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A Note on Wright-type Generalized q-hypergeometric Function

Kuldipkumar K. Chaudhary[1], **Snehal B. Rao**[1]✉

[1] Maharaja Sayajirao University of Baroda, Gujarat, India
✉ snehal.b.rao-appmath@msubaroda.ac.in

Abstract. In 2001, Virchenko et al. published a paper on a new generalization of Gauss hypergeometric function, namely Wright-type generalized hypergeometric function. Present work aims to define the q-analogue generalized hypergeometric function, which reduces to generalized hypegeometric function by letting q tends to one, and study some new properties. Convergence of the series defining generalized q-hypergeometric function and properties including certain differentiation formulae and integral representations have been deduced.

Keywords: basic hypergeometric functions in one variable ${}_r\phi_s$, q-gamma functions, q-beta functions and integrals, q-calculus and related topics

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Научная статья

Заметка об обобщенной q-гипергеометрической функции типа Райта

К. К. Чаудхари[1], **С. Б. Рао**[1]✉

[1] Университет Махараджа Саяджирао в Барода, Гуджарат, Индия
✉ snehal.b.rao-appmath@msubaroda.ac.in

Аннотация. В 2001 г. Вирченко и соавторами была опубликована статья о новой обобщенной гипергеометрической функции Гаусса, а именно обобщенной гипергеометрической функции типа Райта. Цель настоящей работы заключается в определении q-аналога обобщенной гипергеометрической функции, которая сводится к обобщенной гипергеометрической функции, когда q стремится к единице, а также в изучении некоторых новых свойств. Выводится сходимость ряда, определяющего обобщенную q-гипергеометрическую функцию, и некоторые свойства, включая определенные формулы дифференцирования и интегральные представления.

Ключевые слова: базовые гипергеометрические функции одной переменной ${}_r\phi_s$, q-гамма функции, q-бета функции и интегралы, q- исчисление и связанные темы

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1. Introduction

The hypergeometric function ${}_2\mathcal{F}_1(\mathfrak{s},\mathfrak{b};\mathfrak{r};\mathfrak{z})$, represented by hypergeometric series is a well-celebrated and historical one, amongst one of the most important special functions, that includes many other special functions as its specific or limiting cases. The hypergeometric function is also observed in various applications within the fields of physics and statistics [6].

Seeing the historical development, the nomenclature “hypergeometric series” was initially employed by Wallis within his literary work titled “Arithmetica Infinitorum” [15]. As a pioneer, these series were studied by Euler (1707-1783), but the first full systematic treatment was given by Gauss [8]. It can be easily seen in [11] that the function $y = {}_2\mathcal{F}_1(\mathfrak{s},\mathfrak{b};\mathfrak{r};\mathfrak{z})$ represents a solution to a linear differential equation of the second order

$$\mathfrak{z}(1-\mathfrak{z})\,y'' + [\mathfrak{r} - (\mathfrak{s}+\mathfrak{b}+1)\,z]\,y' - \mathfrak{s}\mathfrak{b}y = 0;$$

Riemann [11] demonstrated that, the differential equation for ${}_2\mathcal{F}_1(\mathfrak{z})$ hypergeometric function could be delineated by its three regular singularities.

Several mathematicians in a numerous manner have contributed to the study of the rich theory on hypergeometric functions. And in the sequel to study further, Virchenko et al. [14] have introduced the generalization of

${}_2\mathcal{F}_1(\mathfrak{s},\mathfrak{b};\mathfrak{r};\mathfrak{z})$ in terms of ${}_2\mathcal{R}_1(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;\mathfrak{z}) \equiv {}_2\mathcal{R}_1^{\tau}(\mathfrak{z})$, whose special case is ${}_2\mathcal{F}_1(\mathfrak{z})$, for $\tau = 1$.

It should be noted that while taking a limit $q \to 1$ in the basic hypergeometric series ${}_2\phi_1(\mathfrak{z})$, one gets Gauss hypergeometric series ${}_2\mathcal{F}_1(\mathfrak{z})$, and that is the reason why ${}_2\phi_1(\mathfrak{z})$ is known as the q-analogue of hypergeometric function ${}_2\mathcal{F}_1(\mathfrak{z})$. In this sequel of study the present work aims to define the q-analogue ${}_2\mathcal{R}_1^{\tau,q}(\mathfrak{z})$ of ${}_2\mathcal{R}_1^{\tau}(\mathfrak{z})$ and to study its various properties.

1.1. Preliminaries

The following definitions are used to carry out the present research article.

Definition 1. *For $\mathfrak{z} \in \mathbb{C}$, the q-shifted factorial is given by [7; 10]*

$$(\mathfrak{z};q)_\ell = \begin{cases} 1, \ell = 0 \\ \prod\limits_{m=0}^{\ell-1}(1-\mathfrak{z}q^m), \ell \in \mathbb{N} \end{cases}, \quad (0<|q|<1) \tag{1.1}$$

Ernst [6] reintroduced the q-extension of rising factorial for $\mathfrak{z} \in \mathbb{C}$ as below:

$$\langle \mathfrak{z};q\rangle_\ell = \begin{cases} 1, \ell = 0 \\ \prod\limits_{m=0}^{\ell-1}(1-q^{\mathfrak{z}+m}), \ell \in \mathbb{N} \end{cases}, \ (0<|q|<1) \tag{1.2}$$

Here, observe that, $\langle \mathfrak{z};q\rangle_\ell \equiv (q^{\mathfrak{z}};q)_\ell$. For $[\mathfrak{z}]_q \cdot [\mathfrak{z}+1]_q \cdot \ldots \cdot [\mathfrak{z}+\ell-1]_q$, here we adopt the notation $[\mathfrak{z}]_q^\ell$ (namely KS-q-pochhammer symbol), thus

$$[\mathfrak{z}]_q^\ell = [\mathfrak{z}]_q \cdot [\mathfrak{z}+1]_q \cdot \ldots \cdot [\mathfrak{z}+\ell-1]_q, \textit{where } [\mathfrak{z}]_q = \frac{1-q^{\mathfrak{z}}}{1-q}. \tag{1.3}$$

Definition 2. *For $\Re(\mathfrak{s}), \Re(\mathfrak{b}), \Re(\mathfrak{r}) > 0, \mathfrak{z} \in \mathbb{C}; 0<|q|<1$ and $|\mathfrak{z}|<1$, the basic hypergeometric function is expressed as [6]*

$${}_2\phi_1(\mathfrak{s},\mathfrak{b};\mathfrak{r};q,\mathfrak{z}) \equiv {}_2\phi_1\left[\begin{matrix}\mathfrak{s},\mathfrak{b} \\ \mathfrak{r}\end{matrix};q,\mathfrak{z}\right] = \sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s};q\rangle_\ell\langle\mathfrak{b};q\rangle_\ell}{\langle 1;q\rangle_\ell\langle\mathfrak{r};q\rangle_\ell}\mathfrak{z}^\ell. \tag{1.4}$$

Definition 3. *For $|\mathfrak{z}|<1$, the q-binomial series is stated as [6; 7]*

$${}_1\phi_0(a;q,z) \equiv \sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s};q\rangle_\ell}{\langle 1;q\rangle_\ell}\mathfrak{z}^\ell = \frac{(q^{\mathfrak{s}}\mathfrak{z};q)_\infty}{(q;q)_\infty} = \frac{1}{(\mathfrak{z};q)_{\mathfrak{s}}}, \ (0<|q|<1). \tag{1.5}$$

Definition 4. *The q-differentiation operator D_q is defined by [4; 10]*

$$D_q f(z) = \frac{d_q}{d_q\mathfrak{z}}f(\mathfrak{z}) = \frac{f(\mathfrak{z})-f(q\mathfrak{z})}{\mathfrak{z}-q\mathfrak{z}}, \quad (0<|q|<1) \tag{1.6}$$

and $f'(\mathfrak{z})$ can be retrieved by taking $q \to 1$.

Definition 5. *For $0 < |q| < 1$, the q-integral is established by [7; 10]*

$$\int_{\mathfrak{a}}^{\mathfrak{b}} f(\mathfrak{z})\, d_q\mathfrak{z} = \int_0^{\mathfrak{b}} f(\mathfrak{z})\, d_q\mathfrak{z} - \int_0^{\mathfrak{a}} f(\mathfrak{z})\, d_q\mathfrak{z}, \tag{1.7}$$

where

$$\int_0^{\alpha} f(\mathfrak{z})\, d_q\mathfrak{z} = \alpha(1-q)\sum_{\ell=0}^{\infty} f\left(\alpha q^{\ell}\right) q^{\ell}. \tag{1.8}$$

Definition 6. *For $\Re(\mathfrak{z}) > 0$, the q-gamma function is defined as [6; 7]*

$$\Gamma_q(\mathfrak{z}) = \frac{\langle 1; q\rangle_\infty}{\langle \mathfrak{z}; q\rangle_\infty}(1-q)^{1-\mathfrak{z}} \equiv \frac{(q;q)_\infty}{(q^{\mathfrak{z}};q)_\infty}(1-q)^{1-\mathfrak{z}}, \quad (0<|q|<1) \tag{1.9}$$

$\Gamma_q(\mathfrak{z})$ has poles at $\mathfrak{z} = 0, -1, -2, \ldots$ Andrews [2] derived that $\Gamma_q(\mathfrak{z}) \to \Gamma(\mathfrak{z})$ as $q \to 1$.

Definition 7. *The q-beta function is defined by [6; 7]*

$$\beta_q(\mathfrak{z}_1, \mathfrak{z}_2) = (1-q)\sum_{\ell=0}^{\infty} \frac{\langle \ell+1; q\rangle_\infty}{\langle n+\mathfrak{z}_2; q\rangle_\infty} q^{\ell \mathfrak{z}_1}, \tag{1.10}$$

where $0 < |q| < 1$ and $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C}$ with $\Re(\mathfrak{z}_1), \Re(\mathfrak{z}_2) > 0$.

Definition 8. *The q-Stirling's asymptotic formula is given as [9]*

$$\Gamma_q(\mathfrak{z}) \sim (1+q)^{\frac{1}{2}}\Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-\mathfrak{z}} e^{\mu_q(\mathfrak{z})}, \tag{1.11}$$

where $0 < |q| < 1$ and $\mu_q(\mathfrak{z}) = \frac{\theta q^{\mathfrak{z}}}{1-q-q^{\mathfrak{z}}}$ $(0 < \theta < 1)$.

2. Methodology

We show the convergence of our newly introduced generalized basic (q-) hypergeometric function using q-stirling's formula and Cauchy's nth root test. In addition, we derived some results using q-pochhammer symbol, q-binomial series, q-differentiation, q-integration, q-gamma function and below mentioned results;

1) For $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C}$ and $0 < |q| < 1$ [5];

$$[\mathfrak{z}_1 + \mathfrak{z}_2]_q = [\mathfrak{z}_1]_q + q^{\mathfrak{z}_1}[\mathfrak{z}_2]_q = q^{\mathfrak{z}_2}[\mathfrak{z}_1]_q + [\mathfrak{z}_2]_q. \tag{2.1}$$

2) If $\mathfrak{z} \in \mathbb{C}$ and $0 < |q| < 1$ then [5]

$$[-\mathfrak{z}]_q = -q^{-\mathfrak{z}}[\mathfrak{z}]_q. \tag{2.2}$$

3) The relation between q-gamma and q-beta function is given by [7]

$$\beta_q(\mathfrak{z}_1, \mathfrak{z}_2) = \frac{\Gamma_q(\mathfrak{z}_1)\,\Gamma_q(\mathfrak{z}_2)}{\Gamma_q(\mathfrak{z}_1 + \mathfrak{z}_2)}, \quad (0 < |q| < 1). \tag{2.3}$$

3. Main Results

3.1. Wright-type generalized q-hypergeometric function and its convergence

Within this section, we present the q-version of the Wright-type generalized hypergeometric function $({}_2\mathcal{R}_1^{\tau}(\mathfrak{z}))$ as

$${}_2\mathcal{R}_1{}^{\tau,q}(\mathfrak{z}) \equiv {}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}) = \frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b})} \sum_{\ell=0}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q(\mathfrak{b} + \tau\ell)}{\Gamma_q(\mathfrak{r} + \tau\ell)} \mathfrak{z}^\ell, \tag{3.1}$$

where $\Re(\mathfrak{s}), \Re(\mathfrak{b}), \Re(\mathfrak{r}); 0 < |q| < 1, \tau \in \mathbb{R}^+$ and $|\mathfrak{z}| < 1$.

Special Cases

1) Clearly, (3.1) is special case of ${}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; \mathfrak{z})$ as $q \to 1$.

2) By taking $\tau = 1$, the q-analogue of ${}_2\mathcal{R}_1^{\tau}(\mathfrak{z})$ reduces to (1.4).

3) If $\mathfrak{r} = \mathfrak{b}$ then ${}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z})$ reduces to the q-binomial series (1.5).

4) If $\mathfrak{s} = 1$ and $\mathfrak{r} = \mathfrak{b}$ then (3.1) reduces to the geometric series.

In this section, there discussed a convergence of (3.1) for $|\mathfrak{z}| < 1$.
To show the convergence,let us begin with

$${}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}) = \frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{s})\Gamma_q(\mathfrak{b})} \sum_{\ell=0}^{\infty} \frac{\Gamma_q(\mathfrak{s} + \ell)\Gamma_q(\mathfrak{b} + \tau\ell)}{\Gamma_q(1 + \ell)\Gamma_q(\mathfrak{r} + \tau\ell)} \mathfrak{z}^\ell.$$

Now, let $v_\ell = \frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{s})\Gamma_q(\mathfrak{b})} \frac{\Gamma_q(\mathfrak{s}+\ell)\Gamma_q(\mathfrak{b}+\tau\ell)}{\Gamma_q(1+\ell)\Gamma_q(\mathfrak{r}+\tau\ell)} \mathfrak{z}^\ell$.
By q-stirling formula (1.11), we have

$$v_\ell \sim \frac{(1-q)^{\frac{1}{2}}}{(1+q)^{\frac{1}{2}}\Gamma_{q^2}\left(\frac{1}{2}\right)} \frac{\exp\left(\mu_q(\mathfrak{s} + \ell) + \mu_q(\mathfrak{b} + \tau\ell) + \mu_q(\mathfrak{r})\right)}{\exp\left(\mu_q(\mathfrak{r} + \tau\ell) + \mu_q(1 + \ell) + \mu_q(\mathfrak{s}) + \mu_q(\mathfrak{b})\right)} |\mathfrak{z}^\ell|,$$

Hence,

$$\sqrt[\ell]{v_\ell} \sim \left| \frac{(1-q)^{\frac{1}{2}}}{(1+q)^{\frac{1}{2}}\Gamma_{q^2}\left(\frac{1}{2}\right)} \frac{\exp\left(\mu_q\left(\mathfrak{s}+\ell\right)+\mu_q\left(\mathfrak{b}+\tau\ell\right)+\mu_q\left(\mathfrak{r}\right)\right)\mathfrak{z}^\ell}{\exp\left(\mu_q\left(\mathfrak{r}+\tau\ell\right)+\mu_q\left(1+\ell\right)+\mu_q\left(\mathfrak{s}\right)+\mu_q\left(\mathfrak{b}\right)\right)} \right|^{\frac{1}{\ell}}$$

Now, taking limit $\ell \to \infty$, we get $\frac{1}{R} = \lim_{\ell\to\infty} \sqrt[\ell]{v_\ell} \sim |\mathfrak{z}|$.
Therefore, we can conclude that v_ℓ converges when $|\mathfrak{z}| < 1$.

3.2. Some important results on ${}_2\mathcal{R}_1^{\tau,q}(\mathfrak{z})$

Theorem 1. *If* $\mathfrak{s}, \mathfrak{b}; \mathfrak{r} \in \mathbb{C}, \Re(\mathfrak{s}), \Re(\mathfrak{b}), \Re(\mathfrak{r}) > 0; \tau \in \mathbb{R}^+, 0 < |q| < 1$ *and* $|\mathfrak{z}| < 1$ *then*

$$\frac{d_q}{d_q\mathfrak{z}} {}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}\right) = \frac{\Gamma_q\left(\mathfrak{r}\right)}{\Gamma_q\left(\mathfrak{b}\right)} \sum_{\ell=1}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q\left(\mathfrak{b}+\tau\ell\right)}{\Gamma_q\left(\mathfrak{r}+\tau\ell\right)} [\ell]_q \mathfrak{z}^{\ell-1} \tag{3.2}$$

and

$$\int_{\kappa_1}^{\kappa_2} {}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}\right)\, d_q\mathfrak{z} = \frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b})} \sum_{\ell=0}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q(\mathfrak{b}+\tau\ell)}{\Gamma_q(\mathfrak{r}+\tau\ell)} \frac{\left(\kappa_2^{\ell+1} - \kappa_1^{\ell+1}\right)}{[\ell+1]_q} \tag{3.3}$$

Proof. To prove (3.2), let us consider

$$\begin{aligned} \frac{d_q}{d_q\mathfrak{z}} {}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}\right) &= \frac{d_q}{d_q\mathfrak{z}} \left(\frac{\Gamma_q\left(\mathfrak{r}\right)}{\Gamma_q\left(\mathfrak{b}\right)} \sum_{\ell=1}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q\left(\mathfrak{b}+\tau\ell\right)}{\Gamma_q\left(\mathfrak{r}+\tau\ell\right)} \mathfrak{z}^\ell \right) \\ &= \frac{\Gamma_q\left(\mathfrak{r}\right)}{\Gamma_q\left(\mathfrak{b}\right)} \sum_{\ell=1}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q\left(\mathfrak{b}+\tau\ell\right)}{\Gamma_q\left(\mathfrak{r}+\tau\ell\right)} [\ell]_q \mathfrak{z}^{\ell-1}, \end{aligned}$$

which is (3.2).
By using q-integral formula (1.7), we can prove (3.3) as follows:

$$\begin{aligned} \int_{\kappa_1}^{\kappa_2} {}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}\right)\, d_q\mathfrak{z} = \kappa_2\left(1-q\right) &\sum_{\ell=0}^{\infty} {}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \kappa_2 q^\ell\right) q^\ell \\ - \kappa_1\left(1-q\right) &\sum_{\ell=0}^{\infty} {}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \kappa_1 q^\ell\right) q^\ell, \end{aligned}$$

which on further simplification reduces to $\frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b})} \sum_{\ell=0}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q(\mathfrak{b}+\tau\ell)}{\Gamma_q(\mathfrak{r}+\tau\ell)} \frac{\left(\kappa_2^{\ell+1} - \kappa_1^{\ell+1}\right)}{[\ell+1]_q}$.

□

Remark 1. By taking $\kappa_1 = 0$ and $\kappa_2 = \mathfrak{z}$ in (3.3), we get the integral as below:

$$\int_0^{\mathfrak{z}} {}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}\right) d_q\mathfrak{z} = \frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(b)} \sum_{\ell=0}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q(\mathfrak{b}+\tau\ell)}{\Gamma_q(\mathfrak{r}+\tau\ell)} \frac{\left(\mathfrak{z}^{\ell+1}\right)}{[\ell+1]_q}. \tag{3.4}$$

Theorem 2. *If* $\mathfrak{s}, \mathfrak{b}, \mathfrak{r} \in \mathbb{C}, \Re(\mathfrak{s}), \Re(\mathfrak{b}), \Re(\mathfrak{r}) > 0; \tau \in \mathbb{R}^+, 0 < |q| < 1$ *and* $|\mathfrak{z}| < 1$ *then*

$$\left(q^{\mathfrak{b}}[\mathfrak{r}-1]_q - q^{\mathfrak{r}-1}[\mathfrak{b}]_q\right) {}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}) = [\mathfrak{r}-\mathfrak{b}]_q {}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}+1; \tau; q, \mathfrak{z}\right) + q^{\mathfrak{r}-\mathfrak{b}}[\mathfrak{b}]_q {}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}+\imath; \mathfrak{r}+1; \tau; q, z\right), \tag{3.5}$$

$$[\mathfrak{r}]_q {}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}\right) = q^{\mathfrak{b}}[\mathfrak{r}-\mathfrak{b}]_q {}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}-1; \tau; q, \mathfrak{z}\right) + q^{\mathfrak{r}-\mathfrak{b}}[\mathfrak{b}]_q {}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}+1; \mathfrak{c}+1; \tau; q, \mathfrak{z}\right), \tag{3.6}$$

and

$${}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}\right) = {}_2\mathcal{R}_1\left(\mathfrak{s}+1, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}\right) - \frac{q^{\mathfrak{s}}\mathfrak{z}\Gamma_q\left(\mathfrak{r}\right)\Gamma_q\left(\mathfrak{b}+\tau\right)}{\Gamma_q\left(\mathfrak{b}\right)\Gamma_q\left(\mathfrak{r}+\tau\right)} {}_2\mathcal{R}_1\left(\mathfrak{s}+1, \mathfrak{b}+\tau; \mathfrak{r}+\tau; \tau; q, \mathfrak{z}\right). \tag{3.7}$$

Proof. To prove (3.5), let us begin with

$$\begin{aligned}
q^{\mathfrak{b}}[\mathfrak{r}-1]_q \cdot {}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}-1; \tau; q, \mathfrak{z}\right) - q^{\mathfrak{r}-1}[\mathfrak{b}]_q \cdot {}_2\mathcal{R}_1\left(\mathfrak{s}, \mathfrak{b}+1; \mathfrak{r}; \tau; q, \mathfrak{z}\right)&\\
= q^{\mathfrak{b}}[\mathfrak{r}-1]_q \left(\frac{\Gamma_q\left(\mathfrak{r}-1\right)}{\Gamma_q\left(\mathfrak{b}\right)} \sum_{\ell=0}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q\left(\mathfrak{b}+\tau\ell\right)}{\Gamma_q\left(\mathfrak{r}-1+\tau\ell\right)} \mathfrak{z}^\ell\right)&\\
- q^{\mathfrak{r}-1}[\mathfrak{b}]_q \left(\frac{\Gamma_q\left(\mathfrak{r}\right)}{\Gamma_q\left(\mathfrak{b}+1\right)} \sum_{\ell=0}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q\left(\mathfrak{b}+1+\tau\ell\right)}{\Gamma_q\left(\mathfrak{r}+\tau\ell\right)} \mathfrak{z}^\ell\right)&\\
= \frac{\Gamma_q\left(\mathfrak{r}\right)}{\Gamma_q\left(\mathfrak{b}\right)} \sum_{\ell=0}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \mathfrak{z}^\ell \frac{\Gamma_q\left(\mathfrak{b}+\tau\ell\right)}{\Gamma_q\left(\mathfrak{r}-1+\tau\ell\right)} \left(q^{\mathfrak{b}} - q^{\mathfrak{r}-1} \frac{[\mathfrak{b}+\tau\ell]_q}{[\mathfrak{r}-1+\tau\ell]_q}\right)&\\
= \frac{\Gamma_q\left(\mathfrak{r}\right)}{\Gamma_q\left(\mathfrak{b}\right)} \sum_{\ell=0}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \mathfrak{z}^\ell \frac{\Gamma_q\left(\mathfrak{b}+\tau\ell\right)}{\Gamma_q\left(\mathfrak{r}+\tau\ell\right)} \left(q^{\mathfrak{b}}[\mathfrak{r}-1]_q - q^{\mathfrak{r}-1}[\mathfrak{b}]_q\right),&
\end{aligned}$$

which leads us to (3.5).
To prove (3.6), considering the right hand side using (2.1), we get

$$\begin{aligned}
&[\mathfrak{r}-\mathfrak{b}]_q\, {}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r}+1;\tau;q,\mathfrak{z}\right)+q^{\mathfrak{r}-\mathfrak{b}}[\mathfrak{b}]_q {}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b}+1;\mathfrak{c}+1;\tau;q,\mathfrak{z}\right)\\
&=\frac{\Gamma_q\left(\mathfrak{r}\right)}{\Gamma_q\left(\mathfrak{b}\right)}\sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s};q\rangle_\ell}{\langle 1;q\rangle_\ell}\frac{\Gamma_q\left(\mathfrak{b}+\tau\ell\right)}{\Gamma_q\left(\mathfrak{r}+\tau\ell\right)}\mathfrak{z}^\ell\left(\frac{[\mathfrak{r}-\mathfrak{b}]_q[\mathfrak{r}]_q}{[\mathfrak{r}+\tau\ell]_q}+\frac{q^{\mathfrak{r}-\mathfrak{b}}[\mathfrak{b}]_q[\mathfrak{r}]_q}{[\mathfrak{b}]_q}\frac{[\mathfrak{b}+\tau\ell]_q}{[\mathfrak{r}+\tau\ell]_q}\right)\\
&=\frac{\Gamma_q\left(\mathfrak{r}\right)}{\Gamma_q\left(\mathfrak{b}\right)}\sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s};q\rangle_\ell}{\langle 1;q\rangle_\ell}\frac{\Gamma_q\left(\mathfrak{b}+\tau\ell\right)}{\Gamma_q\left(\mathfrak{r}+\tau\ell\right)}\mathfrak{z}^\ell\frac{[\mathfrak{r}]_q}{[\mathfrak{r}+\tau\ell]_q}\left([\mathfrak{r}-\mathfrak{b}]_q+q^{\mathfrak{r}-\mathfrak{b}}[\mathfrak{b}+\tau\ell]_q\right)\\
&=\frac{\Gamma_q\left(\mathfrak{r}\right)}{\Gamma_q\left(\mathfrak{b}\right)}\sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s};q\rangle_\ell}{\langle 1;q\rangle_\ell}\frac{\Gamma_q\left(\mathfrak{b}+\tau\ell\right)}{\Gamma_q\left(\mathfrak{r}+\tau\ell\right)}\mathfrak{z}^\ell\frac{[\mathfrak{r}]_q}{[\mathfrak{r}+\tau\ell]_q}[\mathfrak{r}+\tau\ell]_q,
\end{aligned}$$

which coincides with left hand side of (3.6).
Identity (3.7) is obtained by considering

$$\begin{aligned}
&\frac{1}{\mathfrak{z}}\left({}_2\mathcal{R}_1\left(\mathfrak{s}+1,\mathfrak{b};\mathfrak{r};\tau;q,\mathfrak{z}\right)-{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;q,\mathfrak{z}\right)\right)\\
&\quad=\frac{\Gamma_q\left(\mathfrak{r}\right)}{\Gamma_q\left(\mathfrak{b}\right)}\sum_{\ell=1}^{\infty}\frac{\langle\mathfrak{s}+1;q\rangle_{\ell-1}\Gamma_q\left(\mathfrak{b}+\tau\ell\right)}{\langle 1;q\rangle_{\ell-1}\left(1-q^\ell\right)\Gamma_q\left(\mathfrak{r}+\tau\ell\right)}\mathfrak{z}^{\ell-1}\left(1-q^{\mathfrak{s}+\ell}-\left(1-q^{\mathfrak{s}}\right)\right)\\
&\quad=\frac{\Gamma_q\left(\mathfrak{r}\right)}{\Gamma_q\left(\mathfrak{b}\right)}\sum_{\ell=1}^{\infty}\frac{\langle\mathfrak{s}+1;q\rangle_{\ell-1}\Gamma_q\left(\mathfrak{b}+\tau+\tau\ell-\tau\right)}{\langle 1;q\rangle_{\ell-1}\left(1-q^\ell\right)\Gamma_q\left(\mathfrak{r}+\tau+\tau\ell-\tau\right)}\mathfrak{z}^{\ell-1}q^{\mathfrak{s}}\left(1-q^\ell\right),
\end{aligned}$$

Which reduces to (3.7).
Moreover, In (3.5), (3.6), (3.7), letting $q\to 1$ we have the well known properties of Generalized Hypergeometric function, established by Virchenko [14] as shown in (3.8), (3.9), (3.10) respectively.
For $\mathfrak{s},\mathfrak{b},\mathfrak{r}\in\mathbb{C};\Re(\mathfrak{s}),\Re(\mathfrak{b});\Re(\mathfrak{r})>0;\tau\in\mathbb{R}^+$ and $|\mathfrak{z}|<1$ [14]

$$\begin{aligned}
&(\mathfrak{r}-\mathfrak{b}-1){}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;\mathfrak{z}\right)\\
&\qquad=(\mathfrak{r}-1)\,{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r}-1;\tau;z\right)+\mathfrak{b}\,{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b}+1;\mathfrak{r};\tau;\mathfrak{z}\right), \qquad (3.8)
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{r}\cdot{}_2\mathcal{R}_1(\mathfrak{s},\mathfrak{b},\mathfrak{r};\tau;\mathfrak{z})\\
&\qquad=(\mathfrak{r}-\mathfrak{b})\,{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b},\mathfrak{r}+1;\tau;\mathfrak{z}\right)+\mathfrak{b}\cdot{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b}+1;\mathfrak{c}+1;\tau;\mathfrak{z}\right), \quad (3.9)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma\left(\mathfrak{b}\right)\Gamma\left(\mathfrak{r}+\tau\right){}_2\mathcal{R}_1(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;\mathfrak{z})&=\Gamma\left(\mathfrak{b}\right)\Gamma\left(\mathfrak{r}+\tau\right){}_2\mathcal{R}_1\left(\mathfrak{s}+1,\mathfrak{b};\mathfrak{r};\tau;\mathfrak{z}\right)\\
&\quad-\mathfrak{z}\Gamma\left(\mathfrak{r}\right)\Gamma\left(\mathfrak{b}+\tau\right){}_2\mathcal{R}_1\left(\mathfrak{s}+1,\mathfrak{b}+\tau;\mathfrak{r}+\tau;\tau;\mathfrak{z}\right). \quad (3.10)
\end{aligned}$$

□

Theorem 3. *If* $\mathfrak{s}, \mathfrak{b}, \mathfrak{r} \in \mathbb{C}, \Re(\mathfrak{s}), \Re(\mathfrak{r}) > 0, \Re(\mathfrak{b}) > 1; 0 < |q| < 1, \tau \in \mathbb{R}^+$ *and* $|\mathfrak{z}| < 1$ *then*

$$
\begin{aligned}
{}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}^\tau) = & {}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}-1; \mathfrak{r}; \tau; q, \mathfrak{z}^\tau) \\
& + \frac{q^{(\mathfrak{b}-1)}\mathfrak{z}}{[\mathfrak{b}-1]_q} \frac{d_q}{d_q\mathfrak{z}} \left({}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}-1; \mathfrak{r}; \tau; q, \mathfrak{z}^\tau)\right).
\end{aligned} \tag{3.11}
$$

Proof. To obtain the result, clearly

$$
\begin{aligned}
& \frac{1}{\mathfrak{z}}\left({}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}^\tau) - {}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}-1; \mathfrak{r}; \tau; q, \mathfrak{z}^\tau)\right) \\
& \quad = \frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b}-1)} \sum_{\ell=0}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q(\mathfrak{b}-1+\tau\ell)}{\Gamma_q(\mathfrak{r}+\tau\ell)} \mathfrak{z}^{\tau\ell-1} \left\{ \frac{[\mathfrak{b}-1+\tau\ell]_q - [\mathfrak{b}-1]_q}{[\mathfrak{b}-1]_q} \right\} \\
& \quad = \frac{q^{(\mathfrak{b}-1)}}{[\mathfrak{b}-1]_q} \frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b}-1)} \sum_{\ell=0}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q(\mathfrak{b}-1+\tau\ell)}{\Gamma_q(\mathfrak{r}+\tau\ell)} [\tau\ell]_q \mathfrak{z}^{\tau\ell-1},
\end{aligned}
$$

which leads us to (3.11). Observe that taking $q \to 1$ in (3.11), it reduces to the new identity involving derivative of ${}_2\mathcal{R}_1^{\tau,q}(\mathfrak{z})$ as below.

$$
{}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; \mathfrak{z}^\tau) = {}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}-1; \mathfrak{r}; \tau; \mathfrak{z}^\tau) + \frac{\mathfrak{z}}{\mathfrak{b}-1} \frac{d}{d\mathfrak{z}} \left({}_2R_1(\mathfrak{s}, \mathfrak{b}-1; \mathfrak{r}; \tau; \mathfrak{z}^\tau)\right).
$$

Which is a particular case of (3.11). One can verify that above expression has the similar proof as (3.11). □

Theorem 4. *For* $\Re(\mathfrak{s}), \Re(b) > 0; 0 < |q| < 1, \tau \in \mathbb{R}^+$*;*

1) If $\Re(\mathfrak{r}) > 0$ *and* $|\mathfrak{z}| < 1$ *then*

$$
\begin{aligned}
q^{-\mathfrak{r}}[\mathfrak{r}]_q \cdot {}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}^\tau) = & q^{-\mathfrak{r}}[\mathfrak{r}]_q \cdot {}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}+1; \tau; q, \mathfrak{z}^\tau) \\
& + \mathfrak{z} \cdot \frac{d_q}{d_q\mathfrak{z}} \left({}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}+1; \tau; q, \mathfrak{z}^\tau)\right).
\end{aligned} \tag{3.12}
$$

2) If $\Re(\mathfrak{r}) > m$ *and* $|\omega\mathfrak{z}^\tau| < 1$ *then*

$$
\begin{aligned}
& \left(\frac{d_q}{d_q\mathfrak{z}}\right)^m [\mathfrak{z}^{\mathfrak{r}-1} {}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \omega\mathfrak{z}^\tau)] \\
& \qquad = \frac{z^{\mathfrak{r}-m-1} \cdot \Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{r}-m)} {}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}-m; \tau; q, \omega\mathfrak{z}^\tau).
\end{aligned} \tag{3.13}
$$

Proof. Let us start with

$$\begin{aligned}
&\mathfrak{z}\cdot\frac{d_q}{d_q\mathfrak{z}}\left({}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r}+1;\tau;q,\mathfrak{z}^{\tau}\right)\right)\\
&\qquad=\mathfrak{z}\cdot\frac{\Gamma_q\left(\mathfrak{r}+1\right)}{\Gamma_q\left(\mathfrak{b}\right)}\sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s};q\rangle_\ell}{\langle 1;q\rangle_\ell}\frac{\Gamma_q\left(\mathfrak{b}+\tau\ell\right)}{\Gamma_q\left(\mathfrak{r}+1+\tau\ell\right)}[\tau\ell]_q\mathfrak{z}^{\tau\ell-1}\\
&\qquad=\frac{\Gamma_q\left(\mathfrak{r}+1\right)}{\Gamma_q\left(\mathfrak{b}\right)}\sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s};q\rangle_\ell}{\langle 1;q\rangle_\ell}\frac{\Gamma_q\left(\mathfrak{b}+\tau\ell\right)}{\Gamma_q\left(\mathfrak{r}+1+\tau\ell\right)}\left[q^{-\mathfrak{r}}[\mathfrak{r}+\tau\ell]_q+[-\mathfrak{r}]_q\right]\mathfrak{z}^{\tau\ell}
\end{aligned}$$

Using the result (2.2), we have

$$\begin{aligned}
&\mathfrak{z}\cdot\frac{d_q}{d_q\mathfrak{z}}\left({}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r}+1;\tau;q,\mathfrak{z}^{\tau}\right)\right)\\
&\qquad=q^{-\mathfrak{r}}[\mathfrak{r}]_q{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;q,\mathfrak{z}^{\tau}\right)-q^{-\mathfrak{r}}[\mathfrak{r}]_q{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r}+1;\tau;q,\mathfrak{z}^{\tau}\right),
\end{aligned}$$

which leads us to (3.12).
To prove (3.13), let us begin with

$$\begin{aligned}
&\left(\frac{d_q}{d_q\mathfrak{z}}\right)^m\left[\mathfrak{z}^{\mathfrak{r}-1}{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;q,\omega\mathfrak{z}^{\tau}\right)\right]\\
&\qquad=\frac{\Gamma_q\left(\mathfrak{r}\right)}{\Gamma_q\left(\mathfrak{b}\right)}\sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s};q\rangle_\ell}{\langle 1;q\rangle_\ell}\frac{\Gamma_q\left(\mathfrak{b}+\tau\ell\right)}{\Gamma_q\left(\mathfrak{r}+\tau\ell\right)}\omega^{\ell}\mathfrak{z}^{\tau\ell+\mathfrak{r}-m-1}\prod_{\iota=1}^{m}[\tau\ell+\mathfrak{r}-\iota]_q\\
&\qquad=\mathfrak{z}^{\mathfrak{r}-m-1}\frac{\Gamma_q\left(\mathfrak{r}\right)}{\Gamma_q\left(\mathfrak{b}\right)}\sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s};q\rangle_\ell}{\langle 1;q\rangle_\ell}\frac{\Gamma_q\left(\mathfrak{b}+\tau\ell\right)}{\Gamma_q\left(\mathfrak{r}+\tau\ell-m\right)}\omega^{\ell}\mathfrak{z}^{\tau\ell}.
\end{aligned}$$

Further simplification of above expression immediately reduces to (3.13). In particular, letting $q\to 1$ in (3.12), (3.13), we get the well known properties (3.14), (3.15) of the generalized hypergeometric function as in [12]. For $\mathfrak{s},\mathfrak{b},\mathfrak{r}\in\mathbb{C};\Re(\mathfrak{s}),\Re(\mathfrak{b}),\Re(\mathfrak{r})>0$, and $\tau\in\mathbb{N}$ [12];

1) If $|\mathfrak{z}|<1$ then

$$ {}_2\mathcal{R}_1(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;\mathfrak{z})={}_2\mathcal{R}_1(\mathfrak{s},\mathfrak{b};\mathfrak{r}+1;\tau;\mathfrak{z})+\frac{\tau\mathfrak{z}}{\mathfrak{r}}\cdot\frac{d}{d\mathfrak{z}}\left({}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r}+1;\tau;\mathfrak{z}\right)\right). \tag{3.14}$$

2) If $|\omega\mathfrak{z}^{\tau}|<1$ then

$$\begin{aligned}
&\left(\frac{d}{d\mathfrak{z}}\right)^m\left[\mathfrak{z}^{\mathfrak{r}-1}{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;\omega\mathfrak{z}^{\tau}\right)\right]\\
&\qquad\qquad=\frac{\Gamma\left(\mathfrak{r}\right)\cdot\mathfrak{z}^{\mathfrak{r}-m-1}}{\Gamma\left(\mathfrak{r}-m\right)}{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r}-m;\tau;\omega\mathfrak{z}^{\tau}\right).
\end{aligned}\tag{3.15}$$

□

Theorem 5. *If* $\mathfrak{s}, \mathfrak{b}, \mathfrak{r} \in \mathbb{C}, \Re(\mathfrak{s}), \Re(\mathfrak{b}) > 0, \Re(\mathfrak{r}) > 1; 0 < |q| < 1, \tau \in \mathbb{R}^+$ *and* $|\alpha \mathfrak{x}^\tau| < 1$ *then*

$$\frac{d_q}{d_q\mathfrak{x}}\left(\mathfrak{x}^{(\mathfrak{b}-1)}{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;q,\alpha\mathfrak{x}^\tau\right)\right)$$
$$= \mathfrak{x}^{\mathfrak{b}-2}\left[q^{\mathfrak{b}-\mathfrak{r}}[\mathfrak{c}-1]_q {}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r}-1;\tau;q,\alpha\mathfrak{x}^\tau\right)+[\mathfrak{b}-\mathfrak{r}]_q {}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;q,\alpha\mathfrak{x}^\tau\right)\right] \tag{3.16}$$

Proof. To prove this, consider the left hand side of (3.16) as

$$\frac{d_q}{d_q\mathfrak{x}}\left(\mathfrak{x}^{(\mathfrak{b}-1)}{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;q,\alpha\mathfrak{x}^\tau\right)\right)$$
$$= \frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b})}\sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s};q\rangle_\ell}{\langle 1;q\rangle_\ell}\frac{\Gamma_q(\mathfrak{b}+\tau\ell)}{\Gamma_q(\mathfrak{r}+\tau\ell)}\alpha^\ell[\tau\ell+\mathfrak{b}-1]_q\mathfrak{x}^{\tau\ell+\mathfrak{b}-2}$$
$$= \mathfrak{x}^{b-2}\frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b})}\sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s};q\rangle_\ell}{\langle 1;q\rangle_\ell}\Gamma_q(\mathfrak{b}+\tau\ell)\,\alpha^\ell\frac{q^{\mathfrak{b}-\mathfrak{r}}[\mathfrak{r}-1+\tau\ell]_q+[\mathfrak{b}-\mathfrak{r}]_q}{\Gamma_q(\mathfrak{r}+\tau\ell)}\mathfrak{x}^{\tau\ell}$$

Using the result (2.1), we have

$$\frac{d_q}{d_q\mathfrak{x}}(\mathfrak{x}^{(\mathfrak{b}-1)}{}_2\mathcal{R}_1(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;q,\alpha\mathfrak{x}^\tau))$$
$$= \mathfrak{x}^{\mathfrak{b}-2}q^{\mathfrak{b}-\mathfrak{r}}\frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b})}\sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s};q\rangle_\ell}{\langle 1;q\rangle_\ell}\frac{\Gamma_q(\mathfrak{b}+\tau\ell)}{\Gamma_q(\mathfrak{r}+\tau\ell-1)}\alpha^\ell\mathfrak{x}^{\tau\ell}$$
$$+ \mathfrak{x}^{\mathfrak{b}-2}[\mathfrak{b}-\mathfrak{r}]_q\frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b})}\sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s};q\rangle_\ell}{\langle 1;q\rangle_\ell}\frac{\Gamma_q(\mathfrak{b}+\tau\ell)}{\Gamma_q(\mathfrak{r}+\tau\ell)}\alpha^\ell\mathfrak{x}^{\tau\ell}.$$

Which is the required proof. □

Remark 2. Taking limit $q \to 1$ in above expression, we get a known identity as (3.17).
If $\mathfrak{s}, \mathfrak{b}, \mathfrak{r} \in \mathbb{C}; \Re(\mathfrak{s}), \Re(\mathfrak{b}) > 0, \Re(\mathfrak{r}) > 1, \tau > 0$ and $|\alpha\mathfrak{x}^\tau| < 1$ then [13]

$$\frac{d}{d\mathfrak{x}}\left(\mathfrak{x}^{(\mathfrak{b}-1)}{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b},\mathfrak{r};\tau;\alpha\mathfrak{x}^\tau\right)\right)$$
$$= \mathfrak{x}^{\mathfrak{b}-2}\left[(\mathfrak{r}-1)\,{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r}-1;\tau;\alpha\mathfrak{x}^\tau\right)+(\mathfrak{b}-\mathfrak{r})\,{}_2\mathcal{R}_1\left(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;\alpha\mathfrak{x}^\tau\right)\right] \tag{3.17}$$

Lemma 1. *If* $\mathfrak{z} \in \mathbb{C}$ *and* $0 < |q| < 1$ *then* $\langle\mathfrak{z};q\rangle_\ell = (1-q^\mathfrak{z})\,\langle\mathfrak{z}+1;q\rangle_{\ell-1}$.

Proof. We can prove this Lemma by considering

$$\langle\mathfrak{z};q\rangle_\ell = (1-q^\mathfrak{z})\left(1-q^{\mathfrak{z}+1}\right)\ldots\left(1-q^{\mathfrak{z}+\ell-1}\right)$$
$$= (1-q^\mathfrak{z})\left[\left(1-q^{(\mathfrak{z}+1)}\right)\left(1-q^{(\mathfrak{z}+1)+1}\right)\ldots\left(1-q^{(\mathfrak{z}+1)+(\ell-2)}\right)\right],$$

which completes the proof of above Lemma 1. □

Theorem 6. *For $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$, if $\Re(\mathfrak{s}) > 0, \Re(\mathfrak{b}), \Re(\mathfrak{r}) > n\tau; 0 < |q| < 1, \tau \in \mathbb{R}^+$ and $|\mathfrak{z}| < 1$ then*

$$\frac{d_q^m}{d_q\mathfrak{z}^m}{}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}-n\tau; \mathfrak{r}-n\tau; \tau; q, \mathfrak{z}) = [\mathfrak{s}]_q^m \frac{\Gamma_q(\mathfrak{r}-n\tau)\Gamma_q(\mathfrak{b}-(n-m)\tau)}{\Gamma_q(\mathfrak{b}-n\tau)\Gamma_q(\mathfrak{r}-(n-m)\tau)} \cdot {}_2\mathcal{R}_1(\mathfrak{s}+m, \mathfrak{b}-(n-m)\tau; \mathfrak{r}-(n-m)\tau; \tau; q, \mathfrak{z}). \quad (3.18)$$

Proof. We will prove this theorem by using mathematical induction. Let us begin by considering

$${}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}-n\tau; \mathfrak{r}-n\tau; \tau; q, \mathfrak{z}) = \frac{\Gamma_q(\mathfrak{r}-n\tau)}{\Gamma_q(\mathfrak{b}-n\tau)} \sum_{\ell=0}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q(\mathfrak{b}-n\tau+\tau\ell)}{\Gamma_q(\mathfrak{r}-n\tau+\tau\ell)} \mathfrak{z}^\ell.$$

By taking q-derivative of the above equation with respect to $\mathfrak{z}$, we get

$$\frac{d_q}{d_q\mathfrak{z}}{}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}-n\tau; \mathfrak{r}-n\tau; \tau; q, \mathfrak{z}) = \frac{\Gamma_q(\mathfrak{r}-n\tau)}{\Gamma_q(\mathfrak{b}-n\tau)} \sum_{\ell=1}^{\infty} \frac{\langle \mathfrak{s}; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q(\mathfrak{b}-n\tau+\tau\ell)}{\Gamma_q(\mathfrak{r}-n\tau+\tau\ell)} [\ell]_q \mathfrak{z}^{\ell-1}.$$

Thanks to the Lemma 1, we obtain the above expression in the form of

$$\frac{d_q}{d_q\mathfrak{z}}{}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}-n\tau; \mathfrak{r}-n\tau; \tau; q, \mathfrak{z}) = [\mathfrak{s}]_q \frac{\Gamma_q(\mathfrak{r}-n\tau)\Gamma_q(\mathfrak{b}-(n-1)\tau)}{\Gamma_q(\mathfrak{b}-n\tau)\Gamma_q(\mathfrak{r}-(n-1)\tau)} \cdot {}_2\mathcal{R}_1(\mathfrak{s}+1, \mathfrak{b}-(n-1)\tau; \mathfrak{r}-(n-1)\tau; \tau; q, \mathfrak{z}). \quad (3.19)$$

Thus, (3.18) is true for $m = 1$. Following the similar arguments and by taking q-derivative of (3.19) with respect to $\mathfrak{z}$, we get

$$\frac{{d_q}^2}{d_q\mathfrak{z}^2}{}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}-n\tau; \mathfrak{r}-n\tau; \tau; q, \mathfrak{z}) = [\mathfrak{s}]_q^2 \frac{\Gamma_q(\mathfrak{r}-n\tau)\Gamma_q(\mathfrak{b}-(n-1)\tau)}{\Gamma_q(\mathfrak{b}-n\tau)\Gamma_q(\mathfrak{r}-(n-2)\tau)} \cdot {}_2\mathcal{R}_1(\mathfrak{s}+2, \mathfrak{b}-(n-2)\tau; \mathfrak{r}-(n-2)\tau; \tau; q, \mathfrak{z}).$$

Thus, (3.18) holds valid for $m = 2$.
Now, let us assume that (3.18) is true when $m = j, j \in \mathbb{N}$. Therefore, we can write that,

$$\frac{d_q^j}{d_q\mathfrak{z}^j}{}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}-n\tau; \mathfrak{r}-n\tau; \tau; q, \mathfrak{z}) = [\mathfrak{s}]_q^j \frac{\Gamma_q(\mathfrak{r}-n\tau)\Gamma_q(\mathfrak{b}-(n-j)\tau)}{\Gamma_q(\mathfrak{b}-n\tau)\Gamma_q(\mathfrak{r}-(n-j)\tau)} \cdot {}_2\mathcal{R}_1(\mathfrak{s}+j, \mathfrak{b}-(n-j)\tau; \mathfrak{r}-(n-j)\tau; \tau; q, \mathfrak{z}).$$

Now, consider the $(j+1)^{th}$ derivative as follows:

$$\frac{d_q^{j+1}}{d_q^{j+1}\mathfrak{z}}{}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}-n\tau; \mathfrak{r}-n\tau; \tau; q, \mathfrak{z}) = \frac{d_q}{d_q z}[\mathfrak{s}]_q^j \frac{\Gamma_q(\mathfrak{r}-n\tau)\Gamma_q(\mathfrak{b}-(n-j)\tau)}{\Gamma_q(\mathfrak{b}-n\tau)\Gamma_q(\mathfrak{r}-(n-j)\tau)} \cdot {}_2\mathcal{R}_1(\mathfrak{s}+j, \mathfrak{b}-(n-j)\tau; \mathfrak{r}-(n-j)\tau; \tau; q, \mathfrak{z}).$$

Use of Lemma 1, yields,

$$\frac{d_q^{j+1}}{d_q^{j+1}\mathfrak{z}}{}_2\mathcal{R}_1(\mathfrak{s},\mathfrak{b}-n\tau;\mathfrak{r}-n\tau;\tau;q,\mathfrak{z})$$

$$=[\mathfrak{s}]_q^j\frac{\Gamma_q(\mathfrak{r}-n\tau)}{\Gamma_q(\mathfrak{b}-n\tau)}\cdot\left(\sum_{\ell=1}^{\infty}\frac{\left(1-q^{\mathfrak{s}+j}\right)\langle\mathfrak{s}+(j+1);q\rangle_{\ell-1}}{\left(1-q^{\ell}\right)\langle 1;q\rangle_{\ell-1}}\right.$$

$$\left.\cdot\frac{\Gamma_q(\mathfrak{b}-(n-(j+1))\tau+\tau(\ell-1))}{\Gamma_q(\mathfrak{r}-(n-(j+1))\tau+\tau(\ell-1))}[\ell]_q\mathfrak{z}^{\ell-1}\right)$$

$$=[\mathfrak{s}]_q^{j+1}\frac{\Gamma_q(\mathfrak{r}-n\tau)\Gamma_q(\mathfrak{b}-(n-(j+1))\tau)}{\Gamma_q(\mathfrak{b}-n\tau)\Gamma_q(\mathfrak{r}-(n-(j+1))\tau)}$$

$$\cdot\left({}_2\mathcal{R}_1(\mathfrak{s}+(j+1),\mathfrak{b}-(n-(j+1))\tau;\mathfrak{r}-(n-(j+1))\tau;\tau;q,\mathfrak{z})\right).$$

As a result, (3.18) holds true for $m=j+1$, whenever it is true for $m=j$. Thus by principle of mathematical induction we can say that, (3.18) is true $\forall m\in\mathbb{Z}^+$. □

Theorem 7. *If* $\Re(\mathfrak{s})>1,\Re(\mathfrak{b}-\tau),\Re(\mathfrak{r}-\tau)>0;0<|q|<1,\tau\in\mathbb{R}^+$ *and* $|\mathfrak{z}|<1$ *then*

$$\int_0^{\mathfrak{z}}{}_2\mathcal{R}_1(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;q,\mathfrak{z})d_q\mathfrak{z}$$

$$=\frac{1}{[\mathfrak{s}-1]_q}\frac{\Gamma_q(\mathfrak{r})\Gamma_q(\mathfrak{b}-\tau)}{\Gamma_q(\mathfrak{b})\Gamma_q(\mathfrak{r}-\tau)}\left({}_2\mathcal{R}_1(\mathfrak{s}-1,\mathfrak{b}-\tau;\mathfrak{r}-\tau;\tau;q,\mathfrak{z})-1\right).\quad(3.20)$$

Proof. To prove this theorem, let us begin by considering the integral as in (3.4):

$$\int_0^{\mathfrak{z}}{}_2\mathcal{R}_1(\mathfrak{s},\mathfrak{b};\mathfrak{r};\tau;q,\mathfrak{z})\,d_q\mathfrak{z}=\frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b})}\sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s};q\rangle_\ell}{\langle 1;q\rangle_\ell}\frac{\Gamma_q(\mathfrak{b}+\tau\ell)}{\Gamma_q(\mathfrak{r}+\tau\ell)}\frac{\mathfrak{z}^{\ell+1}}{[\ell+1]_q}$$

$$=\frac{1}{[\mathfrak{s}-1]_q}\frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b})}\frac{\Gamma_q(\mathfrak{b}-\tau)}{\Gamma_q(\mathfrak{r}-\tau)}$$

$$\cdot\left(\frac{\Gamma_q(\mathfrak{r}-\tau)}{\Gamma_q(\mathfrak{b}-\tau)}\sum_{\ell=-1}^{\infty}\frac{\langle\mathfrak{s}-1;q\rangle_{\ell+1}}{\langle 1;q\rangle_{\ell+1}}\frac{\Gamma_q((\mathfrak{b}-\tau)+\tau(\ell+1))}{\Gamma_q((\mathfrak{r}-\tau)+\tau(\ell+1))}\mathfrak{z}^{\ell+1}-1\right)$$

$$=\frac{1}{[\mathfrak{s}-1]_q}\frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b})}\frac{\Gamma_q(\mathfrak{b}-\tau)}{\Gamma_q(\mathfrak{r}-\tau)}$$

$$\cdot\left(\frac{\Gamma_q(\mathfrak{r}-\tau)}{\Gamma_q(\mathfrak{b}-\tau)}\sum_{\ell=0}^{\infty}\frac{\langle\mathfrak{s}-1;q\rangle_\ell}{\langle 1;q\rangle_\ell}\frac{\Gamma_q((\mathfrak{b}-\tau)+\tau\ell)}{\Gamma_q((\mathfrak{r}-\tau)+\tau\ell)}\mathfrak{z}^{\ell}-1\right),$$

which completes the proof of this theorem. □

Theorem 8 (Integral Representation of ${}_2\mathcal{R}_1^{\tau,q}(\mathfrak{z})$)**.** *If* $\Re(\mathfrak{s}) > 0, \Re(\mathfrak{r} - \mathfrak{b}) > 0; 0 < |q| < 1, \tau \in \mathbb{R}^+$ *and* $|\mathfrak{z}| < 1$ *then*

$$ {}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}) = \frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b})\Gamma_q(\mathfrak{r} - \mathfrak{b})} \int_0^1 t^{\frac{\mathfrak{b}-1}{\tau}} \cdot \frac{\left((t^{1/\tau})q; q\right)_{\mathfrak{r}-\mathfrak{b}-1}}{(\mathfrak{z}t; q)_{\mathfrak{s}}} d_q t. \tag{3.21} $$

Proof. To prove this theorem, let us initiate by using the definition of q-gamma (1.9) in (3.1), we get

$$ {}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}) = \frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b})} \sum_{\ell=0}^{\infty} \frac{\langle \mathfrak{s}; q \rangle_\ell}{\langle 1; q \rangle_\ell} \mathfrak{z}^\ell \frac{\left(q^{\mathfrak{r}+\tau\ell}; q\right)_\infty}{\left(q^{\mathfrak{b}+\tau\ell}; q\right)_\infty} (1-q)^{\mathfrak{r}-\mathfrak{b}} $$

Further simplification and on implementation of q-binomial theorem as in (1.5), we get

$$ \begin{aligned} {}_2\mathcal{R}_1(\mathfrak{s}, \mathfrak{b}; \mathfrak{r}; \tau; q, \mathfrak{z}) &= \frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b})}(1-q)^{\mathfrak{r}-\mathfrak{b}} \sum_{m=0}^{\infty} \frac{\langle \mathfrak{r} - \mathfrak{b}; \mathfrak{q} \rangle_m}{\langle 1; q \rangle_m} \left(q^{m\mathfrak{b}}\right) \frac{1}{(\mathfrak{z}q^{m\tau}; q)_{\mathfrak{s}}} \\ &= \frac{\Gamma_q(\mathfrak{r})}{\Gamma_q(\mathfrak{b})}(1-q)^{\mathfrak{r}-\mathfrak{b}} \frac{\langle \mathfrak{r} - \mathfrak{b}; q \rangle_\infty}{\langle 1; q \rangle_\infty} \sum_{m=0}^{\infty} \left(q^{m\mathfrak{b}}\right) \frac{\left(q^{m+1}; q\right)_{\mathfrak{r}-\mathfrak{b}-1}}{(\mathfrak{z}q^{m\tau}; q)_{\mathfrak{s}}} \end{aligned} $$

Which leads us to (3.21). □

4. Conclusion

Present theoretical investigation intended to study and develop certain basic results involving q-derivative and q-integral operators on the newly defined q-hypergeometric function, thus providing better insights and understanding about the nature of the function.

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Об авторах

Чаудхари Кульдипкумар К., аспирант, кафедра прикладной математики, факультет технологии и инженерии, Университет Махараджа Саяджирао в Барода, Вадодара–390001, Гуджарат, Индия, kuldip.c-appmathphd@msubaroda.ac.in, https://orcid.org/0000-0001-7812-1089

Рао Снехал Б., ассистент профессора, кафедра прикладной математики, факультет технологии и инженерии, Университет Махараджа Саяджирао в Барода, Вадодара–390001, Гуджарат, Индия, snehal.b.rao-appmath@msubaroda.ac.in https://orcid.org/0000-0002-5672-4927

About the authors

Kuldipkumar K. Chaudhary, Research Scholar, Department of Applied Mathematics, Faculty of Technology & Engineering, the Maharaja Sayajirao University of Baroda, Vadodara–390001, Gujarat, India, kuldip.c-appmathphd@msubaroda.ac.in https://orcid.org/0000-0001-7812-1089

Snehal B. Rao, Assistant Prof., Department of Applied Mathematics, Faculty of Technology & Engineering, the Maharaja Sayajirao University of Baroda, Vadodara–390001, Gujarat, India, snehal.b.rao-appmath@msubaroda.ac.in https://orcid.org/0000-0002-5672-4927

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