_n) \leq a + b.$$

Then, $a_n \to a$ and $b_n \to b$ as $n \to \infty$.

Notations for the time-discretization. Let $\tau > 0$ be a constant of time-step size, and let $\{t_i\}_{i=0}^\infty \subset [0,\infty)$ be time sequence defined as $t_i = i\tau$, $i = 0,1,2,\ldots$. Let X be a Hilbert space. Then, for any sequence $\{[t_i, z_i]\}_{i=0}^\infty \subset [0,\infty) \times X$, we define three interpolations $\overline{z}_\tau, \underline{z}_\tau$ and z_τ by:

$$\overline{z}_\tau(t) := z_i, \ \underline{z}_\tau(t) := z_{i-1}, \ z_\tau(t) := \frac{t - t_{i-1}}{\tau} z_i + \frac{t_i - t}{\tau} z_{i-1},$$
$$\text{for } t \in [t_{i-1}, t_i), \text{ and for } i = 1,2,3,\ldots.$$

Note that $\overline{z}_\tau, \underline{z}_\tau \in L^\infty_{\rm loc}([0,\infty); X)$ and $z_\tau \in W^{1,2}_{\rm loc}([0,\infty); X)$. Here, the following estimates can be obtained by use of Young's and Hölder's inequality for $t \geq 0$ and $\tau > 0$:

$$\int_0^t (\partial_t z_\tau(r), \overline{z}_\tau(r))_X\,dr$$
$$\geq \frac{1}{2}\bigl(|\overline{z}_\tau(t)|_X^2 - |z_0|_X^2\bigr) - \tau^{\frac{1}{2}} |\overline{z}_\tau|_{L^\infty(0,t;X)} |\partial_t z_\tau|_{L^2(0,t+\tau;X)}. \qquad (2.1)$$

Meanwhile, for any $\zeta \in L^2_{\rm loc}([0,\infty); X)$, we denote by $\{\zeta_i\}_{i=0}^\infty \subset X$ the sequence of time-discretization data of ζ, defined as:

$$\zeta_0 := 0 \text{ in } X, \text{ and } \ \zeta_i := \frac{1}{\tau}\int_{t_{i-1}}^{t_i} \zeta(t)\,dt \ \text{ in } X, \ \text{ for } i = 1,2,3,\ldots.$$

Clearly, the time-interpolations $\overline{\zeta}_\tau, \underline{\zeta}_\tau$ for the above $\{\zeta_i\}_{i=0}^\infty$ fulfill that:

$$\overline{\zeta}_\tau \to \zeta \text{ and } \underline{\zeta}_\tau \to \zeta \text{ in } L^2_{\rm loc}([0,\infty); X), \text{ as } \tau \downarrow 0.$$

Time-discretization scheme for (S)$_\varepsilon$. The proof of Theorem 1 is based on time-discretization scheme. In this light, we recall a known result (cf. [3, Theorem 1]). Let $\tau \in (0,1)$ be a time-step size, and let $\{t_i\}_{i=0}^{\infty}$ be a time-sequence. The index $\varepsilon \in [0,\infty)$ is given arbitrarily. Now, the time-discretization scheme of (S)$_\varepsilon$, denoted by (AP)$_\tau$, is described as follows:

(AP)$_\tau$: To find $\{[\eta_i,\theta_i]\}_{i=1}^{\infty} \subset [V]^2$ satisfying the following variational formulas:

$$\begin{cases} \left(\dfrac{1}{\tau}(\eta_i - \eta_{i-1}) + g(\mathcal{T}_M \eta_i) + \alpha'(\mathcal{T}_M \eta_i)\gamma_\varepsilon(\nabla\theta_i), \varphi\right)_H + (\nabla\eta_i, \nabla\varphi)_{[H]^N} \\ \qquad + \dfrac{\mu^2}{\tau}(\nabla(\eta_i - \eta_{i-1}), \nabla\varphi)_{[H]^N} = (u_i, \varphi)_H, \text{ for any } \varphi \in V. \end{cases}$$

$$\begin{cases} \dfrac{1}{\tau}(\alpha_0(\mathcal{T}_M \eta_{i-1})(\theta_i - \theta_{i-1}), \theta_i - \psi)_H + \displaystyle\int_\Omega \alpha_M(\eta_{i-1})\gamma_\varepsilon(\nabla\theta_i)\,dx \\ \quad + \dfrac{\nu^2}{\tau}(\nabla(\theta_i - \theta_{i-1}), \nabla(\theta_i - \psi))_{[H]^N} \\ \quad \leq \displaystyle\int_\Omega \alpha_M(\eta_{i-1})\gamma_\varepsilon(\nabla\psi)\,dx + (v_i, \theta_i - \psi)_H, \quad \text{for any } \psi \in V, \end{cases}$$

for $i = 1, 2, 3, \ldots,$ where $[\eta_0, \theta_0]$ is the initial data as in (A4).

Here, $\mathcal{T}_M : r \in \mathbb{R} \mapsto \mathcal{T}_M(r) := M \wedge (-M \vee r)$ is a truncation operator with a large constant $M > 0$, fixed later. α_M is a primitive of $\alpha' \circ \mathcal{T}_M$ such that $\alpha_M = \alpha$ on $[-M, M]$. Finally, for $i = 1, 2, 3, \ldots$, let $[u_i, v_i] \in [H]^2$ be the time-discretization data of $[u_0^{\mathrm{ex}}, v_0^{\mathrm{ex}}]$ which is the zero-extension of $[u, v]$.

Based on the above, the following lemma can be obtained through slight modifications of [3, Theorem 1].

Lemma 1. *There exists a sufficiently small constant $\tau_* \in (0,1)$ such that for any $\tau \in (0, \tau_*)$, (AP)$_\tau$ admits a unique solution $\{[\eta_i, \theta_i]\}_{i=1}^{\infty}$, satisfying*

$$\begin{aligned} &\frac{C_0}{\tau}\left(|\eta_i - \eta_{i-1}|_V^2 + |\theta_i - \theta_{i-1}|_V^2\right) + \mathcal{F}_\varepsilon^M(\eta_i, \theta_i) \\ &\quad \leq \mathcal{F}_\varepsilon^M(\eta_{i-1}, \theta_{i-1}) + \frac{\tau}{2}|u_i|_H^2 + \frac{\tau}{2\delta_*}|v_i|_H^2, \quad \textit{for } i = 1, 2, 3, \ldots, \textit{ where} \end{aligned} \tag{2.2}$$

where

$$\mathcal{F}_\varepsilon^M(\eta, \theta) := \frac{1}{2}\int_\Omega |\nabla\eta|^2\,dx + \int_\Omega G_M(\eta)\,dx + \int_\Omega \alpha_M(\eta)\gamma_\varepsilon(D\theta),$$

and G_M is a non-negative primitive of $g \circ \mathcal{T}_M$.

3. Proof of Theorem 1 and 2

3.1. Proof of Theorem 1

Let $\varepsilon \in [0, \infty)$ be fixed. Throughout this subsection, we use the notations $m_\tau(t) := \left\lfloor \frac{t}{\tau} \right\rfloor, n_\tau(t) := \left\lceil \frac{t}{\tau} \right\rceil$. Before the proof, we prepare two lemmas, concerned with comparison principle for pseudo-parabolic equations.

Lemma 2. *(cf. [3, Lemma 5.4]) Let $\eta^1, \eta^2 \in W^{1,2}(0,T;V)$, $\eta_0^1, \eta_0^2 \in V$, $\widetilde{\theta} \in L^2(0,T;V)$, $\widetilde{u} \in L^2(0,T;H)$, and*

$$\begin{cases} (-1)^{i-1}\big(\partial_t \eta^i - \Delta_N(\eta^i + \mu^2 \partial_t \eta^i) + g(\mathcal{T}_M \eta^i) + \alpha'(\mathcal{T}_M \eta^i)\gamma_\varepsilon(\nabla \widetilde{\theta})\big) \\ \qquad \leq (-1)^{i-1}\widetilde{u}, \ \ a.e.\ \ in\ Q, \\ \eta^i(0) = \eta_0^i, \ \ in\ H. \end{cases}$$

Then, there exists a constant $C_1 > 0$ such that

$$\big|[\eta^1 - \eta^2]^+(t)\big|_V^2 \leq C_1 \big|[\eta_0^1 - \eta_0^2]^+\big|_V^2, \ \ \textit{for any } t \in [0,T].$$

Lemma 3. *We assume that $\theta^1, \theta^2 \in W^{1,2}(0,T;V)$, $\theta_0^1, \theta_0^2 \in V$, $\tilde{\eta} \in W^{1,2}(0,T;H) \cap L^\infty(Q)$, and*

$$\begin{aligned} &(\alpha_0(\tilde{\eta}(t))\partial_t \theta^i(t), \theta^i(t) - \psi)_H + \int_\Omega \alpha(\tilde{\eta}(t))\gamma_\varepsilon(\nabla \theta^i(t))\,dx \\ &\quad + (\nabla \partial_t \theta^i(t), \nabla(\theta^i(t) - \psi))_{[H]^N} \leq \int_\Omega \alpha(\tilde{\eta}(t))\gamma_\varepsilon(\nabla \psi)\,dx, \\ &\qquad \textit{for any } \psi \in V, \ \textit{and for a.e. } t \in (0,T). \end{aligned} \tag{3.1}$$

Then, there exists a constant $C_2 > 0$ such that:

$$\big|[\theta^1 - \theta^2]^+(t)\big|_V^2 \leq C_2 \big|[\theta_0^1 - \theta_0^2]^+\big|_V^2, \ \ \textit{for any } t \in [0,T].$$

Proof. By putting $\psi = (\theta^1 \wedge \theta^2)(t)$ if $i = 1$, and $\psi = (\theta^1 \vee \theta^2)(t)$ if $i = 2$, and taking the sum of two inequalities, we have

$$\begin{aligned} &\big(\alpha_0(\tilde{\eta}(t))\partial_t(\theta^1 - \theta^2)(t), [\theta^1 - \theta^2]^+(t)\big)_H + \frac{\nu^2}{2}\frac{d}{dt}\big(|\nabla[\theta^1 - \theta^2]^+(t)|_{[H]^N}^2\big) \\ &\qquad \leq 0, \ \ \text{for a.e. } t \in (0,T). \end{aligned} \tag{3.2}$$

Also, in light of (A3), the continuous embedding from V to $L^4(\Omega)$ under $N \leq 4$, and the generalized chain rule in BV-theory (cf. [1, Theorem 3.99]), it is observed that:

$$
\begin{aligned}
\big(\alpha_0(\tilde{\eta}(t))\partial_t(\theta^1-\theta^2)(t),[\theta^1-\theta^2]^+(t)\big)_H &\geq \frac{1}{2}\frac{d}{dt}\big(|\sqrt{\alpha_0(\tilde{\eta})}[\theta^1-\theta^2]^+(t)|_H^2\big)\\
&\quad -\frac{1}{2}|\alpha_0'(\tilde{\eta})|_{L^\infty(Q)}|\partial_t\tilde{\eta}(t)|_H\big|[\theta^1-\theta^2]^+(t)\big|^2_{L^4(\Omega)} \qquad (3.3)\\
\geq \frac{1}{2}\frac{d}{dt}\big(|\sqrt{\alpha_0(\tilde{\eta})}[\theta^1-\theta^2]^+(t)|_H^2\big)&\\
-\frac{(C_V^{L^4})^2}{2}|\alpha_0'(\tilde{\eta})|_{L^\infty(Q)}|\partial_t\tilde{\eta}(t)|_H\big|[\theta^1-\theta^2]^+(t)\big|^2_V,& \text{ for a.e. } t\in(0,T),
\end{aligned}
$$

where $C_V^{L^4}$ is the constant of the continuous embedding from V to $L^4(\Omega)$. Now, according to (3.2) and (3.3), we can arrive the following Gronwall type inequality:

$$
\begin{aligned}
&\frac{d}{dt}J_0(t)\leq\frac{(C_V^{L^4})^2}{\delta_*\wedge\nu^2}|\alpha_0'(\tilde{\eta})|_{L^\infty(Q)}|\partial_t\tilde{\eta}(t)|_H J_0(t), \text{ for a.e. } t\in(0,T), \text{ with,}\\
&\qquad J_0(t):=\big|\sqrt{\alpha_0(\tilde{\eta}(t))}[\theta^1-\theta^2]^+(t)\big|_H^2+\nu^2\big|\nabla[\theta^1-\theta^2](t)\big|^2_{[H]^N},
\end{aligned}
$$

and thus, we conclude Lemma 3. □

Proof of Theorem 1 By (A1), (A3) and (A4), let us set the constant M so large that:

$$
\begin{cases} M\geq\max\{|\eta_0|_{L^\infty(\Omega)},|u|_{L^\infty(Q)}\}, \text{ and}\\ g(M)\geq|u|_{L^\infty(Q)}, g(-M)\leq-|u|_{L^\infty(Q)}.\end{cases} \qquad (3.4)
$$

Now, Lemma 1 yields the following boundedness:

- $\partial_t\eta_\tau,\partial_t\theta_\tau\in L^2(0,\infty;V)$ for $\tau\in(0,\tau_*)$, and $\sup\{|\partial_t\eta_\tau|_{L^2(0,\infty;V)}\vee|\partial_t\theta_\tau|_{L^2(0,\infty;V)}\,|\,\tau\in(0,\tau_*)\}<\infty$.
- $\{\eta_\tau\,|\,\tau\in(0,\tau_*)\}$, $\{\theta_\tau\,|\,\tau\in(0,\tau_*)\}$ is bounded in $W^{1,2}(0,T;V)$,
- $\{\overline{\eta}_\tau\,|\,\tau\in(0,\tau_*)\}$, $\{\underline{\eta}_\tau\,|\,\tau\in(0,\tau_*)\}$, $\{\overline{\theta}_\tau\,|\,\tau\in(0,\tau_*)\}$, $\{\underline{\theta}_\tau\,|\,\tau\in(0,\tau_*)\}$ is bounded in $L^\infty(0,T;V)$.

Hence, by applying Aubin's type compactness theory (cf. [16, Corollary 4]), we can obtain a sequence $\{\tau_n\}\subset(0,\tau_*)$; $\tau_n\downarrow 0$ and a pair of functions $[\eta,\theta]\in[W^{1,2}(0,T;V)]^2$ such that:

$$
\begin{aligned}
&\eta_n:=\eta_{\tau_n}\to\eta,\ \theta_n:=\theta_{\tau_n}\to\theta \text{ in } C([0,T];H)\\
&\qquad\text{and weakly in } W^{1,2}(0,T;V), \text{ as } n\to\infty.
\end{aligned} \qquad (3.5)
$$

and in particular,

$$
[\eta(0),\theta(0)]=\lim_{n\to\infty}[\eta_n(0),\theta_n(0)]=[\eta_0,\theta_0] \text{ in } [H]^2. \qquad (3.6)
$$

Additionally, taking into account the boundedness of $[\partial_t \eta_n, \partial_t \theta_n]$, we can compute:

$$\begin{aligned}\big(|(\overline{\eta}_{\tau_n} - \eta_n)|_V \vee |(\underline{\eta}_{\tau_n} - \eta_n)|_V \vee |(\overline{\theta}_{\tau_n} - \theta_n)|_V \vee |(\underline{\theta}_{\tau_n} - \theta_n)|_V\big)(t) \\ \leq \int_{(t_{i-1},t_i)\cap(0,T)} \big(|\partial_t \eta_n(r)|_V \vee |\partial_t \theta_n(r)|_V\big)\, dr \\ \leq \tau_n^{\frac{1}{2}} \big(|\partial_t \eta_n|_{L^2(0,T;V)} \vee |\partial_t \theta_n|_{L^2(0,T;V)}\big), \\ \text{for } t \in (t_{i-1}, t_i) \cap (0,T),\ i = 1,2,3,\dots,n_\tau,\ \text{and } \tau \in (0,\tau_*).\end{aligned} \tag{3.7}$$

Hence, we obtain the following convergences:

$$\begin{aligned}\overline{\eta}_n := \overline{\eta}_{\tau_n} \to \eta,\ \underline{\eta}_n := \underline{\eta}_{\tau_n} \to \eta,\ \text{and } \overline{\theta}_n := \overline{\theta}_{\tau_n} \to \theta, \underline{\theta}_n := \underline{\theta}_{\tau_n} \to \theta, \\ \text{in } L^\infty(0,T;H) \text{ and weakly-}* \text{ in } L^\infty(0,T;V),\ \text{as } n \to \infty,\end{aligned} \tag{3.8}$$

and in particular,

$$\begin{aligned}\overline{\eta}_n(t) \to \eta(t),\ \underline{\eta}_n(t) \to \eta(t),\ \overline{\theta}_n(t) \to \theta(t),\ \underline{\theta}_n(t) \to \theta(t) \\ \text{in } H \text{ and weakly in } V,\ \text{for any } t \in [0,T].\end{aligned} \tag{3.9}$$

Now, let us show the pair of functions $[\eta, \theta]$ is a solution to the system $(\mathrm{S})_\varepsilon$. The initial condition is easily confirmed by (3.6). Let us check that $[\eta, \theta]$ solves the variational inequalities (S1) and (S2). From Lemma 1, the sequences given in (3.5) and (3.8) fulfill that

$$\begin{aligned}\int_I ((\partial_t \eta_n + g(\mathcal{T}_M \overline{\eta}_n) + \alpha'(\mathcal{T}_M \overline{\eta}_n(r)) \gamma_\varepsilon(\nabla \overline{\theta}_n(r))), w(r))_H\, dr \\ + \int_I (\nabla(\overline{\eta}_n + \mu^2 \partial_t \eta_n)(r), \nabla w(r))_{[H]^N}\, dr = \int_I (\overline{u}_{\tau_n}(r), w(r))_H\, dr,\end{aligned} \tag{3.10}$$

for any $w \in L^2(0,T;V)$, and

$$\begin{aligned}\int_I (\alpha_0(\mathcal{T}_M \underline{\eta}_n(r)) \partial_t \theta_n(r), (\overline{\theta}_n - \omega)(r))_H\, dr \\ + \int_I \int_\Omega \alpha_M(\underline{\eta}_n(r)) \gamma_\varepsilon(\nabla \overline{\theta}_n(r))\, dx dr \\ + \nu^2 \int_I (\nabla \partial_t \theta_n(r), \nabla(\overline{\theta}_n - \omega)(r))_{[H]^N}\, dr \\ \leq \int_I \int_\Omega \alpha_M(\underline{\eta}_n(r)) \gamma_\varepsilon(\nabla \omega(r))\, dx dr + \int_I (\overline{v}_{\tau_n}(r), (\overline{\theta}_n - \omega)(r))_H\, dr,\end{aligned} \tag{3.11}$$

for any $\omega \in L^2(0,T;V)$, and any open interval $I \subset (0,T)$ and $n = 1,2,3,\dots$. Here, set $I = (0,t)$ for $t \in (0,T]$, and $w = (\overline{\eta}_n - \eta)$ in (3.10).

Then, using (2.1), (3.5) and (3.8), it is observed that:

$$
\begin{aligned}
&\varlimsup_{n\to\infty}\Big(\int_0^t |\nabla\overline{\eta}_n(r)|^2_{[H]^N}\,dr+\frac{\mu^2}{2}\big(|\nabla\overline{\eta}_n(t)|^2_{[H]^N}-|\nabla\eta_0|^2_{[H]^N}\big)\Big)\\
&\leq \varlimsup_{n\to\infty}\left(\int_0^t |\nabla\overline{\eta}_n(r)|^2_{[H]^N}\,dr+\mu^2\int_0^t (\nabla\partial_t\eta_n(r),\nabla\overline{\eta}_n(r))_{[H]^N}\,dr\right) \qquad (3.12)\\
&\leq -\lim_{n\to\infty}\int_0^t \big((\partial_t\eta_n+g(\mathcal{T}_M\overline{\eta}_n)+\alpha'(\mathcal{T}_M\overline{\eta}_n)\gamma_\varepsilon(\nabla\overline{\theta}_n))(r),(\overline{\eta}_n-\eta)(r)\big)_H\,dr\\
&\quad+\lim_{n\to\infty}\int_0^t (\nabla(\overline{\eta}_n+\mu^2\partial_t\eta_n)(r),\nabla\eta(r))_{[H]^N}\,dr\\
&\quad+\lim_{n\to\infty}\int_0^t (\overline{u}_{\tau_n}(r),(\overline{\eta}_n-\eta)(r))_H\,dr\\
&=\int_0^t |\nabla\eta(r)|^2_{[H]^N}\,dr+\mu^2\int_0^t (\nabla\partial_t\eta(r),\nabla\eta(r))_{[H]^N}\,dr\\
&=\int_0^t |\nabla\eta(r)|^2_{[H]^N}\,dr+\frac{\mu^2}{2}\big(|\nabla\eta(t)|^2_{[H]^N}-|\nabla\eta_0|^2_{[H]^N}\big).
\end{aligned}
$$

Moreover, if we take $\omega=\theta$ in (3.11), then having in mind (2.1), (3.5) and (3.8), we see that:

$$
\begin{aligned}
&\varlimsup_{n\to\infty}\Big(\int_0^t\int_\Omega \alpha_M(\underline{\eta}_n(r))\gamma_\varepsilon(\nabla\overline{\theta}_n(r))\,dxdr+\frac{\nu^2}{2}(|\nabla\overline{\theta}_n(t)|^2_{[H]^N}-|\nabla\theta_0|^2_{[H]^N})\Big)\\
&\leq \varlimsup_{n\to\infty}\int_0^t\Big(\int_\Omega \alpha_M(\underline{\eta}_n(r))\gamma_\varepsilon(\nabla\overline{\theta}_n(r))\,dx+\nu^2(\nabla\partial_t\theta_n(r),\nabla\overline{\theta}_n(r))_{[H]^N}\Big)\,dr\\
&\leq \lim_{n\to\infty}\int_0^t (-\alpha_0(\mathcal{T}_M\underline{\eta}_n(r))\partial_t\theta_n(r)+\overline{v}_{\tau_n}(r),(\overline{\theta}_n-\theta)(r))_H\,dr \qquad (3.13)\\
&\quad+\lim_{n\to\infty}\int_0^t\Big(\int_\Omega \alpha_M(\underline{\eta}_n(r))\gamma_\varepsilon(\nabla\theta(r))\,dx+\nu^2(\nabla\partial_t\theta_n(r),\nabla\theta(r))_{[H]^N}\Big)dr\\
&=\int_0^t\int_\Omega \alpha_M(\eta(r))\gamma_\varepsilon(\nabla\theta(r))\,dxdr+\nu^2\int_0^t (\nabla\partial_t\theta(r),\nabla\theta(r))_{[H]^N}\,dr\\
&=\int_0^t\int_\Omega \alpha_M(\eta(r))\gamma_\varepsilon(\nabla\theta(r))\,dxdr+\frac{\nu^2}{2}\big(|\nabla\theta(t)|^2_{[H]^N}-|\nabla\theta_0|^2_{[H]^N}\big).
\end{aligned}
$$

On the other hand, from (3.8), (3.9) and Fatou's lemma, we have:

- $\varliminf_{n\to\infty}|\nabla\overline{\eta}_n(t)|^2_{[H]^N}\geq|\nabla\eta(t)|^2_{[H]^N}$, $\varliminf_{n\to\infty}|\nabla\overline{\theta}_n(t)|^2_{[H]^N}\geq|\nabla\theta(t)|^2_{[H]^N}$,
- $\varliminf_{n\to\infty}\int_0^t|\nabla\overline{\eta}_n(r)|^2_{[H]^N}\,dr\geq\int_0^t|\nabla\eta(r)|^2_{[H]^N}\,dr$, and (3.14)
- $\varliminf_{n\to\infty}\int_0^t\int_\Omega \alpha_M(\underline{\eta}_n(r))\gamma_\varepsilon(\nabla\overline{\theta}_n(r))\,dxdr\geq\int_0^t\int_\Omega \alpha_M(\eta(r))\gamma_\varepsilon(\nabla\theta(r))\,dxdr.$

Now, by the fact(∗) in Section 2, (3.9), (3.12)–(3.14), and the uniform convexity of $[H]^N$, we can derive the following convergences as $n \to \infty$:

$$|\nabla \overline{\eta}_n(t)|_{[H]^N} \to |\nabla \eta(t)|_{[H]^N}, \ |\nabla \overline{\theta}_n(t)|_{[H]^N} \to |\nabla \theta(t)|_{[H]^N}, \tag{3.15}$$
and therefore, $\overline{\eta}_n(t) \to \eta(t)$, $\overline{\theta}_n(t) \to \theta(t)$ in V, for all $t \in [0,T]$.

Moreover, from (3.7), (3.15), the boundedness of $[\partial_t \eta_\tau, \partial_t \theta_\tau]$ in $L^2(0,T;H)$, and Lebesgue's dominated convergence theorem, we find the following convergences as $n \to \infty$:

$$\begin{cases} \bullet \ \eta_n(t), \underline{\eta}_n(t) \to \eta(t), \ \theta_n(t), \underline{\theta}_n(t) \to \theta(t) \text{ in } V, \\ \bullet \ \overline{\eta}_n \to \eta, \ \overline{\theta}_n \to \theta \text{ in } L^2(0,T;V), \text{ with} \\ \quad |\overline{\eta}_n - \eta|_{L^2(0,T;V)}, |\overline{\theta}_n - \theta|_{L^2(0,T;V)} \to 0, \end{cases} \tag{3.16}$$

Easily, in view of (3.5), (3.8), (3.15) and (3.16), when $w = \varphi$ in V in (3.10) and $\omega = \psi$ in V in (3.11), we let $n \to \infty$ and observe that for any open interval $I \subset (0,T)$:

$$\int_I \big((\partial_t \eta + (g(\mathcal{T}_M \eta)) + \alpha'(\mathcal{T}_M \eta)\gamma_\varepsilon(\nabla\theta))(r), \varphi\big)_H \, dr \\ + \int_I (\nabla(\eta + \mu^2 \partial_t \eta)(r), \nabla\varphi)_{[H]^N} \, dr = \int_I (u(r), \varphi)_H \, dr.$$

and

$$\int_I (\alpha_0(\mathcal{T}_M \eta(r))\partial_t \theta(r), \theta(r) - \psi)_H \, dr + \int_I \int_\Omega \alpha_M(\eta(r))\gamma_\varepsilon(\nabla\theta(r)) \, dxdr \\ + \nu^2 \int_I (\nabla \partial_t \theta(r), \nabla(\theta(r) - \psi))_{[H]^N} \, dr \\ \leq \int_I \int_\Omega \alpha_M(\eta(r))\gamma_\varepsilon(\nabla\psi) \, dxdr + \int_I (v(r), \theta(r) - \psi)_H \, dr.$$

Since the interval $I \subset (0,T)$ is arbitrary, the limiting pair $[\eta, \theta]$ fulfills (S1) and (S2) if $|\eta|_{L^\infty(Q)} \leq M$.

Here, let us confirm the L^∞-boundedness for the limiting function η, and for θ when $v \equiv 0$. Take into account (A2) and (3.4), it follows that:

$$\begin{cases} \partial_t M - \Delta(M + \mu^2 \partial_t M) + g(M) + \alpha'(M)|\nabla\theta(t)| \geq u(t), \\ \partial_t(-M) - \Delta(-M + \mu^2 \partial_t(-M)) + g(-M) + \alpha'(-M)|\nabla\theta(t)| \leq u(t), \end{cases}$$
$$\text{a.e. on } \Omega, \text{ and for a.e. } t \in (0,T).$$

Hence, applying Lemma 2 with

$$\begin{cases} [\eta^1, \eta^2, \tilde{\theta}, \tilde{u}] = [\eta, M, \theta, u] \text{ in } L^2(0,T;H), \ [\eta_0^1, \eta_0^2] = [\eta_0, M] \text{ in } H, \text{ and} \\ [\eta^1, \eta^2, \tilde{\theta}, \tilde{u}] = [-M, \eta, \theta, u] \text{ in } L^2(0,T;H), \ [\eta_0^1, \eta_0^2] = [-M, \eta_0] \text{ in } H, \end{cases}$$

one can see that $|\eta(t)|_{L^\infty(Q)} \leq M$. Also, since any constant function satisfies the variational inequality (3.1), if we suppose $\theta_0 \in L^\infty(\Omega)$ and $v \equiv 0$, then applying Lemma 3 under:

$$\begin{cases} [\theta^1, \theta^2, \tilde{\eta}] = [\theta, |\theta_0|_{L^\infty(\Omega)}, \eta] \text{ in } L^2(0,T;H), \\ [\theta_0^1, \theta_0^2] = [\theta_0, |\theta_0|_{L^\infty(\Omega)}] \text{ in } H, \text{ and} \\ [\theta^1, \theta^2, \tilde{\eta}] = [-|\theta_0|_{L^\infty(\Omega)}, \theta, \eta] \text{ in } L^2(0,T;H), \\ [\theta_0^1, \theta_0^2] = [-|\theta_0|_{L^\infty(\Omega)}, \theta_0] \text{ in } H, \end{cases}$$

we obtain that $|\theta|_{L^\infty(Q)} \leq |\theta_0|_{L^\infty(\Omega)}$. Therefore, $[\eta, \theta]$ fulfills (S0)–(S2).

Next, we proceed to verify the energy inequality (1.2). Fix $s, t \in [0, T]$; $s < t$. For any $n = 1, 2, 3, \dots$, by summing up the both side of (2.2) for $i = n_{\tau_n}(s), n_{\tau_n}(s) + 1, \dots, n_{\tau_n}(t)$, it is observed that:

$$\begin{aligned} & C_0 \int_s^t \Big(|\partial_t \eta_n(r)|_V^2 + |\partial_t \theta_n(r)|_V^2 \Big)\, dr + \mathcal{F}_\varepsilon^M(\overline{\eta}_n(t), \overline{\theta}_n(t)) \qquad (3.17) \\ & \leq C_0 \int_{m_{\tau_n}(s)\tau_n}^{n_{\tau_n}(t)\tau_n} \Big(|\partial_t \eta_n(r)|_V^2 + |\partial_t \theta_n(r)|_V^2 \Big)\, dr + \mathcal{F}_\varepsilon^M(\overline{\eta}_n(t), \overline{\theta}_n(t)) \\ & \leq \mathcal{F}_\varepsilon^M(\underline{\eta}_n(s), \underline{\theta}_n(s)) + \frac{1}{2} \int_{m_{\tau_n}(s)\tau_n}^{n_{\tau_n}(t)\tau_n} \left(|u_0^{\mathrm{ex}}(r)|_H^2 + \frac{1}{\delta_*} |v_0^{\mathrm{ex}}(r)|_H^2 \right) dr. \end{aligned}$$

On this basis, owing to the convergences (3.5), (3.15), (3.16), the L^∞-boundedness of η, the estimate (3.17), and Lebesgue's dominated convergence theorem, letting $n \to \infty$ yields that:

$$\begin{aligned} & C_0 \int_s^t \Big(|\partial_t \eta(r)|_V^2 + |\nabla \partial_t \theta(r)|_V^2 \Big)\, dr + \mathcal{F}_\varepsilon(\eta(t), \theta(t)) \\ & \leq \varliminf_{n\to\infty} C_0 \int_s^t \Big(|\partial_t \eta_n(r)|_V^2 + |\nabla \partial_t \theta_n(r)|_V^2 \Big)\, dr + \lim_{n\to\infty} \mathcal{F}_\varepsilon^M(\overline{\eta}_n(t), \overline{\theta}_n(t)) \\ & \leq \lim_{n\to\infty} \mathcal{F}_\varepsilon^M(\underline{\eta}_n(s), \underline{\theta}_n(s)) + \frac{1}{2} \lim_{n\to\infty} \int_{m_{\tau_n}^s \tau_n}^{n_{\tau_n}^t \tau_n} \left(|u_0^{\mathrm{ex}}(r)|_H^2 + \frac{1}{\delta_*} |v_0^{\mathrm{ex}}(r)|_H^2 \right) dr \\ & = \mathcal{F}_\varepsilon(\eta(s), \theta(s)) + \frac{1}{2} \int_s^t \left(|u(r)|_H^2 + \frac{1}{\delta_*} |v(r)|_H^2 \right) dr. \end{aligned}$$

and hence the solution $[\eta, \theta]$ fulfills the energy inequality.

Now, our remaining task is the verification of uniqueness of solution. The proof of uniqueness is given in [3, Main Theorem 2], and as in [3, Remark 3.2], the points lie in the pseudo-parabolic regularity $\theta \in W^{1,2}(0,T;V)$ and the continuous embedding from V to $L^4(\Omega)$. So, In this article, we introduce the outline of proof. Let $[\eta^k, \theta^k]$, $k = 1, 2$, be the solutions to $(\mathrm{S})_\varepsilon$ corresponding to the same initial value $[\eta_0, \theta_0]$ and forcings $[u, v]$. Let us set $M_0 := |\eta^1|_{L^\infty(Q)} \vee |\eta^2|_{L^\infty(Q)}$, take the difference between the variational identities for η^k, $k = 1, 2$, and put $\varphi := (\eta^1 - \eta^2)(t)$. Then, by using (A1), (A2), convexity of α and Hölder's and Young's inequality, we can compute as follows:

$$\frac{1}{2}\frac{d}{dt}\bigl(|(\eta^1 - \eta^2)(t)|_H^2 + \mu^2|\nabla(\eta^1 - \eta^2)(t)|_{[H]^N}^2\bigr) \tag{3.18}$$
$$\leq |g'|_{L^\infty(-M_0,M_0)}|(\eta^1 - \eta^2)(t)|_H^2 + \frac{|\alpha'|_{L^\infty(-M_0,M_0)}}{2}\bigl(|(\eta^1 - \eta^2)(t)|_H^2 + |\nabla(\theta^1 - \theta^2)(t)|_{[H]^N}^2\bigr),$$
$$\text{for a.e. } t \in (0, T).$$

On the other hand, we consider putting $\psi = \theta^2$ in the variational inequality for θ^1, and $\psi = \theta^1$ in the one for θ^2, and adding the both sides of two inequalities. Then, using (A2), continuous embedding from V to $L^4(\Omega)$, and generalized chain rule in BV-theory, we can compute as follows:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\bigl(|\sqrt{\alpha_0(\eta^1(t))}(\theta^1 - \theta^2)(t)|_H^2 + \nu^2|\nabla(\theta^1 - \theta^2)(t)|_{[H]^N}^2\bigr)\\
&\leq \frac{|\alpha'|_{L^\infty(-M_0,M_0)}}{2}\bigl(|(\eta^1 - \eta^2)(t)|_H^2 + |\nabla(\theta^1 - \theta^2)(t)|_{[H]^N}^2\bigr)\\
&\quad + \frac{\bigl(C_V^{L^4}\bigr)^2|\alpha_0'|_{L^\infty(-M_0,M_0)}}{2}(|\partial_t\eta^1(t)|_H + |\partial_t\theta^2(t)|_V)\cdot\\
&\quad \cdot\bigl(|(\eta^1 - \eta^2)(t)|_H^2 + |(\theta^1 - \theta^2)(t)|_V^2\bigr), \text{ for a.e. } t \in (0, T).
\end{aligned} \tag{3.19}$$

Therefore, putting

$$\begin{aligned}
J(t) &:= |(\eta^1 - \eta^2)(t)|_H^2 + \mu^2|\nabla(\eta^1 - \eta^2)(t)|_{[H]^N}^2\\
&\quad + |\sqrt{\alpha_0(\eta^1(t))}(\theta^1 - \theta^2)(t)|_H^2 + \nu^2|\nabla(\theta^1 - \theta^2)(t)|_{[H]^N}^2, \text{ for } t \in [0, T],\\
C_3 &:= \frac{2\bigl(|\alpha'|_{L^\infty(-M_0,M_0)} + |g'|_{L^\infty(-M_0,M_0)} + (C_V^{L^4})^2|\alpha_0'|_{L^\infty(-M_0,M_0)}\bigr)}{1 \wedge \delta_* \wedge \nu^2},
\end{aligned}$$

it is deduced from (3.18) and (3.19) that:

$$\frac{d}{dt}J(t) \leq C_3\bigl(|\partial_t\eta^1(t)|_H + |\partial_t\theta^2(t)|_V + 1\bigr)J(t), \quad \text{a.e. } t \in (0, T). \tag{3.20}$$

(3.20) implies that the solution to $(\mathrm{S})_\varepsilon$ is unique, and thus, we complete the proof of Theorem 1. □

3.2. Proof of Theorem 2

By setting a large constant $M^* > 0$ such that

$$M^* \geq \sup_{n\in\mathbb{N}}(|\eta_{0,n}|_{L^\infty(\Omega)} \vee |u_n|_{L^\infty(Q)}), \text{ and}$$
$$g(M^*) \geq \sup_{n\in\mathbb{N}}|u_n|_{L^\infty(Q)}, \; g(-M^*) \leq \sup_{n\in\mathbb{N}} -|u_n|_{L^\infty(Q)},$$

and applying the same argument in Theorem 1, we obtain $|\eta_n|_{L^\infty(Q)} \leq M^*$ for $n = 1, 2, 3, \dots$. Also, thanks to Theorem 1, the sequence $\{[\eta_n, \theta_n]\}_{n=1}^\infty$ satisfies the following energy-inequality for $n = 1, 2, 3, \dots$:

$$C_0 \int_0^T \left(|\partial_t \eta_n(r)|_V^2 + |\partial_t \theta_n(r)|_V^2\right) dr + \mathcal{F}_{\varepsilon_n}(\eta_n(T), \theta_n(T)) \\ \leq \mathcal{F}_{\varepsilon_n}(\eta_{0,n}, \theta_{0,n}) + \frac{1}{2} \int_0^T \left(|u_n(r)|_H^2 + \frac{1}{\delta_*}|v_n(r)|_H^2\right) dr. \qquad (3.21)$$

Now, taking into account (1.3), (3.21) and L^∞-boundedness of η_n, we can derive the following boundedness:

– $\{\eta_n\}_{n=1}^\infty$ is bounded in $W^{1,2}(0, T; V)$ and in $L^\infty(Q)$,
– $\{\theta_n\}_{n=1}^\infty$ is bounded in $W^{1,2}(0, T; V)$,

By virtue of Aubin's type compactness theory and variational techniques used in (3.12)–(3.16), we can find a subsequence $\{n_k\} \subset \{n\}$; $n_k \uparrow \infty$, and a pair of functions $[\bar{\eta}, \bar{\theta}] \in [W^{1,2}(0, T; V) \cap L^\infty(Q)] \times W^{1,2}(0, T; V)$ fulfilling the following convergences as $k \to \infty$:

- $\eta_{n_k} \to \bar{\eta}$ in $C([0, T]; H), L^2(0, T; V),$ weakly in $W^{1,2}(0, T; V)$,

  and weakly-$*$ in $L^\infty(Q)$, (3.22)
- $\theta_{n_k} \to \bar{\theta}$ in $C([0, T]; H), L^2(0, T; V),$ weakly in $W^{1,2}(0, T; V)$,
- $\eta_{n_k}(t) \to \bar{\eta}(t), \theta_{n_k}(t) \to \bar{\theta}(t)$ in $V,$ for all $t \in [0, T]$.

In particular, by (1.3) and (3.22), we see that:

$$[\bar{\eta}(0), \bar{\theta}(0)] = \lim_{k \to \infty} [\eta_{n_k}(0), \theta_{n_k}(0)] \\ = \lim_{k \to \infty} [\eta_{0,n_k}, \theta_{0,n_k}] = [\eta_0, \theta_0] \text{ in } [H]^2.$$

Additionally, we can check the pair $[\bar{\eta}, \bar{\theta}]$ satisfies the variational inequalities (S1) and (S2) by letting $k \to \infty$ in the following (3.23) and (3.24):

$$\int_I ((\partial_t \eta_{n_k} + g(\eta_{n_k}))(t), \varphi)_H \, dt + \int_I (\nabla(\eta_{n_k} + \mu^2 \partial_t \eta_{n_k})(t), \nabla \varphi)_{[H]^N} \, dt \\ + \int_I \int_\Omega \alpha'(\eta_{n_k}(t)) \varphi \, \gamma_{\varepsilon_{n_k}}(\nabla \theta_{n_k}(t)) \, dx dt = \int_I (u_{n_k}(t), \varphi)_H \, dt, \qquad (3.23)$$

for any $\varphi \in V$, and

$$\int_I (\alpha_0(\eta_{n_k}(t)) \partial_t \theta_{n_k}(t), \theta_{n_k}(t) - \psi)_H \, dt + \int_I \int_\Omega \alpha(\eta_{n_k}(t)) \gamma_{\varepsilon_{n_k}}(\nabla \theta_{n_k}(t)) \, dx dt \\ + \nu^2 \int_I (\nabla \partial_t \theta_{n_k}(t), \nabla(\theta_{n_k}(t) - \psi))_{[H]^N} \, dt \\ \leq \int_I \int_\Omega \alpha(\eta_{n_k}(t)) \gamma_{\varepsilon_{n_k}}(\nabla \psi) \, dx dt + \int_I (v_{n_k}(t), \theta_{n_k}(t) - \psi)_H \, dt, \quad (3.24)$$

for any $\psi \in V$ and any open interval $I \subset (0,T)$. Therefore, the pair $[\bar{\eta}, \bar{\theta}]$ is a solution to $(\mathrm{S})_\varepsilon$, and we see that $[\bar{\eta}, \bar{\theta}]$ coincides the unique solution $[\eta, \theta]$.

Finally, by the uniqueness of the limit, the convergence (1.4) is verified. Thus, we complete the proof of Theorem 2.

References

1. Ambrosio L., Fusco N., Pallara D. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Mathematical Monographs. New York, The Clarendon Press, Oxford University Press, 2000. https://doi.org/10.1093/oso/9780198502456.001.0001
2. Antil H., Kubota S., Shirakawa K., Yamazaki N. Constrained optimization problems governed by PDE models of grain boundary motions. *Adv. Nonlinear Anal.*, 2022, vol. 11, no. 1, pp. 1249–1286. https://doi.org/10.1515/anona-2022-0242
3. Antil H., Mizuno D., Shirakawa K. Well-posedness of a pseudo-parabolic KWC system in materials science. *SIAM J. Math. Anal.*, 2024, vol. 56, no. 5, pp. 6422–6455. https://doi.org/10.1137/24M163952X
4. Berselli L.C., Bisconti L. On the structural stability of the Euler-Voigt and Navier-Stokes-Voigt models. *Nonlinear Anal.*, 2012, vol. 75, no. 1, pp. 117–130. https://doi.org/10.1016/j.na.2011.08.011
5. Cao Y., Lunasin E.M., Titi E.S. Global well-posedness of the three-dimensional viscous and inviscid simplified Bardina turbulence models. *Commun. Math. Sci.*, 2006, vol. 4, no. 4, pp. 823–848. http://projecteuclid.org/euclid.cms/1175797613
6. Ito A., Kenmochi N., Yamazaki N. A phase-field model of grain boundary motion. *Appl. Math.*, 2008, vol. 53, no. 5, pp. 433–454. https://doi.org/10.1007/s10492-008-0035-8
7. Kobayashi R., Warren J.A., Carter W.C. A continuum model of grain boundaries. *Phys. D*, 2000, vol. 140, no. 1–2, pp. 141–150. https://doi.org/10.1016/S0167-2789(00)00023-3
8. Kobayashi R., Warren J.A., Carter W.C. Grain boundary model and singular diffusivity. *GAKUTO Internat. Ser. Math. Sci. Appl.*, 2000, vol. 14, pp. 283–294.
9. Kubota S., Shirakawa K. Periodic solutions to Kobayashi–Warren–Carter systems. *Adv. Math. Sci. Appl.*, 2023, vol. 32, pp. 511–553.
10. Moll S., Shirakawa K. Existence of solutions to the Kobayashi–Warren–Carter system. *Calc. Var. Partial Differential Equations*, 2014, vol. 51, no. 3–4, pp. 621–656. https://doi.org/10.1007/s00526-013-0689-2
11. Moll S., Shirakawa K., Watanabe H. Energy dissipative solutions to the Kobayashi–Warren–Carter system. *Nonlinearity*, 2017, vol. 30, no. 7, pp. 2752–2784. https://doi.org/10.1088/1361-6544/aa6eb4
12. Moll S., Shirakawa K., Watanabe H. Kobayashi-Warren-Carter type systems with nonhomogeneous Dirichlet boundary data for crystalline orientation. *Nonlinear Anal.*, 2022, vol. 217, paper no. 112722. https://doi.org/10.1016/j.na.2021.112722
13. Nakayashiki R., Shirakawa K. Kobayashi-Warren-Carter system of singular type under dynamic boundary condition. *Discrete Contin. Dyn. Syst. Ser. S*, 2023, vol. 16, no. 12, pp. 3746–3783. https://doi.org/10.3934/dcdss.2023162
14. Oskolkov A.P. The uniqueness and solvability in the large of boundary value problems for the equations of motion of aqueous solutions of polymers. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 1973, vol. 38, pp. 98–136.
15. Oskolkov A.P. On the theory of unsteady flows of Kelvin-Voigt fluids. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 1982, vol. 115, pp. 191–202.

16. Simon J. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 1987, vol. 146, pp. 65–96. https://doi.org/10.1007/BF01762360
17. Yamazaki N. Global attractors for non-autonomous phase-field systems of grain boundary motion with constraint. *Adv. Math. Sci. Appl.*, 2013, vol. 23, no. 1, pp. 267–296.

Об авторах

Мизуно Дайки, аспирант, Университет Тиба, Тиба, 263-8522, Япония, d-mizuno@chiba-u.jp

About the authors

Daiki Mizuno, Postgraduate, Chiba University, Chiba, 263-8522, Japan, d-mizuno@chiba-u.jp

Поступила в редакцию / Received 25.07.2024
Поступила после рецензирования / Revised 03.09.2024
Принята к публикации / Accepted 06.09.2024