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Integration of the Loaded Negative Order Nonlinear Schrodinger Equation in the Class of Periodic Functions

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Abstract. In this paper, we consider the loaded negative order nonlinear Schrodinger equation (NSE) in the class of periodic functions. It is shown that the loaded negative order nonlinear Schrodinger equation can be integrated by the inverse spectral problem method. The evolution of the spectral data of the Dirac operator with a periodic potential associated with the solution of the loaded negative order nonlinear Schrodinger equation is determined. The results obtained make it possible to apply the inverse problem method to solve the loaded negative order nonlinear Schrodinger equation in the class of periodic ones. Important corollaries are obtained about the analyticity and period of the solution concerning the spatial variable.

Keywords: loaded negative order nonlinear Schrodinger equation, soliton, Dirac operator, inverse spectral problem, Dubrovin's system of equations, trace formulas

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Научная статья

Интегрирование нелинейного уравнения Шредингера отрицательного порядка с нагруженным членом в классе периодических функций

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Аннотация. Рассматривается нелинейное уравнение Шредингера отрицательного порядка с нагруженным членом в классе периодических функций. Показано, что такое уравнение может быть проинтегрировано методом обратной спектральной задачи. Определена эволюция спектральных данных оператора Дирака с периодическим потенциалом, связанного с решением нелинейного уравнения Шредингера отрицательного порядка с нагруженным членом. Полученные результаты позволяют применить метод обратной задачи для решения нелинейного уравнения Шредингера отрицательного порядка с нагруженным членом в классе периодических функций. Получены важные следствия об аналитичности и о периоде решения по пространственной переменной.

Ключевые слова: нагруженное нелинейное уравнение Шредингера отрицательного порядка, солитон, оператор Дирака, обратная спектральная задача, система уравнений Дубровина, формулы следов

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1. Introduction

One of the representatives of the class of completely integrable nonlinear partial differential equations, which has great applied significance, is the nonlinear Schrodinger equation (NSE). The complete integrability of this equation by the inverse problem method, in the classes of periodic and finite-zone functions, was first established in [1; 8].

In [7; 26], other integrable nonlinear evolution equations with loaded terms in the class of periodic functions were studied using the inverse spectral problem method. The use of the (G'/G)-expansion method for integrating the loaded Korteweg-de Vries equation (KdV), the loaded modified Korteweg-de Vries equation (mKdV) and the loaded Burger's equation is discussed in [2; 20; 23].

In 1991, J.M. Verosky [25], when studying symmetries and negative powers of the recursive operator, derived the following KdV equation of negative order:

$$\left(\frac{p_{xx}}{p}\right)_t + 2pp_x = 0\,. \tag{1.1}$$

S.Y. Lou [15] introduced additional symmetries based on the invertibility of the recursive operator for the KdV equation, and in particular showed that the negative order KdV equation is equivalent to the system of equations

$$\begin{cases} q_t = 2pp_x, \\ p_{xx}+qp = 0. \end{cases} \tag{1.2}$$

In [19], using the inverse scattering problem method, the KdV equation of negative order was integrated in the class of rapidly decreasing functions.

The KdV equation of negative order with a self-consistent source in the class of periodic functions was studied in [12;21;22], and in [6] the negative-even mKdV hierarchy and its soliton solutions were studied. Mixed positive and negative hierarchies were studied in [5;13].

Hierarchies of the negative-order mKdV equation were studied using recurrent methods in [17].

In [9], breather solutions of the following mKdV equation of negative order were studied

$$\left(\frac{q_{xt}}{q}\right)_x +(2q^2)_t= 0\,,$$

or

$$\rho_{xx} = q^2, q_{xt} + 2q\rho_{xt} + \alpha q = 0.$$

It is easy to see that by replacing q by iq and using transformation$\mu = \rho_x + \frac{\alpha}{2}t$ the mKdV equation of negative order will take the following simpler form

$$\begin{cases} q_{xt} = -2q\mu_t, \\ \mu_x = -q^2 \end{cases} t > 0, \quad x \in R.$$

The mKdV equation of negative order in the class of periodic functions was studied in [24], and in [18], the negative order mKdV equation was integrated in the class of rapidly decreasing functions using the inverse scattering problem method.

In this paper, the inverse spectral problem method is applied to the integration of the loaded negative order Schrodinger equation (NSE) in the class periodic functions.

We consider the following loaded negative order nonlinear Schrodinger equation

$$\begin{cases} p_{xt} = 2\mu p - q_x - \gamma(t)(p^2(0,t) + q^2(0,t))q_x, \\ q_{xt} = 2\mu q + p_x + \gamma(t)(p^2(0,t) + q^2(0,t))p_x \quad t > 0, \quad x \in R \\ \mu_x = 2qq_t + 2pp_t \end{cases} \tag{1.3}$$

with conditions

$$q(x,t)|_{t=0} = q_0(x), \; p(x,t)|_{t=0} = p_0(x), \; \mu(x,t)|_{x=0} = \mu_0(t), \tag{1.4}$$

where, $\mu_0(t) \in C^1[0, \infty)$, $p_0(x)$,$q_0(x)$ and $\gamma(t) \in C[0, \infty)$ - given real functions with period π, and the function $\gamma(t)$ is bounded. It is required to find the real functions $p(x,t), q(x,t)$ and $\mu(x,t)$ that are periodic in variable x, where

$$p(x+\pi,t) \equiv p(x,t), q(x+\pi,t) \equiv q(x,t), \mu(x+\pi,t) \equiv \mu(x,t), t \geq 0, \quad x \in R,$$

and satisfying smoothness conditions:

$$\begin{array}{l} p(x,t) \in C_x^1(t>0) \cap C_t^1(t>0) \cap C(t \geq 0), \\ q(x,t) \in C_x^1(t>0) \cap C_t^1(t>0) \cap C(t \geq 0), \\ \mu(x,t) \in C_x^1(t>0) \cap C_t^1(t>0) \cap C(t \geq 0). \end{array} \tag{1.5}$$

When studying problem (1.3)-(1.5), we use the spectral problem for the following Dirac operator

$$L(t)y \equiv B\frac{dy}{dx} + \Omega(x,t)y = \lambda y,\ x \in R \tag{1.6}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \ \Omega(x,t) = \begin{pmatrix} p(x,t) & q(x,t) \\ q(x,t) & -p(x,t) \end{pmatrix},\ \ y = \begin{pmatrix} y_1(x,t) \\ y_2(x,t) \end{pmatrix}.$$

The purpose of this work is to give a procedure for constructing a solution $(q(x,t), p(x,t), \mu(x,t))$ to problem (1.3)-(1.5), within the framework of the inverse spectral problem for the Dirac equation (1.6).

2. Direct and inverse spectral problems for the Dirac operator with a periodic coefficient

In this section, we present some basic information concerning the inverse spectral problem for the Dirac operator with a periodic coefficient [3; 4; 10; 11; 14; 16]. Let us consider the system of Dirac equations on the entire straight-line

$$Ly \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, x \in R \tag{2.1}$$

where $p(x)$ and $q(x)$ are real continuous functions from class $C^1(R)$, which has period π, and a complex parameter λ.

Let us denote by

$$c(x,\lambda) = (c_1(x,\lambda), c_2(x,\lambda))^T \quad \text{and} \quad s(x,\lambda) = (s_1(x,\lambda), s_2(x,\lambda))^T$$

solutions to equation (2.1) satisfying the initial conditions $c(0,\lambda) = (1,\ 0)^T$ and $s(0,\lambda) = (0,\ 1)^T$. Function $\Delta(\lambda) = c_1(\pi,\ \lambda) + s_2(\pi,\ \lambda)$ is called the Lyapunov function or the Hill discriminant for the Dirac operator (2.1). The spectrum of the operator (2.1) consists of the following set

$$E = \{\lambda \in R: \ \ -2 \leq \Delta(\lambda) \leq 2\ \} = R \setminus \left\{ \bigcup_{n=-\infty}^{\infty} (\lambda_{2n-1}, \lambda_{2n}) \right\}.$$

The intervals $(\lambda_{2n-1},\ \lambda_{2n})$, $n\in Z$ are called lacunes.

The roots of the equation $s_1(\pi,\lambda)=0$ we denote by ξ_n, $n\in Z$. The numbers ξ_n, $n\in Z$ coincide with the eigenvalues of the Dirichlet problem $y_1(0)=0, y_1(\pi)=0$ for system (2.1) and the relations $\xi_n\in[\lambda_{2n-1},\lambda_{2n}]$, $n\in Z$ are satisfied.

Numbers $\xi_n\in[\lambda_{2n-1},\lambda_{2n}]$, $n\in Z$ and signs

$$\sigma_n = sign\left\{s_2(\pi,\xi_n)-c_1(\pi,\xi_n)\right\},$$

$n\in Z$ are called spectral parameters of problem (2.1). Spectral parameters ξ_n, σ_n, $n\in Z$ and spectrum boundaries λ_n, $n\in Z$ are called spectral data of problem (2.1). Finding the spectral data of problem (2.1) is called a direct problem, and recovering the coefficients $p(x)$ and $q(x)$ from spectral data is called an inverse problem.

If in the problem (2.1) we consider $p(x+\tau)$ and $q(x+\tau)$ instead of $p(x)$ and $q(x)$, then the spectrum of the resulting problem does not depend on the parameter τ: $\lambda_n(\tau)\equiv\lambda_n$, $n\in Z$, and the spectral parameters depend on the parameter τ:$\xi_n(\tau)$, $\sigma_n(\tau)$, $n\in Z$. These spectral parameters satisfy an analog of the Dubrovin's system of equations:

$$\frac{d\xi_n}{d\tau}=(-1)^{n-1}\sigma_n(\tau)h_n(\xi(\tau))\left\{2\xi_n(\tau)+\sum_{k=-\infty}^{\infty}(\lambda_{2k-1}+\lambda_{2k}-2\xi_k(\tau))\right\},n\in Z,$$

where

$$h_n(\xi)=\sqrt{(\xi_n(\tau)-\lambda_{2n-1})(\lambda_{2n}-\xi_n(\tau))}\cdot \sqrt{\prod_{\substack{k=-\infty\\ k\neq n}}^{\infty}\frac{(\lambda_{2k-1}-\xi_n(\tau))(\lambda_{2k}-\xi_n(\tau))}{(\xi_k(\tau)-\xi_n(\tau))^2}}.$$

The sign $\sigma_n(\tau)$ — changes to the opposite at each collision of $\xi_n(\tau)$ with the boundaries of its lacuna $[\lambda_{2n-1},\lambda_{2n}]$.

Dubrovin's system of equations, as well as the following trace formulas $p(\tau)=\sum_{k=-\infty}^{\infty}\left(\frac{\lambda_{2k-1}+\lambda_{2k}}{2}-\xi_k(\tau)\right)$, $q(\tau)=\sum_{n=-\infty}^{\infty}(-1)^{n-1}\sigma_n(\tau)h_n(\xi(\tau))$ give the method for solving the inverse problem.

Lemma 1. *If the vector function* $(y_1,\ y_2)^T$ *is a solution to the system* (2.1), *then the following identities hold:*

$$2y_2y_1=\frac{1}{2\lambda}[y_2^2-y_1^2]'+\frac{1}{\lambda}q(y_1^2+y_2^2) \tag{2.2}$$

$$y_1^2-y_2^2=\frac{1}{\lambda}[y_1y_2]'+\frac{1}{\lambda}p(y_1^2+y_2^2) \tag{2.3}$$

$$\frac{1}{2}[y_2^2+y_1^2]'=q(y_1^2-y_2^2)-2py_1y_2. \tag{2.4}$$

3. Evolution of spectral parameters

The main result of this work is the following theorem.

Theorem 1. *Let $(p(x,t), q(x,t), \mu(x,t))$ be the solution to problem* (1.3)-(1.5)*. Then the spectrum of operator* (1.6) *does not depend on the parameter t, and the spectral parameters $\xi_n = \xi_n(t),\ n \in Z\backslash\{0\}$ satisfy an analogue of the Dubrovin system of equations:*

$$\dot{\xi}_n = \frac{1}{\xi_n}(-1)^n \sigma_n(t) h_n(\xi) \times$$

$$\times \left\{ q_t(0,t) + \mu(0,t) + (-p(0,t) - \xi_n)(1 + \gamma(t)(p^2(0,t) + q^2(0,t))) \right\}. \quad (3.1)$$

The signs $\sigma_n(t) = \pm 1$ change each time a point $\xi_n(t)$ collides with the boundaries of its lacuna $[\lambda_{2n-1},\ \lambda_{2n}]$. Moreover, the following initial conditions are satisfied

$$\xi_n(t)|_{t=0} = \xi_n^0, \quad \sigma_n(t)|_{t=0} = \sigma_n^0, \quad n \in Z\backslash\{0\} \quad (3.2)$$

where $\xi_n^0,\ \sigma_n^0,\ n \in Z\backslash\{0\}$ are the spectral parameters of the Dirac operator with coefficients $p_0(x), q_0(x)$.

Proof. Let us denote by $y_n(x,t) = (y_{n,1}(x,t), y_{n,2}(x,t))^T$, $n \in Z$, the orthonormal eigenvector functions of the Dirichlet problem for equation (1.6), corresponding to the eigenvalues of $\xi_n(t),\ n \in Z$.

Differentiating the identity $\xi_n(t) = (L(t)y_n, y_n)$ with respect to t and using the symmetry of the operator $L(t)$, we have

$$\dot{\xi}_n = (\dot{\Omega}(x,t)y_n, y_n). \quad (3.3)$$

Using an explicit dot product

$$(y,z) = \int_0^\pi [y_1(x)\bar{z}_1(x) + y_2(x)\bar{z}_2(x)]\, dx,\ y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},\ z = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix},$$

we rewrite equality (3.3) in the form

$$\dot{\xi}_n = \int_0^\pi [(y_{n,1}^2 - y_{n,2}^2)p_t + 2y_{n,1}y_{n,2}\, q_t]dx. \quad (3.4)$$

Using formula (2.2) and (2.3) we obtain the following equality

$$\dot{\xi}_n = \frac{1}{\xi_n}\int_0^\pi (y_{n,1}y_{n,2})' p_t dx + \frac{1}{\xi_n}\int_0^\pi (y_{n,2}^2 + y_{n,1}^2) p p_t dx +$$

$$\frac{1}{2\xi_n}\int_0^\pi (y_{n,2}^2 - y_{n,1}^2)' q_t dx + \frac{1}{\xi_n}\int_0^\pi (y_{n,2}^2 + y_{n,1}^2) q q_t dx, \quad n \in Z\backslash\{0\}, \quad (3.5)$$

The equality (3.5) can be rewritten as

$$\dot{\xi}_n = \frac{1}{2\xi_n}[y_{n,2}^2(\pi,t) - y_{n,2}^2(0,t)]q_t(0,t) - \frac{1}{\xi_n}\int_0^\pi (y_{n,1}y_{n,2})p_{xt}dx -$$

$$\frac{1}{2\xi_n}\int_0^\pi (y_{n,2}^2 - y_{n,1}^2)q_{xt}dx + \frac{1}{\xi_n}\int_0^\pi (y_{n,2}^2 + y_{n,1}^2)(pp_t + qq_t)dx.$$

From the system of equation (1.3) we have

$$pp_t + qq_t = \frac{\mu_x}{2},\ p_{xt} = 2p\mu - q_x \ -\gamma(t)(p^2(0,t) + q^2(0,t))q_x,$$

$$q_{xt} = 2q\mu + p_x + \gamma(t)(p^2(0,t) + q^2(0,t))p_x. \tag{3.6}$$

Hence,

$$\dot{\xi}_n = \frac{1}{2\xi_n}[y_{n,2}^2(\pi,t) - y_{n,2}^2(0,t)]q_t(0,t) -$$

$$\frac{2}{\xi_n}\int_0^\pi y_{n,1}y_{n,2}p\mu dx - \frac{1}{\xi_n}\int_0^\pi (y_{n,2}^2 - y_{n,1}^2)q\mu dx +$$

$$+\frac{1}{\xi_n}\int_0^\pi (y_{n,1}y_{n,2})q_x dx - \frac{1}{2\xi_n}\int_0^\pi (y_{n,2}^2 - y_{n,1}^2)p_x dx -$$

$$+\gamma(t)(p^2(0,t) + q^2(0,t))\int_0^\pi \left\{\frac{1}{\xi_n}(y_{n,1}y_{n,2})q_x - \frac{1}{2\xi_n}(y_{n,2}^2 - y_{n,1}^2)p_x\right\}dx +$$

$$+\frac{1}{2\xi_n}\int_0^\pi (y_{n,2}^2 + y_{n,1}^2)\mu_x dx \tag{3.7}$$

We integrate the last integral by parts

$$I_1 = \frac{1}{2\xi_n}\int_0^\pi (y_{n,2}^2 + y_{n,1}^2)\mu_x dx =$$

$$= \frac{1}{2\xi_n}[y_{n,2}^2(\pi,t) - y_{n,2}^2(0,t)]\mu(0,t) - \frac{1}{2\xi_n}\int_0^\pi (y_{2,n}^2 + y_{1,n}^2)'\mu dx. \tag{3.8}$$

Based on (2.4), equality (3.8) will take the form

$$I_1 = \frac{1}{2\xi_n}[y_{n,2}^2(\pi,t) - y_{n,2}^2(0,t)]\mu(0,t) +$$

$$+\frac{1}{\xi_n}\int_0^\pi (y_{2,n}^2 - y_{1,n}^2)q\mu dx + \frac{2}{\xi_n}\int_0^\pi y_{1,n}y_{2,n}p\mu dx. \tag{3.9}$$

Now consider the third and fourth integrals in equality (3.7):

$$I_2 = \frac{1}{\xi_n}\int_0^\pi (y_{n,1}y_{n,2})q_x dx - \frac{1}{2\xi_n}\int_0^\pi (y_{n,2}^2 - y_{n,1}^2)p_x dx =$$

$$-\frac{1}{2\xi_n}[y_{n,2}^2(\pi,t)-y_{n,2}^2(0,t)]p(0,t)+$$

$$+\frac{1}{\xi_n}\int_0^\pi \left((y_{n,2}p-y_{n,1}q)y'_{n,2}-(y_{n,1}p+y_{n,2}q)y'_{n,1}\right)dx. \qquad (3.10)$$

From equation (1.6) the following equalities follow:

$$\begin{cases} y'_{n,1}+\xi_n y_{n,2}=q\,y_{n,1}-p\,y_{n,2} \\ \xi_n y_{n,1}-y'_{n,2}=p\,y_{n,1}+q\,y_{n,2}\,. \end{cases}$$

Using these identities, we get

$$I_2=\frac{1}{\xi_n}\int_0^\pi (y_{n,1}y_{n,2})q_x dx-\frac{1}{2\xi_n}\int_0^\pi (y_{n,2}^2-y_{n,1}^2)p_x dx=$$

$$=\frac{1}{2\xi_n}[y_{n,2}^2(\pi,t)-y_{n,2}^2(0,t)](-p(0,t)-\xi_n) \qquad (3.11)$$

Now let's calculate the fifth integral in equality (3.7):

$$I_3=\int_0^\pi \left\{\frac{1}{\xi_n}(y_{n,1}y_{n,2})q_x-\frac{1}{2\xi_n}(y_{n,2}^2-y_{n,1}^2)p_x\right\}dx=$$

$$=\frac{1}{2\xi_n}[y_{n,2}^2(\pi,t)-y_{n,2}^2(0,t)](-p(0,t)-\xi_n) \qquad (3.12)$$

From (3.7), (3.9), (3.11) and (3.12) we deduce that

$$\dot{\xi}_n(t)=\frac{1}{2\xi_n}[y_{n,2}^2(\pi,t)-y_{n,2}^2(0,t)]\times$$
$$\times\left\{q_t(0,t)+\mu(0,t)+(-p(0,t)-\xi_n)(1+\gamma(t)(p^2(0,t)+q^2(0,t)))\right\},$$
$$n\in Z\backslash\{0\}. \qquad (3.13)$$

Let us denote by $s(x,\lambda,t)=(s_1(x,\lambda,t),\ s_2(x,\lambda,t))^T$ the solution of equation (1.6) satisfying the initial conditions $s(0,\lambda,t)=(0,\ 1)^T$. From equality

$$\int_0^\pi [s_1^2(x,\lambda,t)+s_2^2(x,\lambda,t)]dx$$

$$=s_1(\pi,\lambda,t)\frac{\partial s_2(\pi,\lambda,t)}{\partial\lambda}-s_2(\pi,\lambda,t)\frac{\partial s_1(\pi,\lambda,t)}{\partial\lambda}$$

we find a formula for the norm of the eigenvector function $s(x,\xi_n(t),t)$ of the Dirichlet problem (1.6), (2.4), corresponding to the eigenvalue $\xi_n(t)$:

$$c_n^2(t)=\int_0^\pi [s_1^2(x,\xi_n(t),t)+s_2^2(x,\xi_n(t),t)]dx \ =$$

$$=-\frac{\partial s_1(\pi,\xi_n(t),t)}{\partial\lambda}s_2(\pi,\xi_n(t),t) \qquad (3.14)$$

Using equality $y_n(x,t)=\frac{1}{c_n(t)}s(x,\xi_n(t),t)$ and (3.13) we get

$$y_{n,2}^2(\pi,t)-y_{n,2}^2(0,t)=\frac{s_2^2(\pi,\xi_n(t),t)-1}{c_n^2(t)}=$$

$$=-\frac{s_2(\pi,\xi_n(t),t)-\frac{1}{s_2(\pi,\xi_n(t),t)}}{\frac{\partial s_1(\pi,\xi_n(t),t)}{\partial\lambda}} \tag{3.15}$$

Substituting values $x=\pi$ and $\lambda=\xi_n(t)$ into identity

$$c_1(x,\lambda,t)s_2(x,\lambda,t)-c_2(x,\lambda,t)s_1(x,\lambda,t)=1,$$

we find

$$c_1(\pi,\xi_n(t),t)=\frac{1}{s_2(\pi,\xi_n(t),t)}. \tag{3.16}$$

Taking into account the equality (3.16) and the following identity

$$[c_1(\pi,\lambda,t)-s_2(\pi,\lambda,t)]^2=(\Delta^2(\lambda)-4)-4c_2(\pi,\lambda,t)s_1(\pi,\lambda,t),$$

we write

$$s_2(\pi,\xi_n(t),t)-\frac{1}{s_2(\pi,\xi_n(t),t)}=\sigma_n(t)\sqrt{\Delta^2(\xi_n(t))-4}, \tag{3.17}$$

where

$\Delta(\lambda)=c_1(\pi,\lambda,t)+s_2(\pi,\lambda,t), \sigma_n(t)=sign\left\{s_2(\pi,\xi_n(t),t)-c_1(\pi,\xi_n(t),t)\right\}.$

From (3.15) and (3.17) we deduce

$$y_{n,2}^2(\pi,t)-y_{n,2}^2(0,t)=-\frac{\sigma_n(t)\sqrt{\Delta^2(\xi_n(t))-4}}{\frac{\partial s_1(\pi,\xi_n(t),t)}{\partial\lambda}}. \tag{3.18}$$

Using the following expansions

$$\Delta^2(\lambda)-4=-4\pi^2\prod_{k=-\infty}^{\infty}\frac{(\lambda-\lambda_{2k-1})(\lambda-\lambda_{2k})}{a_k^2}, s_1(\pi,\lambda,t)=\pi\prod_{k=-\infty}^{\infty}\frac{\xi_k-\lambda}{a_k},$$

where $a_0=1$ and $a_k=k$ when $k\neq 0$, the equality (3.18) can be rewritten as follows:

$$y_{n,2}^2(\pi,t)-y_{n,2}^2(0,t)=2(-1)^n\sigma_n(t)h_n(\xi). \tag{3.19}$$

At the same time, we used the equality

$$sign\left\{-\frac{\pi}{a_n}\prod_{\substack{k=-\infty\\ k\neq n}}^{\infty}\frac{\xi_k-\xi_n}{a_k}\right\}=(-1)^{n-1}.$$

Substituting expression (3.19) into identity (3.13) we derive (3.1).

If we replace Dirichlet boundary conditions with periodic $y(\pi) = y(0)$ or antiperiodic $y(\pi) = -y(0)$ boundary conditions, then instead of equation (3.13) we have $\dot{\lambda}_n = 0$. This means that the eigenvalues λ_n, $n \in Z$ of the periodic and anti-periodic problems do not depend on the parameter t. □

Corollary 1. *If we consider $p(x,t)$ and $q(x,t)$ instead of $p(x+\tau,t)$ and $q(x+\tau,t)$, then the eigenvalues of the periodic and antiperiodic problem do not depend on the parameters τ, t, and the eigenvalues ξ_n of the Dirichlet problem and the signs σ_n depend on τ, t: $\xi_n = \xi_n(\tau,t)$, $\sigma_n = \sigma_n(\tau,t) = \pm 1$, $n \in Z$. In this case, the signs $\sigma_n(\tau,t) = \pm 1$ change, when a point $\xi_n(\tau,t)$ collides with the boundaries of its lacuna $[\lambda_{2n-1}, \lambda_{2n}]$, and system* (3.1) *will take the form*

$$\frac{\partial \xi_n}{\partial t} = \frac{1}{\xi_n}(-1)^n \sigma_n(\tau,t) h_n(\xi) \times$$

$$\times \left\{ q_t(\tau,t) + \mu(\tau,t) + (-p(\tau,t) - \xi_n(\tau,t))(1 + \gamma(t)(p^2(0,t) + q^2(0,t))) \right\},$$

$$n \in Z \backslash \{0\}, \tag{3.20}$$

$$\xi_n(\tau,t)|_{t=0} = \xi_n^0(\tau), \quad \sigma_n(\tau,t)|_{t=0} = \sigma_n^0(\tau), n \in Z\backslash\{0\} \tag{3.21}$$

Using the trace formula

$$q(\tau,t) = \sum_{n=-\infty}^{\infty} (-1)^{n-1} \sigma_n(\tau,t) h_n(\xi) \tag{3.22}$$

$$p(\tau,t) = \sum_{n=-\infty}^{\infty} \left(\frac{\lambda_{2n-1} - \lambda_{2n}}{2} - \xi_n(\tau,t) \right) \tag{3.23}$$

and the equality $\mu_x = 2pp_t + 2qq_t$, we get

$$q_t(\tau,t) = \sum_{n=-\infty}^{\infty} (-1)^{n-1} \sigma_n(\tau,t) \frac{\partial h_n(\xi)}{\partial t} \tag{3.24}$$

$$p_t = -\sum_{n=-\infty}^{\infty} \frac{\partial \xi_n(\tau,t)}{\partial t}, \tag{3.25}$$

$$\mu(\tau,t) = \mu_0(t) + 2\int_0^{\tau} (p(s,t)p_t(s,t) + q(s,t)q_t(s,t))ds \tag{3.26}$$

Corollary 2. *This theorem provides a method for solving problem* (1.3)-(1.5). *To do this, we first find the spectral data λ_n, $\xi_n^0(\tau)$, $\sigma_n^0(\tau)$, $n \in Z$ of the Dirac operator corresponding to the potential $q_0(x+\tau)$, $p_0(x+\tau)$. Next, by solving the Cauchy problem* (3.20)-(3.21) *when $\tau = 0$ we find $\xi_n(0,t)$ and $\sigma_n(0,t)$, $n \in Z$. Using these data we will find $q(0,t)$, $p(0,t)$. After this, we* *substitute the expression for* $q(0,t)$, $p(0,t)$, *into equation* (3.20) *and solving the Cauchy problem for an arbitrary* τ *value, we find* $\xi_n(\tau,t)$ *and* $\sigma_n(\tau,t)$, $n \in Z$. *After this, using the trace formulas* (3.22) *and* (3.23), *we find the solutions* $p(x,t)$ *and* $q(x,t)$ *of problem* (1.3)-(1.5), *and then from formula* (3.26) *we determine* $\mu(x,t)$.

Corollary 3. *Using the results of work [10], we conclude that if the initial functions* $p_0(x)$ *and* $q_0(x)$ *are real analytical functions, then the components of the solution* $p(x,t)$ *and* $q(x,t)$ *are real analytical functions in* x.

Corollary 4. *If the number* $\frac{\pi}{2}$ *is the period for the initial function* $p_0(x)$ *and* $q_0(x)$, *then all the roots of the equation* $\Delta(\lambda)+2=0$ *are twice multiple. Since the Lyapunov function corresponding to the coefficients* $p(x,t)$ *and* $q(x,t)$ *coincides with* $\Delta(\lambda)$, *then according to Borg's converse theorem [11], the number* $\frac{\pi}{2}$ *is also a period for both the solution* $p(x,t)$ *and* $q(x,t)$ *in the variable* x.

Corollary 5. *If the number* $\frac{\pi}{2}$ *is an antiperiod for the initial function* $p_0(x)$ *and* $q_0(x)$, *then all the roots of the equation* $\Delta(\lambda)-2=0$ *twice multiple. Since the Lyapunov function corresponding to the coefficients* $p(x,t)$ *and* $q(x,t)$ *coincides with* $\Delta(\lambda)$, *then [3] it follows that the number is also an antiperiod for solutions* $p(x,t)$ *and* $q(x,t)$ *in the variable* x.

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