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Hyperbolic Volumes of Two Bridge Cone-Manifolds

Alexander D. Mednykh[1,2], **Aydos B. Qutbaev**[1,2,3]✉

[1] Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russian Federation
[2] Novosibirsk State University, Novosibirsk, Russian Federation
[3] Nukus State Pedagogical Institute named after Ajiniyaz, Nukus, Karakalpakstan, Uzbekistan
✉ aydosqutbaev@gmail.com

Abstract. In this paper we investigate the existence of hyperbolic, Euclidean and spherical structures on cone-manifolds with underlying space 3-sphere and with singular set a given two-bridge knot. For two-bridge knots with 8 crossings we present trigonometric identities involving the length of singular geodesics and cone angles of such cone-manifolds. Then these identities are used to produce exact integral formulae for the volume of the corresponding cone-manifold modeled in the hyperbolic space.

Keywords: cone-manifold, orbifold, two-bridge knot, volume, geodesic length

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Научная статья

Гиперболические объемы конических многообразий для двумостовых узлов

А. Д. Медных[1,2], **А. Б. Кутбаев**[1,2,3]✉

[1] Институт математики им. С. Л. Соболева СО РАН, Новосибирск, Российская Федерация
[2] Новосибирский государственный университет, Новосибирск, Российская Федерация
[3] Нукусский государственный педагогический институт им. Ажинияза, Нукус, Каракалпакстан, Узбекистан

✉ aydosqutbaev@gmail.com

Аннотация. Исследуется существование гиперболических, евклидовых и сферических структур на трехмерной сфере, сингулярным множеством которых служит двумостовый узел. Для двумостовых узлов с 8 пересечениями представляются тригонометрические тождества, включающие длину сингулярных геодезических и конических углов таких конических многообразий. Затем эти тождества используются для получения точных интегральных формул для объема соответствующего конического многообразия, смоделированного в гиперболическом пространстве.

Ключевые слова: коническое многообразие, орбифолд, двумостовый узел, объем, геодезическая длина

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1. Introduction

In 1975 R.Riley found examples of hyperbolic structures on some knot and link complements in the 3-sphere. Seven of them, so called excellent knots, were described in [13]. One more, very important case of the figure eight knot was investigated in his manuscript [3; 14]. Later, in the spring of 1977, W.P.Thurston announced an existence theorem for Riemannian metrics of constant negative curvature on 3-manifolds. In particular, it turned out that the knot complement of a simple knot (excepting for torus and satellite) admits a complete hyperbolic structure. This fact allowed us to consider knot theory as a part of the geometry and theory of discrete groups.

Starting from Alexander's work [2], polynomial invariants became a convenient instrument for study of knots. A lot of different kinds of such polynomials were discovered in the last two decades of the twentieth century. Among them we mention the Jones, Kaufmann bracket and HOMFLY-PT polynomials, complex distance polynomial, A-polynomial etc.; for more information see [4; 6; 7; 11; 15]. This relates the knot theory with algebra and algebraic geometry. The algebraic technique is used to find the most important geometrical characteristics of knots such as the volume, length of shortest geodesics and others.

The geometry of knots is completely defined by the geometry of their fundamental polyhedra. The results related to the volume and other basic geometric invariants of polyhedra in the spaces of constant curvature were given in [1; 16].

In this work, we investigate the existence of hyperbolic structure on cone-manifolds whose underlying space is the 3-sphere and whose singular set is a given knot with 8 crossings. We present trigonometrical identities involving the lengths of singular geodesics and cone angles of such cone-manifolds. These identities will be used to produce exact integral formulae for volumes of the corresponding cone-manifolds. This type of results were obtained by the first author in [9] for two-bridge knot cone manifolds with not more than seven crossings.

2. Preliminaries

Let us present some basic concepts on the theory of manifolds from [5].

An *n-dimensional cone-manifold* is a manifold, M, which can be triangulated so that the link of each simplex is piecewise linear homeomorphic to a standard sphere and M is equipped with a complete path metric such that the restriction of the metric to each simplex is isometric to a geodesic simplex of constant curvature K. The cone-manifold is *hyperbolic* if K is -1.

The *singular locus* Σ of a cone-manifold M consists of the points with no neighbourhood isometric to a ball in a Riemannian manifold. It follows that

- Σ is a union of totally geodesic closed simplices of dimension $n-2$.
- At each point of Σ in an open $(n-2)$-simplex, there is a *cone angle* which is the sum of dihedral angles of n-simplices containing the point.
- $M-\Sigma$ has a smooth Riemannian metric of constant curvature K, but this metric is *incomplete* if $\Sigma \neq \emptyset$.

In this paper, we will deal only with cone-manifolds whose underlying space M is the 3-manifold sphere $\mathbb{S}^3$ and the singular set Σ is a knot.

The main tool for volume calculation is the following formula [10].

Theorem 1 (**Schläfli**). *Suppose that C_t is a smooth 1-parameter family of (curvature K) cone-manifold structures on a $n-$manifold, with singular locus Σ of a fixed topological type. Then the derivative of the volume of C_t satisfies*

$$(n-1)KdV(C_t) = \sum V_{n-2}(\sigma)d\theta(\sigma),$$

where the sum is taken over all components σ of the singular locus Σ and $\theta(\sigma)$ is the cone angle along σ, $V_{n-2}(\cdot)$ stands for $n-2$ — dimensional volume.

Since we will deal only with 3-dimensional cone-manifolds of constant curvature K, the Schläfli formula in this case reduces to

$$KdV = \frac{1}{2}\sum_i l_{\alpha_i} d\alpha_i,$$

where the sum is taken over all components of the singular set Σ with lengths l_{α_i} and cone angles α_i.

Let M be a hyperbolic 3-dimensional cone-manifold whose singular set Σ_α is a knot with cone angle $\alpha, 0 < \alpha \leq 2\pi$. Following [8], choose the canonical longitude-meridian pair (l, m) in the fundamental group $\pi_1(M \setminus \Sigma_\alpha)$ in such a way that m is an oriented boundary of meridian disc of Σ_α and a longitude curve l is nullhomologous outside of Σ_α. Let $h : \pi_1(M \setminus \Sigma_\alpha) \to PSL(2,\mathbb{C})$ be the holonomy map of $M \setminus \Sigma_\alpha$. Then h admits two liftings to $SL(2,\mathbb{C})$. The image of l in $SL(2,\mathbb{C})$ under these two liftings is the same since l is nullhomologous outside the singular set. Thus up to conjugation in $SL(2,\mathbb{C})$,

$$h(m) = \pm\begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2}\end{bmatrix}, h(l) = \begin{bmatrix} e^{\gamma_\alpha/2} & 0 \\ 0 & e^{-\gamma_\alpha/2}\end{bmatrix}$$

where $\gamma_\alpha = l_\alpha + i\phi_\alpha$, l_α is the length of Σ_α, and $\phi_\alpha, -2\pi \leq \phi_\alpha < 2\pi$, is the angle of the lifted holonomy of Σ_α. For the sake of simplicity, we will refer to $\gamma_\alpha = l_\alpha + i\phi_\alpha$ as a *complex length* of the singular geodesics Σ_α.

In the present paper, we make use of the notion of A-polynomial for a manifold M introduced in [4]. An important property of this polynomial is that cone angle α and complex length γ_α of the singular set Σ_α are related by the equation

$$A(L, M) = 0, \textit{ where } L = e^{\gamma/2}, M = e^{i\alpha/2}. \tag{2.1}$$

Also, by the basic properties of A-polynomial we easily obtain the equalities $A(L, M) = A(L^{-1}, M)$ and $A(L, M) = A(L, -M)$.

3. The knot 8_1

Knot 8_1 is a rational knot of a slope $\frac{13}{6}$. It is shown in Figure 1 below.

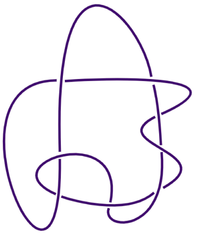

Figure 1. The knot 8_1.

The following theorem describes the geometry of this knot.

Theorem 2. *A cone-manifold* $8_1(\alpha)$ *is hyperbolic for* $0 < \alpha < \alpha_0$, *Euclidean for* $\alpha = \alpha_0$, *and spherical for* $\alpha_0 < \alpha < 2\pi - \alpha_0$, *where* $\alpha_0 = 2.75069...$ *and* $A_0 = \cot \frac{\alpha_0}{2}$ *is a root of the algebraic equation*

$$92051A^{20} + 1408494A^{18} - 386457A^{16} - 1596504A^{14} + 28472022A^{12} - \\ -38140844A^{10} - 102389898A^{8} + 105581736A^{6} - 52317161A^{4} + \\ +11367278A^{2} - 371293 = 0.$$

The proof of the theorem will be done by repeating of the arguments for the proof of theorem 4 given in the next section.

The volume of the cone-manifold $8_1(\alpha)$ modeled in the hyperbolic geometry is given by the following statement.

Theorem 3. *Let* $8_1(\alpha)$ *be a hyperbolic cone-manifold that is* $0 < \alpha < \alpha_0 = 2.75069...$. *Then the volume of* $8_1(\alpha)$ *is given by the formula*

$$Vol(8_1(\alpha)) = 2\int_{A_0}^{A} f(\mathcal{A})d\mathcal{A},$$

where $A_0 = cot(\alpha_0/2)$, $A = cot(\alpha/2)$ *and*

$$f(\mathcal{A}) = \begin{cases} \frac{\ln|\frac{\mathcal{A}-iz_2}{\mathcal{A}+iz_2}|}{1+\mathcal{A}^2}, & \textit{if } 1.3946 \le \mathcal{A} \le 2.38, \\ \frac{\ln|\frac{\mathcal{A}-iz_4}{\mathcal{A}+iz_4}|}{1+\mathcal{A}^2}, & \textit{if } 0.597 \le \mathcal{A} \le 1.3946,\ 2.38 \le \mathcal{A} \le 2.435, \\ \frac{\ln|\frac{\mathcal{A}-iz_6}{\mathcal{A}+iz_6}|}{1+\mathcal{A}^2}, & \textit{otherwise,} \end{cases}$$

where the points $\{1.3946..., 2.38..., 2.435...\}$ *form the intersection points of the corresponding branches of the analytic functions, and* z_i *are defined in the program Wolfram Mathematica 13.2 [9] by the following command*

$$z\,[i\_] := \mathrm{Root}\left[-1 + 45\mathcal{A}^2 - 227\mathcal{A}^4 + 239\mathcal{A}^6 + \#\mathit{1}(420\mathcal{A}^6 - 836\mathcal{A}^4 + 268\mathcal{A}^2 - 12) + \#\mathit{1}^2\left(395\mathcal{A}^6 - 767\mathcal{A}^4 + 353\mathcal{A}^2 - 21\right) + \#\mathit{1}^3(208\mathcal{A}^6 - 432\mathcal{A}^4 - 80\mathcal{A}^2 + 48) + \#\mathit{1}^4\left(69\mathcal{A}^6 - 97\mathcal{A}^4 - 81\mathcal{A}^2 + 85\right) + \#\mathit{1}^5(12\mathcal{A}^6 - 12\mathcal{A}^4 - 60\mathcal{A}^2 - 36) + \#\mathit{1}^6\left(\mathcal{A}^6 + 3\mathcal{A}^4 + 3\mathcal{A}^2 + 1\right)\&, i\right].$$

Remark 1. The function $f(\mathcal{A})$ constructed by the described way will be continous [Figure 2]. Theorem 2 will be proved by the argument given in the next section for the proof of a similar statement for the cone-manifold $8_2(\alpha)$.

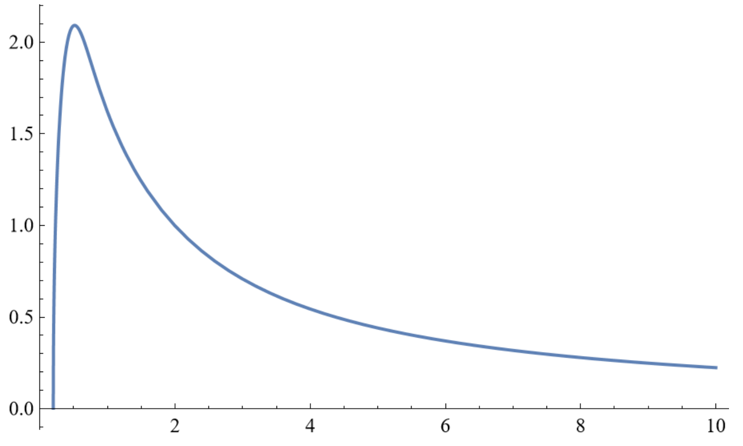

Figure 2. Graph of $f(\mathcal{A})$ in the Theorem 2.

4. The knot 8_2

Knot 8_2 is a rational knot of a slope $\frac{17}{6}$ shown in Figure 3 below.

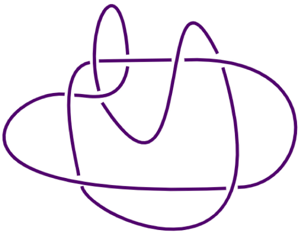

Figure 3. The knot 8_2.

The following theorem describes the geometry of this knot.

Theorem 4. *A cone-manifold $8_2(\alpha)$ is hyperbolic for $0 < \alpha < \alpha_0$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha_0 < \alpha < 2\pi - \alpha_0$, where $\alpha_0 = 2.84713060...$ and $A_0 = \cot\frac{\alpha_0}{2}$ is a root of the algebraic equation*

$$21309911A^{24} + 316656444A^{22} + 1327148086A^{20} + 8929478012A^{18} +$$

$$18155484113A^{16} + 61695875128A^{14} + 15967488340A^{12} + 30130064920A^{10} - 208555548055A^{8} + 167417176332A^{6} - 116692367402A^{4} + 21144510444A^{2} - 410338673 = 0.$$

Proof. By J. Porti [12] there exists a unique positive α_0 such that cone-manifold $8_2(\alpha)$ is hyperbolic for $0 < \alpha < \alpha_0$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha_0 < \alpha < 2\pi - \alpha_0$. Therefore, to prove the theorem it is enough to find α_0. Consider A-polynomial $A_{8_2}(L, M)$ presented in [15]:

$$A_{8_2}(L, M) = L^8M^4 + L^7\left(-5M^{16} + 9M^{14} + 7M^{12} - 3M^{10} - M^4 + 2M^2 - 1\right) + L^6(10M^{28} - 32M^{26} - M^{24} + 56M^{22} + 17M^{20} - 28M^{18} - M^{16} + 14M^{14} - 4M^{12} - 8M^{10} + 7M^8 + 2M^6) + L^5(-10M^{40} + 42M^{38} - 24M^{36} - 87M^{34} + 29M^{32} + 143M^{30} + 33M^{28} - 77M^{26} - 17M^{24} + 29M^{22} - 2M^{20} + M^{18} - 8M^{16} + 5M^{14} - M^{12}) + L^4(5M^{52} - 24M^{50} + 26M^{48} + 36M^{46} - 43M^{44} - 108M^{42} + 47M^{40} + 192M^{38} + 47M^{36} - 108M^{34} - 43M^{32} + 36M^{30} + 26M^{28} - 24M^{26} + 5M^{24} + L^3(-M^{64} + 5M^{62} - 8M^{60} + M^{58} - 2M^{56} + 29M^{54} - 17M^{52} - 77M^{50} + 33M^{48} + 143M^{46} + 29M^{44} - 87M^{42} - 24M^{40} + 42M^{38} - 10M^{36}) + L^2(-2M^{70} + 7M^{68} - 8M^{66} - 4M^{64} + 14M^{62} - M^{60} - 28M^{58} + 17M^{56} + 56M^{54} - M^{52} - 32M^{50} + 10M^{48}) + L(-M^{76} + 2M^{74} - M^{72} - 3M^{66} + 7M^{64} + 9M^{62} - 5M^{60}) + M^{72}.$$

Denote by $\gamma_\alpha = l_\alpha + i\varphi_\alpha$ the complex length of the longitude for the hyperbolic cone-manifold $8_2(\alpha)$ chosen in such a way that $l_\alpha > 0$ and $-2\pi \le \varphi_\alpha < 2\pi$.

We set $L = e^{\gamma_\alpha/2}$ and $M = e^{i\alpha/2}$. Then, by definition of A-polynomial (see [4]), we have $A_{8_2}(L, M) = 0$. Next we will use the following statement, the so called cotangent rule.

Theorem 5. *Let $\gamma_\alpha = l_\alpha + i\varphi_\alpha$ be the complex length of the longitude for the hyperbolic cone-manifold $8_2(\alpha)$ and $\gamma'_\alpha = \gamma_\alpha + 12i\alpha$. Then*

$$\coth\frac{\gamma'_\alpha}{4}\cot\frac{\alpha}{2} = iz,$$

where $z, \Im z < 0$ is a root of the equation

$$-1 + 69A^2 - 627A^4 + 1351A^6 - 14z + 542A^2z - 2970A^4z + 2618A^6z - 22z^2 + 774A^2z^2 - 2482A^4z^2 + 2866A^6z^2 + 178z^3 - 1122A^2z^3 - 1242A^4z^3 + 2106A^6z^3 + 400z^4 - 952A^2z^4 - 288A^4z^4 + 1064A^6z^4 - 154z^5 - 438A^2z^5 + 98A^4z^5 + 382A^6z^5 - 122z^6 - 150A^2z^6 + 66A^4z^6 + 94A^6z^6 - 10z^7 - 6A^2z^7 + 18A^4z^7 + 14A^6z^7 + z^8 + 3A^2z^8 + 3A^4z^8 + A^6z^8 = 0$$

and $A = \cot\frac{\alpha}{2}$.

Proof. Note that

$$\coth\frac{\gamma'_\alpha}{4} = \frac{LM^{12}+1}{LM^{12}-1}, \quad \cot\frac{\alpha}{2} = i\frac{M^2+1}{M^2-1},$$

where $L = e^{\gamma_\alpha/2}$ and $M = e^{i\alpha/2}$. We put

$$z = i\coth\frac{\gamma'_\alpha}{4}\cot\frac{\alpha}{2} = \frac{(L+M^{12})(M^2+1)}{(L-M^{12})(M^2-1)}, \quad A = \cot\frac{\alpha}{2} = i\frac{M^2+1}{M^2-1}.$$

Then

$$L = -(\frac{A+i}{A-i})^6\frac{A+iz}{A-iz}, \quad M = \sqrt{\frac{A+i}{A-i}}.$$

Substituting these identities into equation $A_{82}(L,M)=0$ we get the equation in the theorem in A and z.

Since $\Re(\gamma'_\alpha) > 0$ we have $\Im z < 0$. □

The value of α_0 that we call *a limit of hyperbolicity* is a positive root of discriminant of the equation in Theorem 4. Therefore we have

$$-410338673 + 211445104444A^2 - 116692367402A^4 + 167417176332A^6 -$$
$$208555548055A^8 + 30130064920A^{10} + 15967488340A^{12} + 61695875128A^{14} +$$
$$18155484113A^{16} + 8929478012A^{18} + 1327148086A^{20} +$$
$$316656444A^{22} + 21309911A^{24} = 0.$$

The latest has three positive roots $\{0.148304..., 0.461826..., 1.00073...\}$. The smallest of them $A_0 = 0.148304\ldots$ (corresponding the highest

$$\alpha_0 = 2\,arccot A_0 = 2.84713060\ldots)$$

is the limit of hyperbolicity. □

The volume of the cone-manifold $8_2(\alpha)$ modeled in the hyperbolic geometry is given by the following statement.

Theorem 6. *Let* $8_2(\alpha)$ *be a hyperbolic cone-manifold that is* $0 < \alpha < \alpha_0 = 2.84713060....$ *Then the volume of* $8_2(\alpha)$ *is given by the formula*

$$Vol(8_2(\alpha)) = 2\int_{A_0}^{A} f(\mathcal{A})d\mathcal{A},$$

where $A_0 = cot(\alpha_0/2) = 0.148304..., A = cot(\alpha/2)$ *and*

$$f(\mathcal{A}) = \begin{cases} |\frac{\ln|\frac{\mathcal{A}-iz_4}{\mathcal{A}+iz_4}|}{1+\mathcal{A}^2}|, & \textit{if } \mathcal{A} \le 1, \\ \max|\frac{\ln|\frac{\mathcal{A}-iz_5}{\mathcal{A}+iz_5}|}{1+\mathcal{A}^2}|, |\frac{\ln|\frac{\mathcal{A}-iz_7}{\mathcal{A}+iz_7}|}{1+\mathcal{A}^2}|, & \textit{if } 1 \le \mathcal{A} \le 1.589, \\ |\frac{\ln|\frac{\mathcal{A}-iz_5}{\mathcal{A}+iz_5}|}{1+\mathcal{A}^2}|, & \textit{if } 1.589 \le \mathcal{A} \le 1.7, \\ |\frac{\ln|\frac{\mathcal{A}-iz_7}{\mathcal{A}+iz_7}|}{1+\mathcal{A}^2}|, & \textit{if } \mathcal{A} \ge 1.7 \end{cases}$$

where the points $\{1.00..., 1.589..., 1.7...\}$ *form the intersection points of the corresponding branches of the analytic functions, and* z_i *are defined in the programm Wolfram Mathematica 13.2 by the following command*

$$z\,[i\_, A\_] := \text{Root}\,\big[-1 + 69A^2 - 627A^4 + 1351A^6 + (-14 + 542A^2 - 2970A^4 + 2618A^6)\#1 + \left(-22 + 774A^2 - 2482A^4 + 2866A^6\right)\#1^2 + (178 - 1122A^2 - 1242A^4 + 2106A^6)\#1^3 + \left(400 - 952A^2 - 288A^4 + 1064A^6\right)\#1^4 + \left(-154 - 438A^2 + 98A^4 + 382A^6\right)\#1^5 + \left(-122 - 150A^2 + 66A^4 + 94A^6\right)\#1^6 + \left(-10 - 6A^2 + 18A^4 + 14A^6\right)\#1^7 + \left(1 + 3A^2 + 3A^4 + A^6\right)\#1^8\&, i\big]\,.$$

Proof. Let $V = V(8_2(\alpha))$ and $l_\alpha = \Re(\gamma_\alpha^{'})$ be a real length of the longitude of $8_2(\alpha)$. By the Schläfli formula we have $dV/d\alpha = -l_\alpha/2$. Since the cone-manifold $8_2(\alpha)$ is Euclidean for $\alpha = \alpha_0$, we get $V \to 0$ as $\alpha \to \alpha_0$. Thus, V is a unique solution of the following differential problem

$$\frac{dV}{d\alpha} = -\frac{l_\alpha}{2}, \quad V(\alpha_0) = 0.$$

We show that the function in the theorem satisfies these conditions.

Note that z_i are the roots of the equivalent equation to the $A-$polynomial equation. Hence, by the Cotangent rule (Theorem 5), we have $A\coth\gamma_\alpha^{'}/4 = iz_i$. Since $\frac{dA}{d\alpha} = -\frac{1+A^2}{2}$ we get

$$\frac{dV}{d\alpha} = 2f(A)\frac{dA}{d\alpha} = 2|\frac{\ln|\frac{A-iz_i}{A+iz_i}|}{1+A^2}|(-\frac{1+A^2}{2}) = -|\ln|\frac{A-iz_i}{A+iz_i}|| =$$

$$-|\ln|\frac{A - A\coth\gamma_\alpha^{'}/4}{A + A\coth\gamma_\alpha^{'}/4}|| = -|\ln|\frac{1-\coth\gamma_\alpha^{'}/4}{1+\coth\gamma_\alpha^{'}/4}|| =$$

$$-|\ln|e^{\gamma_\alpha^{'}/2}|| = -|\ln|e^{l_\alpha/2}|| = -\frac{l_\alpha}{2}.$$

Moreover, it is obvious that $V(\alpha_0) = 0$. □

Remark 2. The function $f(\mathcal{A})$ constructed by this way will be continuous [Figure 4].

5. The knot 8_3

Knot 8_3 is a rational knot of slope $\frac{17}{4}$. See Figure 5 below.

One can obtain similar statements on existence of hyperbolic, Euclidean and spherical structures and on the hyperbolic volume of the cone-manifold $8_3(\alpha)$. The proof of next two theorems are similar to the proofs of Theorems of 4 and 6.

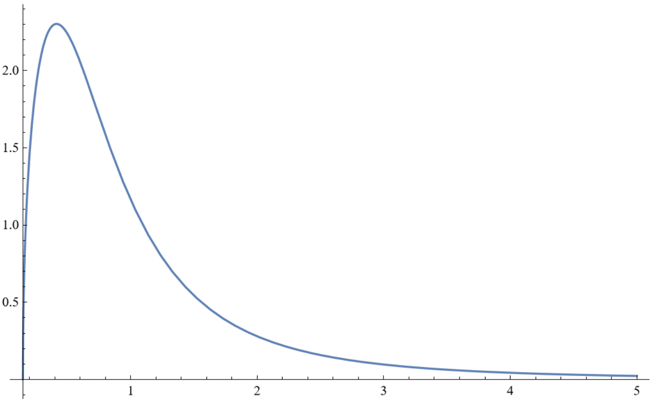

Figure 4. The graph of the function $f(\mathcal{A})$.

Theorem 7. *A cone-manifold $8_3(\alpha)$ is hyperbolic for $0 < \alpha < \alpha_0$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha_0 < \alpha < 2\pi - \alpha_0$, where $\alpha_0 = 2.84764...$ and $A_0 = \cot\frac{\alpha_0}{2}$ is the smallest positive root of the algebraic equation*

$$17A^4 - 46A^2 + 1 = 0.$$

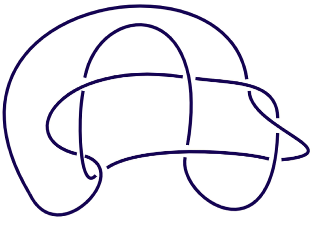

Figure 5. The knot 8_3.

Theorem 8. *Let $8_3(\alpha)$ be a hyperbolic cone-manifold that is $0 < \alpha < \alpha_0 = 2.84764...$. Then the volume of $8_3(\alpha)$ is given by the formula*

$$Vol(8_3(\alpha)) = 2\int_{A_0}^{A} f(\mathcal{A})d\mathcal{A},$$

where $A_0 = cot(\alpha_0/2)$, $A = cot(\alpha/2)$ and

$$f(\mathcal{A}) = \begin{cases} \ln|\frac{\mathcal{A}-iz_5}{\mathcal{A}+iz_5}|, & \text{if } \mathcal{A} \leq 1.638, \\ \ln|\frac{\mathcal{A}-iz_6}{\mathcal{A}+iz_6}|, & \text{if } 1.638 \leq \mathcal{A} \leq 2.33, \\ \ln|\frac{\mathcal{A}-iz_4}{\mathcal{A}+iz_4}|, & \text{otherwise}, \end{cases}$$

where z_i are defined in the program Wolfram Mathematica 13.2 by the following command

$$z\,[i\_] := z/.\,\text{Solve}\,\big[1 - 60\mathcal{A}^2 + 710\mathcal{A}^4 - 2492\mathcal{A}^6 + 833\mathcal{A}^8 - 68z^2 + 1600\mathcal{A}^2z^2 - 5416\mathcal{A}^4z^2 + 1504\mathcal{A}^6z^2 + 396\mathcal{A}^8z^2 + 918z^4 - 3480\mathcal{A}^2z^4 + 564\mathcal{A}^4z^4 + 904\mathcal{A}^6z^4 + 38\mathcal{A}^8z^4 - 612z^6 - 160\mathcal{A}^2z^6 + 504\mathcal{A}^4z^6 + 64\mathcal{A}^6z^6 + 12\mathcal{A}^8z^6 + 17z^8 + 52\mathcal{A}^2z^8 + 54\mathcal{A}^4z^8 + 20\mathcal{A}^6z^8 + \mathcal{A}^8z^8 == 0, z][[i]],$$

and the points $\{1.638..., 2.33...\}$ *are the intersection points of the corresponding branches of the analytic functions.*

6. Example

Let $\alpha \in (0, 2.84764...)$ be the cone-angle of the hyperbolic cone-manifold $8_3(\alpha)$. We calculate the volume of this cone-manifold by the formula in the Theorem 8, whenever $\alpha = \frac{\pi}{n}$ (n is a positive integer) :

α	$Vol(8_3(\alpha))$
$\frac{\pi}{3}$	1.7623439357427506‘
$\frac{\pi}{4}$	3.244234492108506‘
$\frac{\pi}{5}$	3.977450213471493‘
$\frac{\pi}{6}$	4.373182985780311‘

α	$Vol(8_3(\alpha))$
$\frac{\pi}{7}$	4.608302118127839‘
$\frac{\pi}{8}$	4.759000441173507‘
$\frac{\pi}{9}$	4.861339692093855‘
$\frac{\pi}{10}$	4.934022452788975‘

The latter numerical results coincide with the results in the program SnapPy, originally written by Jeffrey Weeks and further modified by Marc Culler, Nathan Dunfield and others.

7. Conclusion

The cone-manifolds whose singular set is a given two bridge knot with 8 crossings and whose underlying space is the 3-sphere are investigated. The existence conditions of these cone-manifolds in hyperbolic, Euclidean and spherical spaces are obtained. Moreover, exact integral volume formulae of corresponding cone-manifold modeled in hyperbolic space are presented.

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Об авторах

Медных Александр Дмитриевич, д-р физ.-мат. наук, проф., Институт математики им. С. Л. Соболева СО РАН, Новосибирск, 630090, Российская Федерация, Новосибирский государственный университет, Новосибирск, 630090, Российская Федерация, smedn@mail.ru, https://orcid.org/0000-0003-3084-1225

About the authors

Alexander D. Mednykh, Dr. Sci. (Phys.-Math.), Prof., Sobolev Institute of Mathematics SB RAS, Novosibirsk, 630090, Russian Federation, Novosibirsk State University, Novosibirsk, 630090, Russian Federation, smedn@mail.ru, https://orcid.org/0000-0003-3084-1225

Кутбаев Айдос Бакберген улы, Институт математики им. С. Л. Соболева СО РАН, Новосибирск, 630090, Российская Федерация, Новосибирский государственный университет, Новосибирск, 630090, Российская Федерация, aydosqutbaev@gmail.com, https://orcid.org/0009-0002-2404-4414

Aydos B. Qutbaev, Sobolev Institute of Mathematics SB RAS, Novosibirsk, 630090, Russian Federation, Novosibirsk State University, Novosibirsk, 630090, Russian Federation, aydosqutbaev@gmail.com, https://orcid.org/0009-0002-2404-4414

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