Серия «Математика»
2024. Т. 48. С. 3—20

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского
государственного
университета

Research article

УДК 519.6, 519.83
MSC 49M05, 91A05
DOI https://doi.org/10.26516/1997-7670.2024.48.3

Algorithm for Solving the Problem of the First Phase in a Game Problem with Arbitrary Situations

Akmal R. Mamatov [1]✉

[1] Samarkand State University named after Sh. Rashidov, Samarkand, Uzbekistan
✉ akmm1964@rambler.ru

Abstract. The game problem of two persons (players) is considered. The two players alternately choose their strategies from the appropriate sets. First, the first player chooses his strategy, then, knowing the strategy of the first player, the second player chooses his strategy. The set of strategies of the second player depends on the strategy of the first player. It is required to determine the following: for any strategy of the first player, does there exist a corresponding strategy of the second player? This problem is solved using a special linear maximin problem with connected variables, the solution of which is reduced to determining the maximum value of the objective function of the problem's dual to it on special strategies. The algorithm for solving the problem considered is given. Two examples that illustrate the algorithm and the results of numerical experiment is given.

Keywords: game problem, first phase problem, dual problem, support, algorithm

For citation: Mamatov A. R. Algorithm for Solving the Problem of the First Phase in a Game Problem with Arbitrary Situations. *The Bulletin of Irkutsk State University. Series Mathematics*, 2024, vol. 48, pp. 3–20.
https://doi.org/10.26516/1997-7670.2024.48.3

Научная статья

Алгоритм решения задачи первой фазы в игровой задаче с произвольными ситуациями

А. Р. Маматов[1]✉

[1] Самаркандский государственный университет им. Ш. Рашидова, Самарканд, Узбекистан
✉ akmm1964@rambler.ru

Аннотация. Рассматривается игровая задача двух лиц (игроков). Два игрока поочередно выбирают свои стратегии из соответствующих множеств. Сначала первый игрок выбирает свою стратегию, затем, зная стратегию первого игрока, второй игрок выбирает свою стратегию. Множество стратегий второго игрока зависит от стратегии первого игрока. Требуется определить: существует ли для любой стратегии первого игрока соответствующая стратегия второго игрока? Данная задача решается с помощью специальной линейной максиминной задачи со связанными переменными, решение которой сводится к определению максимального значения целевой функции двойственной к ней задачи на специальных стратегиях. Приведен алгоритм решения рассматриваемой задачи, два примера, иллюстрирующие работу алгоритма, а также результаты численных экспериментов.

Ключевые слова: игровая задача, задача первой фазы, двойственная задача, опора, алгоритм

Ссылка для цитирования: Mamatov A. R. Algorithm for Solving the Problem of the First Phase in a Game Problem with Arbitrary Situations // Известия Иркутского государственного университета. Серия Математика. 2024. Т. 48. С. 3–20.
https://doi.org/10.26516/1997-7670.2024.48.3

1. Introduction

As is known [4], the problem of the nonemptiness of the set of feasible solutions for the problem of linear programming, can be solved using a special linear programming problem - the problem of the first phase. Analogycal problems could be involved also for game problems with connected variables [1; 2; 7; 12; 13].

Consider the sets

$$X = \{x \mid f_* \leq x \leq f^*\} \text{ and } Y(x) = \{y \mid g_* \leq y \leq g^*, Ax + By = b\},$$

where $x = x(J)$, $f_* = f_*(J)$, $f^* = f^*(J) \in \mathbb{R}^n$, $y = y(K_1)$, $g_* = g_*(K_1)$, $g^* = g^*(K_1) \in \mathbb{R}^l$, $A = A(I, J) \in \mathbb{R}^{m\times n}$, $B = B(I, K_1) \in \mathbb{R}^{m\times l}$, $rankB = m < l, b = b(I) \in \mathbb{R}^m, I = \{1, 2, ..., m\}, J = \{1, 2, ..., n\}, K_1 = \{1, 2, ..., l\}$.

The following game problems with connected variables: game problems with favorable situations [11]; game problems with arbitrary situations [7], [12] can be considered on sets X and $Y(x)$.

When $X = \{x \mid f_* \leq x \leq f^*, Y(x) \neq \varnothing\}$ we can also consider game problems with forbidden situations [1; 2; 13]. Note that the weak linear bilevel programming problem [3; 9; 10; 15; 16] with the same objective functions at the upper and lower levels is a game problem with connected variables (linear maximin (minimax) problem with connected variables). In [3;10] the weak linear bilevel programming problem is reduced to a linear game problem with connected variables. Various algorithms [1–3; 9; 10; 12; 13; 15; 16], have been developed to solve game problems with connected variables, as well as weak problems of linear bilevel programming. The analysis of these algorithms shows that, in a certain sense, it is not possible to avoid enumeration.

It is known [4] that the Duality Theory of linear programming problems is based on the existence theorem, the duality theorem and the duality relations between the solutions of the primal and dual problems that arising from them.

For game problems with favorable situations, sufficient optimality conditions similar to those from duality theory of linear programming, generally speaking, not true. Moreover, here it is not always possible to speak of local optimality [14].

To develop efficient algorithms for solving game problems with arbitrary situations [7; 12], as well as game problems with forbidden situations [1], [2], [13] it is important to find conditions under which $\forall x \in X, Y(x) \neq \varnothing$ or $\exists x^* \in X, Y(x^*) = \varnothing$.

Based on this, in the paper the problem which arises in game problems with connected variables is under research [1; 2; 7; 12; 13],that is the problem of determining whether the set of strategies of the second player is nonempty for any strategy of the first player is studied.

Following the theory of linear programming has been formulated and proved the lemma on the nonemptiness of the set strategies of the second player for any strategy of the first player.

In the work [12] for a game problem with arbitrary situations (with connected variables) [7], a dual algorithm for solving it theoretically substantiated and developed. The algorithm from [12] consists of two parts. The first part of the algorithm defines $x^0 \in X$ such that $Y(x^0) = \varnothing$ or we conclude that $\forall x \in X, Y(x) \neq \varnothing$.

In this paper, an improved version of the first part of the [12] algorithm is also proposed. In this algorithm, based on a new kind of dual problem for a special problem of the first phase of the game with arbitrary situations, as well as special “supporting” properties of the problem, the number of mathematical operations has been reduced significantly.

2. Problem statement. Preliminary information from the theory of game problems with favorable situations

Let there be two players who choose vectors x and y, respectively, from the sets $X, Y(x)$ in turn, first the first player chooses x, then, knowing x, the second player chooses y.

The goal of the first player is to find $\hat{x}$ that gives the maximum value for the function

$$\varphi(x) = \min_{y \in Y(x)} \Psi(x, y), x \in X, \text{i.e. } \varphi(\hat{x}) = \max_{x \in X} \varphi(x),$$

the second player's goal is to find $\hat{y}$ that minimizes the function

$$\Psi(\hat{x}, y), y \in Y(\hat{x}), \text{i.e. } \Psi(\hat{x}, \hat{y}) = \min_{y \in Y(\hat{x})} \Psi(\hat{x}, y).$$

$$\text{Here } \Psi(x, y) = \begin{cases} c'x + d'y, \text{ if } x \in X, y \in Y(x), c \in \mathbb{R}^n, d \in \mathbb{R}^l; \\ +\infty, \text{ if } x \in X, Y(x) = \varnothing. \end{cases}$$

Then we have a maximin problem with connected variables [7; 12]:

$$\varphi(x) = \min_{y \in Y(x)} \Psi(x, y) \to \max_{x \in X}. \tag{2.1}$$

Definition 1. *The vector $x \in X$ is called a strategy (a feasible strategy) of the first player.*

Definition 2. *The vector $y \in Y(x)$ is called a strategy (a feasible strategy) of the second player, corresponding to the first player's strategy x (in short, the second player's x-strategy).*

Note that problem (2.1) refers to games with arbitrary situations. Additionally, note that if when solving problem (2.1) $\hat{x} \in X, \varphi(\hat{x}) < \infty$, then $\hat{x}$ is the solution of the game problem with forbidden situations [1], [2], [13]:

$$\varphi(x) = \min_{y \in Y(x)} (c'x + d'y) \to \max_{x \in \overline{X}}, \overline{X} = \{x \mid f_* \leq x \leq f^*, Y(x) \neq \varnothing\}.$$

As noted earlier, to develop an efficient algorithm for solving the problem (2.1), as well as game problems with forbidden situations, it is important to find conditions under which $\forall x \in X, Y(x) \neq \varnothing$ or $\exists x^* \in X, Y(x^*) = \varnothing$.

Based on the above, consider the following problem.

It is required to determine, for any strategy of the first player $x \in X$, whether there exists a corresponding x-strategy of the second player, or whether there exists a strategy of the first player $x^* \in X$, such that $Y(x^*) = \varnothing$, i.e.

$$(a) \quad \forall x \in X, Y(x) \neq \varnothing? \text{ or } (b) \quad \exists x^* \in X, Y(x^*) = \varnothing? \tag{2.2}$$

The problem (2.2) can be considered from the point of view of linear algebra, as the problem of determining the nonemptiness of the set of solutions of systems of linear inequalities $Y(x) = \{y \mid g_* \le y \le g^*, Ax+By = b\}$ with parameters x for any parameter from a given region X or as a problem of finding the parameter $x^* \in X$ such as the set of solutions of systems of linear inequalities $Y(x^*) = \{y \mid g_* \le y \le g^*, Ax^* + By = b\}$ is empty.

Along with problem (2.2), consider the maximin problem:

$$f(x) = \min_{g_* \le y \le g^*} \sum_{i \in I} |A(i,J)x(J) + B(i,K_1)y(K_1) - b(i)| \to \max_{x \in X}. \quad (2.3)$$

Theorem 1. *[12] The optimal values of the objective functions in problems (2.3) and the problem*

$$F(x) = \min_{(y,\xi,\eta) \in H(x)} (e'\xi + e'\eta) \to \max_{x \in X}, \quad (2.4)$$

$$H(x) = \{(y,\xi,\eta) \mid Ax + By - \xi + \eta = b, g_* \le y \le g^*, \xi \ge 0, \eta \ge 0\},$$

$e' = (1,1,...,1)$, *are identical.*

Problem (2.4) is a game problem with favorable situations, i.e for any $x \in X, H(x) \ne \varnothing$.

Problems (2.3) and (2.4) are related to problem (2.2) by the fact that, based on the solutions of these problems, we can conclude about solving problem (2.2).

The algorithm from [12] consists of two parts. The first part of the algorithm defines $x^0 \in X$ such that $Y(x^0) = \varnothing$ or we conclude that for any $x \in X, Y(x) \ne \varnothing$ (Problem of the first part of the algorithm from [12]). For this purpose, the following problem is considered

$$F(x) = \min_{\overline{y} \in \overline{Y}(x)} \overline{d}'\overline{y} \to \max_{x \in X}, \overline{Y}(x) = \{\overline{y} \mid Ax + \overline{B}\overline{y} = b, \overline{g}_* \le \overline{y} \le \overline{g}^*\}. \quad (2.5)$$

Here $\overline{d} = (\overline{d}_k, k \in \overline{K}), \overline{d}_k = 0, k \in K_1, \overline{d}_k = 1, k \in K_2 \cup K_3; \overline{y} \in \mathbb{R}^{l+2m}$,

$$\overline{B} = (B \vdots -E \vdots E) \in \mathbb{R}^{m \times (l+2m)}, \overline{g}_* = (\overline{g}_{*k}, k \in \overline{K}), \overline{g}_{*k} = g_{*k}, k \in K_1,$$

$$\overline{g}_{*k} = 0, k \in K_2 \cup K_3; \overline{g}^* = (\overline{g}^*_k, k \in \overline{K}), \overline{g}^*_k = g^*_k, k \in K_1, \overline{g}^*_{l+i} = \overline{g}^*_{l+m+i} = \gamma_i,$$

$$\gamma_i = \max_{f_* \le x \le f^*, g_* \le y \le g^*} |A(i,J)x(J) + B(i,K)y(K) - b(i)|, i \in I,$$

$$K_2 = \{l+1, l+2, ..., l+m\}, K_3 = \{l+m+1, l+m+2, ..., l+2m\},$$

$$\overline{K} = K_1 \cup K_2 \cup K_3.$$

The solution of the problem of the first part of the algorithm from [12] is reduced to determining the positive value of the objective function of the following problem

$$\psi(\mu,\sigma,\tau) = \min_{(\lambda,\nu) \in \Lambda(\mu,\sigma,\tau)} (b'\mu + \overline{g}'_*\sigma - \overline{g}^{*'}\tau + f^{*'}\lambda - f'_*\nu) \to \max_{(\mu,\sigma,\tau) \in \overline{\Xi}}, \quad (2.6)$$

$$\overline{\Xi} = \{(\mu, \sigma, \tau) \in \mathbb{R}^{m+2(l+m)} \mid \overline{B}'\mu - \tau + \sigma = \overline{d}; \sigma \geq 0, \tau \geq 0\},$$
$$\Lambda(\mu, \sigma, \tau) = \{(\lambda, \nu) \in \mathbb{R}^{2n} \mid A'\mu - \nu + \lambda = 0; \nu \geq 0, \lambda \geq 0\},$$

or determining that the optimal value of the objective function of a given problem is equal to zero on special classes of strategies of players in this problem, constructed using the support [5] of the internal problem of the problem (2.1).

Problem (2.6) is called dual problem to problem (2.5) [12].

3. Theoretical foundations for developing an algorithm for solving the problem of the first phase

Following the theory of linear programming [4], we formulate the lemma of nonemptiness sets of second player strategies at any strategy of the first player problem (2.1).

Definition 3. *The vector $x \in X$ is called a strategy of the first player, and the vector $(y, \xi, \eta) \in H(x)$ is called a x-strategy of the second player of the problem (2.4).*

Lemma. *If for any strategy of the first player x, $x \in X$ in the problem (2.1) the set of x-strategies of the second player $Y(x)$ is nonempty then in the solution $(x^0, y^0, \xi^0, \eta^0)$ of the problem (2.4) the components ξ^0, η^0 of the strategy of the second player is zero. If in the solution of the problem (2.4) $(x^0, y^0, \xi^0, \eta^0)$ components ξ^0, η^0 strategy of the second player are equal to zero, then for any strategy of the first player x, $x \in X$ in the problem (2.1) the set of x-strategies of the second player $Y(x)$ is nonempty.*

Proof. Let $Y(x) \neq \varnothing \quad \forall x \in X$. Then the optimal value of the objective function of problem (2.4) is equal to zero. Indeed, if the optimal value of the objective function of problem (2.4) is positive for $x^0 \in X$, then according to Theorem 1 from [12] $Y(x^0) = \varnothing$. The optimal value of the objective function of problem (2.4) is equali to zero only for $\xi^0 = 0 \in \mathbb{R}^m, \eta^0 = 0 \in \mathbb{R}^m$.

Let in the solution of the problem (2.4) $(x^0, y^0, \xi^0, \eta^0), \xi^0 = \eta^0 = 0 \in \mathbb{R}^m$. Then, according to Theorem 3 from [12], the optimal value of the objective function (2.3) is equal to zero. Therefore, by Theorem 2 from [12], for any $x \in X, Y(x) \neq \varnothing$. □

Remark 1. Thus, according to Theorem 1 and Lemma, if the optimal value objective function of the problem (2.3) ((2.4)) is equal to zero, then $\forall x \in X, Y(x) \neq \varnothing$, i.e. the problem (2.2) for the case (a) is solved. Taking into account Theorem 1 from [12], and also that for each $x \in X, f(x) = F(x)$ we conclude that if the value of the objective function of the problem (2.3) ((2.4)) is positive for some $x^* \in X$, i.e. $f(x^*) > 0\,(F(x^*) > 0)$, then $Y(x^*) = \varnothing$, i.e. the problem (2.2) for the case (b) is solved.

Recall that to determine both the emptiness and nonemptiness of the set of feasible solutions to a linear programming problem, it is necessary to solve the first phase problem. To determine the nonemptiness of the set of strategies of the second player for some strategy of the first player in a game problem with favorable situations, as mentioned above, in the general case, it is not necessary to solve the problem (2.4). Based on this, we introduce the following definitions.

Definition 4. *Let us call problem (2.2) the problem of the first phase for the problem (2.1).*

Definition 5. *Let us call problem (2.4) the special problem of the first phase for the problem (2.1).*

Definition 6. *The problem of maximizing the function $\psi(\mu, s, t)$ with respect to $(\mu, s, t) \in \Xi$, i.e.*

$$\psi(\mu, s, t) = \min_{(\lambda,\nu)\in\Lambda(\mu,s,t)} (b'\mu + g_*'s - {g^*}'t + {f^*}'\lambda - f_*'\nu) \to \max_{(\mu,s,t)\in\Xi}, \quad (3.1)$$

$$\Xi = \{(\mu, s, t) \mid B'\mu - t + s = 0; s \geq 0, t \geq 0, -e \leq \mu \leq e\},$$

$$\Lambda(\mu, s, t) = \{(\lambda, \nu) \mid A'\mu - \nu + \lambda = 0; \nu \geq 0, \lambda \geq 0\},$$

will be called dual to the problem (2.4).

In this regard, problem (2.4) we will call the primal problem. In the following theorem the connection between the primal and dual problems is established.

Theorem 2. *The optimal values of the objective functions in problems (2.4) and (3.1) are identical.*

Proof. Let us first prove that the optimal value of the objective functions of the problem (2.4) and problem

$$(b - Ax)'\mu + g_*'s - {g^*}'t \to \max_{x\in X,(\mu,s,t)\in\Xi} \quad (3.2)$$

match up. Let $x \in X$. Be the optimal value of the objective function of the internal problem of the problem (2.4)

$$(e'\xi + e'\eta) \to \min_{(y,\xi,\eta)\in H(x)}$$

can be considered as a function of the parameter x:

$$F(x) = \min_{(y,\xi,\eta)\in H(x)} (e'\xi + e'\eta).$$

The optimal value of the objective function of the problem

$$(b - Ax)'\mu + g_*'s - g^{*'}t \to \max_{(\mu,s,t)\in\Xi},$$

which is dual to the inner problem of problem (2.4), can also be considered as a function of the x parameter: $\beta(x) = \max_{(\mu,s,t)\in\Xi}((b-Ax)'\mu+g_*'s-g^{*'}t)$.

According to the duality theory of linear programming [4], for a fixed $x \in X, F(x) = \beta(x)$. This implies

$$\max_{x\in X} F(x) = \max_{x\in X,(\mu,s,t)\in\Xi}((b - Ax)'\mu + g_*'s - g^{*'}t).$$

Similarly, it is proven that the optimal value of the objective functions of problems (3.2) and (3.1) coincide, from which follows the proof of the theorem. □

Definition 7. *Vector $(\mu, s, t) \in \Xi$ is called a strategy of the first player in problem (3.1), and vector $(\lambda, \nu) \in \Lambda(\mu, s, t)$ is called a (μ, s, t)-strategy of the second player in problem (3.1).*

We emphasize that it is necessary to distinguish between the corresponding players participating in problems (2.4) and (3.1). Namely, the first player in problem (2.4) is also not the first player in problem (3.1), and the second player in problem (2.4) is not the second player in problem (3.1).The first player in problem (3.1) is the dual of the second player in problem (2.4), and the second player in problem (3.1) is the dual of the first player in problem (2.4). Problem (3.1) is formulated directly for problem (2.4), and problem (2.6) is formulated for problem (2.5) for being equivalent to problem (2.4), and obtained from problem (2.4) by adding fictitious upper constraints for the strategies of the second player. The addition of fictitious upper constraints entails an increase in the dimension of the dual problem. This naturally affects the efficiency of the numerical solution of the dual problem.

Let x be a strategy of the first player in problem (2.4).

Definition 8. *The strategy (y, ξ, η) of the second player is called the optimal x-strategy in problem (2.4), if it is a solution of the problem*

$$e'\xi + e'\eta \to \min_{(y,\xi,\eta)\in H(x)}. \tag{3.3}$$

Set $K_{op} = K_{op}(x) = K_{op1} \cup K_{op2} \cup K_{op3} \quad (K_{op1} \subset K_1, K_{op2} \subseteq K_2, K_{op3} \subseteq K_3, \mid K_{op} \mid= m)$ is called a support [5] of problem (3.3) if

$$det\overline{B}(I, K_{op}) \neq 0 \quad (\overline{B}(I, \overline{K}) = (B(I, K_1); -e_{k-l}, k \in K_2; e_{k-l-m}, k \in K_3)).$$

Based on the K_{op} support, construct the vectors $\mu(I), \Delta(\overline{K}), \nabla(J)$ as follows:

$$\Delta'(\overline{K}) = \mu'(I)\overline{B}(I, \overline{K}) - \overline{d}'(\overline{K}), \nabla'(J) = \mu'(I)A(I, J), \tag{3.4}$$

$$\mu'(I) = \overline{d}'(K_{op})[\overline{B}(I, K_{op})]^{-1}.$$

We present the optimality conditions for the x-strategy of the second player (y, ξ, η) of problem (2.4), i.e., internal problem of the problem (2.4) (problem (3.3)), which is a linear programming problem.

According to [4–6], the following theorem is valid.

Theorem 3. *The strategy $(y, \xi, \eta) \in H(x)$ is the optimal x-strategy of the second player if and only if there exist a support K_{op} such that, for the vector $\Delta(\overline{K})$ constructed by formula (3.4), the following relations are satisfied:*

$$\begin{gathered}\Delta_k \le 0 \quad \mathit{for} \quad y_k = g_{*k}; \Delta_k \ge 0 \quad \mathit{for} \quad v_k = g^*_k;\\ \Delta_k = 0 \quad \mathit{for} \quad g_{*k} < y_k < g^*_k, k \in K_{n1} = K_1 \backslash K_{op1};\\ \Delta_k \le 0 \ \mathit{for}\ \xi_k = 0; \Delta_k = 0 \ \mathit{for}\ \xi_k > 0, k \in K_{n2} = K_2 \backslash K_{op2}; \qquad (3.5)\\ \Delta_k \le 0 \ \mathit{for}\ \eta_k = 0; \Delta_k = 0 \ \mathit{for}\ \eta_k > 0, k \in K_{n3} = K_3 \backslash K_{op3}.\end{gathered}$$

Proof. Sufficiency. Let the conditions of the theorem be satisfied. Note that $\Delta_{l+k} = -\mu_k - 1, \Delta_{l+m+k} = \mu_k - 1, k = \overline{1, m}$. Hence, $-e \le \mu \le e$. We construct vectors s, t as follows:

$$s_k = 0, t_k = \Delta_k \ \ \mathit{if} \ \ \Delta_k \ge 0; s_k = -\Delta_k, t_k = 0 \ \mathit{if} \ \ \Delta_k < 0, k \in K_1. \qquad (3.6)$$

We have the equality

$$e'\xi + e'\eta = \overline{d}'(K_{op}) \begin{pmatrix} y(K_{op1}) \\ \xi(K_{op2}) \\ \eta(K_{op3}) \end{pmatrix} + \overline{d}'(K_n) \begin{pmatrix} y(K_{n1}) \\ \xi(K_{n2}) \\ \eta(K_{n3}) \end{pmatrix} =$$

$$= \overline{d}'(K_{op})[\overline{B}(I, K_{op})]^{-1}[b - Ax - \overline{B}(I, K_{n1} \cup K_{n2} \cup K_{n3}) \begin{pmatrix} y(K_{n1}) \\ \xi(K_{n2}) \\ \eta(K_{n3}) \end{pmatrix}] +$$

$$+ \overline{d}'(K_n) \begin{pmatrix} y(K_{n1}) \\ \xi(K_{n2}) \\ \eta(K_{n3}) \end{pmatrix} = (b - Ax)'\mu + g'_* s - {g^*}' t. \qquad (3.7)$$

Then, it follows from the duality theory of linear programming [4] that strategy $(y, \xi, \eta) \in H(x)$ is the optimal x-strategy of the second player. The proof of the necessary part of the theorem is similar to the proof of the optimality criterion [6] (Part 1, Ch. 6, §1.). □

The support K_{op} is called the x-optimal support (corresponding to the optimal x-strategy (y, ξ, η) in problem (3.3)), if relation (3.5) are satisfied on the pair $\{(y, \xi, \eta), K_{op}\}$.

Let us present the optimality conditions for the strategy of the first player x in the problem (2.4).

Theorem 4. *Let x be the optimal strategy of the first player in problem (2.4). Then, there is an x-optimal support K_{op} of the problem (3.3), where for the vector $\nabla(J)$, constructed according to formula (3.4), the following relations are fulfilled:*

$$\nabla_j \leq 0 \quad for \quad x_j = f_j^*; \nabla_j \geq 0 \quad for \quad x_j = f_{*j};$$
$$\nabla_j = 0 \quad for \quad f_{*j} < x_j < f_j^*, j \in J. \tag{3.8}$$

Proof. Let x be an optimal strategy of the first player in problem (2.4). Solving problem (3.3) by the adaptive method [5], [6] (Part 3), we obtain an x-optimal strategy (y, ξ, η) and an x-optimal support $K_{op}(x)$. From the support $K_{op}(x)$ we construct the vectors (μ, s, t) according to (3.4), (3.6). Then, according to (3.7) we have:

$$e'\xi + e'\eta = -\nabla' x + b'\mu + g_*' s - {g^*}' t.$$

Since x is an optimal strategy of the first player in problem (2.4), then for a fixed (μ, s, t) it is a solution of the problem

$$-\nabla' x + b'\mu + g_*' s - {g^*}' t \to \max_{x \in X}.$$

The optimality conditions for the vector x for this problem are (3.8). □

Definition 9. *Vectors $\delta(\overline{K})$ and $\overline{\nabla}(J)$,*

$$\delta'(\overline{K}) = \mu'(I)\overline{B}(I, \overline{K}) - \overline{d}'(\overline{K}), \overline{\nabla}'(J) = \mu'(I)A(I, J), \tag{3.9}$$

constructed from the component μ strategies of the first player (μ, s, t) in problem (3.1) are called the costrategies of the first and second players, correspondingly, for problem (2.4).

Note that

$$\delta(\overline{K}) = \Delta(\overline{K}), \overline{\nabla}(J) = \nabla(J) \text{ at } \mu'(I) = \overline{d}'(K_{op})[\overline{B}(I, K_{op})]^{-1}.$$

Definition 10. *Terms*

$$s_k = 0, t_k = \delta_k \text{ } if \text{ } \delta_k \geq 0; s_k = -\delta_k, t_k = 0 \text{ } if \text{ } \delta_k < 0, k \in K_1;$$
$$\nu_j = \overline{\nabla}_j, \lambda_j = 0 \text{ } if \text{ } \overline{\nabla}_j \geq 0; \nu_j = 0, \lambda_j = -\overline{\nabla}_j \text{ } if \text{ } \overline{\nabla}_j < 0; j \in J, \tag{3.10}$$

let us call the matching conditions for the strategies of the players (μ, s, t), (λ, ν) of problem (3.1) with the costrategies $\delta(K), \overline{\nabla}(J)$ of problem (2.4).

In this case, we have:

$$b'\mu + g_*' s - {g^*}' t = \max_{(\mu, \overline{s}, \overline{t}) \in \Xi} (b'\mu + g_*' \overline{s} - {g^*}' \overline{t});$$

$$f^{*'}\lambda - f'_*\nu = \min_{(\overline{\lambda},\overline{\nu})\in\Lambda(\mu,\overline{s},\overline{t})} (f^{*'}\overline{\lambda} - f'_*\overline{\nu}). \tag{3.11}$$

Indeed, let $(\mu, \overline{s}, \overline{t}) \in \Xi, (\overline{\lambda}, \overline{\nu}) \in \Lambda(\mu, \overline{s}, \overline{t})$ are arbitrary strategies of the players in the problem (3.1), $(\mu, s, t) \in \Xi, (\lambda, \nu) \in \Lambda(\mu, s, t)$ are agreed strategies of the players in the problem (3.1). The to the strategies of the first player $(\mu, s, t) \in \Xi, (\mu, \overline{s}, \overline{t}) \in \Xi$ corresponds to the same costrategy $\delta(\overline{K})$. Similar to the strategies of the second player $(\lambda, \nu) \in \Lambda(\mu, s, t)$, $(\overline{\lambda}, \overline{\nu}) \in \Lambda(\mu, \overline{s}, \overline{t})$ also corresponds to the same costrategy $\overline{\nabla}(J)$.

Consequently,

$$B'\mu - t + s = 0, B'\mu - \overline{t} + \overline{s} = 0; A'\mu - \nu + \lambda = 0, A'\mu - \overline{\nu} + \overline{\lambda} = 0.$$

Then we have

$$b'\mu + g'_*s - g^{*'}t = b'\mu - \sum_{k\in K_1,\delta_k<0} g_{*k}\delta_k - \sum_{k\in K_1,\delta_k\geq 0} g^*_k\delta_k =$$
$$= b'\mu - \sum_{k\in K_1,\delta_k<0} g_{*k}(\overline{t}_k - \overline{s}_k) - \sum_{k\in K_1,\delta_k\geq 0} g^*_k(\overline{t}_k - \overline{s}_k) \geq$$
$$\geq b'\mu - \sum_{k\in K_1,\delta_k<0} (g^*_k\overline{t}_k - g_{*k}\overline{s}_k) - \sum_{k\in K_1,\delta_k\geq 0} (g^*_k\overline{t}_k - g_{*k}\overline{s}_k) = b'\mu + g'_*\overline{s} - g^{*'}\overline{t};$$

Similarly, we have $f^{*'}\lambda - f'_*\nu \leq f^{*'}\overline{\lambda} - f'_*\overline{\nu}$.

Therefore, for studying problem (2.6), it is sufficient to consider only the agreed strategies of the players.

Definition 11. *The pair $\beta = (\delta, \overline{\nabla})$ we call a cosituation of problem (2.4).*

Definition 12. *The pair $\{\beta, K_{op}\}$ from the cosituation β and the support K_{op}, we call the support cosituation of the problem (2.4).*

Definition 13. *The vector $(\overline{y}, \overline{\xi}, \overline{\eta})$ satisfying the relation $B\overline{y} - \overline{\xi} + \overline{\eta} = = b - A\overline{x}$ is called a pseudostrategy of the second player in problem (2.4) corresponding to the strategy $\overline{x}$ of the first player (shortly, $\overline{x}$-pseudostrategy).*

Given the support cosituation $\{\beta, K_{op}\}$, we construct the corresponding the strategy of the first player $\overline{x}$ and $\overline{x}$ is a pseudostrategy of the second player $(\overline{y}, \overline{\xi}, \overline{\eta})$ of problem (2.4):

$$\overline{x}_j = f_{*j} \;\; for \;\; \overline{\nabla}_j > 0; \overline{x}_j = f^*_j \;\; for \;\; \overline{\nabla}_j < 0;$$
$$\overline{x}_j = f_{*j} \vee f^*_j \;\; for \; \overline{\nabla}_j = 0, j \in J;$$
$$\overline{y}_k = g_{*k} \;\; for \; \delta_k < 0; \overline{y}_k = g^*_k \;\; for \;\; \delta_k > 0;$$
$$\overline{y}_k = g_{*k} \vee g^*_k \;\; for \;\; \delta_k = 0, k \in K_{n1}; \tag{3.12}$$
$$\overline{\xi}_k = 0 \;\; for \;\; k \in K_{n2}; \overline{\eta}_k = 0 \;\; for \;\; k \in K_{n3};$$

$$\begin{pmatrix} \overline{y}(K_{op1}) \\ \overline{\xi}(K_{op2}) \\ \overline{\eta}(K_{op3}) \end{pmatrix} = [\overline{B}(I, K_{op})]^{-1}[b - A\overline{x} - \overline{B}(I, K_{n1})\overline{y}(K_{n1})].$$

Let us present the conditions under which the value of the objective functions of problems (2.4) and (3.1) for the strategies of the players corresponding to the support cosituation $\{\beta, K_{op}\}$ and consistent with the cosituation β coincide.

Theorem 5. *Let us $\{\beta, K_{op}\}$ the supporting cosituation of problem (2.4), $(\mu, s, t), (\lambda, \nu)$ are players' strategies of the problem (2.6), consistent with cosituation $\beta, \overline{x}, (\overline{y}, \overline{\xi}, \overline{\eta})$ is the strategy of the first player and $\overline{x}$ - pseudostrategy of the second player of the problem (2.6), corresponding to the supporting cosituation $\{\beta, K_{op}\}$. If the relations*

$$\overline{y}_k = g_{*k} \text{ for } \delta_k < 0; \quad \overline{y}_k = g^*_k \text{ for } \delta_k > 0;$$

$$\overline{y}_k \in [g_{*k}, g^*_k] \text{ for } \delta_k = 0, k \in K_{op1};$$

$$\xi_k = 0 \quad \text{for} \quad \delta_k < 0; \quad \xi_k \geq 0 \quad \text{for} \quad \delta_k = 0, k \in K_{op2};$$

$$\eta_k = 0 \quad \text{for} \quad \delta_k < 0; \quad \eta_k \geq 0 \quad \text{for} \quad \delta_k = 0, k \in K_{op3},$$

is satisfied then the equality $F(\overline{x}) = \psi(\mu, s, t)$.

Proof. If the conditions of the theorem are satisfied, similarly to (3.7) we obtain

$$e'\overline{\xi} + e'\overline{\eta} = (b - A\overline{x})'\mu + g'_* s - {g^*}' t.$$

Therefore we have

$$F(\overline{x}) = e'\overline{\xi} + e'\overline{\eta} = b'\mu + g'_* s - {g^*}' t + \mu' A\overline{x} = b'\mu + g'_* s - {g^*}' t + {f^*}'\lambda - f'_*\nu.$$

Considering (3.11), we have

$$\psi(\mu, s, t) = b'\mu + g'_* s - {g^*}' t + {f^*}'\lambda - f'_*\nu.$$

Consequently $F(\overline{x}) = \psi(\mu, s, t)$. □

4. Algorithm for solving the problem of the first phase

Thus, according to the lemma, remark 1 and theorems 2-5, the solution of problem (2.2) for case (b) can be determined, by finding such a support K^0_{op} of problem (3.3), under which the value of the objective function of problem (3.1) are positive on the agreed strategies of the players, constructed using the support K^0_{op} according by formulas (3.9), (3.10) with $\mu' = \overline{d}'(K^0_{op})[\overline{B}(I, K^0_{op})]^{-1}$.

Indeed, first, the value of the objective function of the problem (2.3) is nonnegative, and second, the optimal values of the objective functions of problems (2.3), (2.4), and (3.1) coincide. Therefore, if the value of the objective function of the problem (3.1) is positive for some (μ, s, t), $(\mu, s, t) \in \Xi$, then the value of the objective functions of the problems (2.3), (2.4) are also positive for some $\overline{x} \in X$ and vice versa. Third, for the optimal strategy of the first player, there exists some support K_{op}^0 under which the conditions of Theorems 3 and 4 are satisfied. Fourth, the number of possible supports of problem (3.3) is, of course, no more than $q = C_{l+2m}^m = \frac{(l+2m)!}{m!(l+m)!}$. When studying problem (3.1), it is enough to consider only the agreed strategies of the players in the problem. Based on the above that it suffices to consider only the agreed strategies of the players of problem (3.1) constructed using support K_{op}^0 according by formulas (3.9), (3.10) with $\mu' = \overline{d}'(K_{op}^0)[\overline{B}(I, K_{op}^0)]^{-1}$. The first player's strategy $\overline{x}$ corresponding to the supporting cosituation $\{\beta, K_{op}^0\}$ is a solution to problem (2.2) for case (b), because

$$f(\overline{x}) = F(\overline{x}) = \max_{(\overline{\mu},\overline{s},\overline{t})\in\Xi}((b - A\overline{x})'\overline{\mu} + g_*'\overline{s} - g^{*'}\overline{t}) \geq$$

$$\geq (b - A\overline{x})'\mu + g_*'s - g^{*'}t = b'\mu + g_*'s - g^{*'}t + f^{*'}\lambda - f_*'\nu > 0.$$

Similarly, the solution of problem (2.2) for case (a) can be determined as follows: if the maximum value of the objective function of problem (3.1) is equal to zero on the agreed strategies of the players built using the support K_{op}^0 by formulas (3.9), (3.10) with $\mu' = \overline{d}'(K_{op}^0)[\overline{B}(I, K_{op}^0)]^{-1}$, then $\forall x \in X, Y(x) \neq \varnothing$.

Remark 2. From (3.9), and (3.10) it follows that the value of the objective function of problem (3.1) for $\mu' = \overline{d}'(K_{op})[\overline{B}(I, K_{op})]^{-1}, K_{op} \subset K_1$ equals zero. Note that the elements $k \in K_2, j \in K_3$ for which it is appropriate $k + m = j$ cannot be simultaneously in some support, because the corresponding matrix will be degenerate.

Let us present an algorithm for solving problem (2.2).

Denote by $K_{op}^1, K_{op}^2, ..., K_{op}^q, q = C_{l+2m}^m$, the lexicographic order [8] of possible supports of problem (3.3).

Step 1. Set $z := 1, k1 := 1, z^0 := z, K_{op}^o := K_{op}^z, f := -\infty$. Define and construct the matrix $\overline{B} = (B \vdots - E \vdots E)$.

Step 2. If all elements of the set K_{op}^z from K_1 or the environment of its elements contains elements $k, q1$ such as $k + m = q1, k \in K_2, q1 \in K_3$, then go to step 6.

Step 3. Calculate $\det\overline{B}(I, K_{op}^z)$. If $\det\overline{B}(I, K_{op}^z) = 0$, then go to step 6.

Step 4. For K_{op}^z construct the vector $\mu'(I) = \overline{d}'(K_{op})[\overline{B}(I, K_{op})]^{-1}$. If $\exists i_* \in I, |\mu_{i^*}| > 1$, then go to step 6.

Step 5. Using formulas (3.4), (3.9), (3.10), and (3.12), construct the vectors $(\mu, s, t), (\lambda, \nu), \overline{x}$, and calculate the value $F = b'\mu + g_*'s - g^{*'}t + f^{*'}\lambda - f_*'\nu$. If $F > 0$, then set $k1 := 0, z^o := z, K_{op}^o := K_{op}^z, x^o := \overline{x}, f := +\infty$ and go to step 7.

Step 6. If $z < q$, then set $z := z + 1$ go to step 2.

Step 7. The algorithm stops. If $k1 = 0$, then for x^o $Y(x^o) = \varnothing$. If, $k1 = 1$, then $\forall x \in X, Y(x) \neq \varnothing$.

Example 1. Consider problem (2.2) with the following values of the parameters:

$$m = 2, n = 3, l = 5, f_*' = (-5; -30; 0), f^{*'} = (3; 25; 40),$$

$$g_*' = (-109; -6; -101; -10; -3), g^{*'} = (44; 6; 298; 10; 15),$$

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 6 & 3 & 2 & 3 & 4 \\ 4 & 2 & 1 & 2 & 3 \end{pmatrix}, b = (5; 4).$$

We have $I = \{1, 2\}, J = \{1, 2, 3\}, K_1 = \{1, 2, 3, 4, 5\}, K_2 = \{6, 7\}, K_3 = \{8, 9\}, \overline{K} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, K_{op}^1 = \{1, 2\}, ..., K_{op}^{36} = \{8, 9\}$.

We apply the described algorithm.

Step 1. $z := 1, k1 := 1, z^0 := z, \overline{B} = \begin{pmatrix} 6 & 3 & 2 & 3 & 4 & -1 & 0 & 1 & 0 \\ 4 & 2 & 1 & 2 & 3 & 0 & -1 & 0 & 1 \end{pmatrix}$, $K_{op}^o := \{1, 2\}, f := -\infty$. Step 2. $K_{op}^1 = \{1, 2\}$. Step 6. $z := 2$. ... Step 6. $z := 5$. Step 2. $K_{op}^5 = \{1, 6\}$. Step 3. $\det\overline{B}(I, K_{op}^5) = 4$. Step 4. $\mu' = (-1; 1, 5)$. ...Step 6. $z := 36$. Step 2. Step 3. $\det\overline{B}(I, K_{op}^{36}) = 1$. Step 4. $\mu' = (1; 1)$. Step 5. $F = -1475$. Step 6. Step 7. $k1 = 1, \forall x \in X, Y(x) \neq \varnothing$.

The number of possible supports of problem (3.3) is 36, and the number of calculated determinants of support matrices $(\overline{B}(I, K_{op}^z))$ of problem (3.3) is 23. Out of 23, in 13 cases it was necessary to calculate the value of the objective function of the problem (3.1).

Example 2. Consider example 1 for $f^{*'} = (3; 25; 50)$.

We apply the described algorithm.

Step 1. $z := 1, k1 := 1, z^0 := z, \overline{B} = \begin{pmatrix} 6 & 3 & 2 & 3 & 4 & -1 & 0 & 1 & 0 \\ 4 & 2 & 1 & 2 & 3 & 0 & -1 & 0 & 1 \end{pmatrix}$, $K_{op}^o := \{1, 2\}, f := -\infty$. Step 2. $K_{op}^1 = \{1, 2\}$. Step 6. $z := 2$. Step 2. $K_{op}^2 = \{1, 3\}$. Step 6. $z := 3$. Step 2. $K_{op}^3 = \{1, 4\}$. Step 6. $z := 4$. Step 2. $K_{op}^4 = \{1, 5\}$. Step 6. $z := 5$. Step 2. $K_{op}^5 = \{1, 6\}$. Step 3. $\det\overline{B}(I, K_{op}^5) = 4$. Step 4. $\mu' = (-1; 1, 5)$. Step 6. $z := 6$. Step 2. Step 3. $\det\overline{B}(I, K_{op}^6) = -6$. Step 4. $\mu' = (2/3; -1)$. Step 5. $F = 10,667, f = +\infty$. Step 7. $k1 = 0, f = +\infty, x^0 = (-5; 25; 50), Y(x^0) = \varnothing$.

The number of possible supports of problem (3.3) is 36, and the number of calculated determinants of support matrices $(\overline{B}(I, K_{op}^z))$ of problem (3.3) is 2. Out of 2, in 1 cases it was necessary to calculate the value of the objective function of the problem (3.1).

5. Numerical experiment

The algorithm was implemented in the Simple Fortran 2.26 environment. A numerical experiment was set up on a PC (Windows 7; Intel(R) Celeron(R) CPUN2930 (1.83 GHz); 4GB RAM; system type:32-bit OS). There are two types of generated problems.

a) Elements of Problem (2.2) are generated by a random number generator. The elements of the matrices A, B were chosen from the segment [-10,10]. The coordinates vectors f_*, g_*, were chosen from segment [-10, 0], and the coordinates of vectors f^*, g^*, were chosen from the segment [0,10]. The vector b was assumed to be equal to $b = Ax^0 + By^0$. Here x^0, y^0, vectors, whose coordinates were assumed to be equal $x_j^0 = (f_{*j} + f_j^*)/2$, $j \in \{1, 2, ..., n\}, y_k^0 = (g_{*k} + g_k^*)/2, k \in \{1, 2, ..., l\}$.

b) If after generating the elements of the problem in case a) the first m components of the vectors g_*, g^* redefine as follows:

$$g_i^* = \max_{f_* \le x \le f^*, g_*(K_n) \le y(K_n) \le g^*(K_n)} h_i(b(I) - A(I,J)x(J) - B(I,K_n)y(K_n)),$$

$$g_{*i} = \min_{f_* \le x \le f^*, g_*(K_n) \le y(K_n) \le g^*(K_n)} h_i(b(I) - A(I,J)x(J) - B(I,K_n)y(K_n)),$$

$i = \overline{1, m}; h_i, i$-th row of the matrix $B_{op}^{-1}, i \in I = \{1, 2, ..., m\}, K_n = \{l - m + 1, l - m + 2, ..., l\}$ then problems are formed for which $\forall x \in X, Y(x) \neq \varnothing$.

The resulting problems were solved by the proposed algorithm, and also for comparison with the first part of the algorithm [12].

The results are shown in the table 1, where $p_1 = 9231, p_2 = 1321$, $p_{12} = 12524, p_3 = 10712, p_4 = 1914, p_{34} = 184756, p_5 = 69206565$, $p_6 = 1647237, p_7 = 440630280, p_8 = 9632608$.

The following designations are accepted: N_z determines the type of problem (2.2) generation for given $m, n, l; K_r$ is the number of possible supports of the problem (3.3), for which it was necessary to calculate the determinant of the corresponding matrices; K_z is the number of calculated values of the objective function of the problem (3.1); $k1$ is the outcoming result of the algorithm; K_{r1} is the number of possible supports of the problem (3.3), for which it was necessary to calculate the determinant of the corresponding matrices when solving the problem by the first part of the [12] algorithm (in this case, K_{r1} is also the number of calculated values of the objective function of the problem (2.6)); t is time of solving the problem (2.2) by the proposed algorithm; t_1 is time to solve the problem (2.2) according to the first part of the algorithm [12]; "-" means that problem (2.2) was not solved in 10 hours.

From the values of K_r, K_z, K_{r1}, t, t_1 in the table it follows that the proposed algorithm is more effective than the first part of the algorithm from [12].

The applicability limit of the algorithm (specifically achievable values m,n and l) depends on the generation of parameters problem (2.2). In the numerical experiment carried out, when time was no more than 10 hours, in case b) it was equal to $m = 10$, $n = 20$ and $l = 20$, in case a)it was equal to $m = 3000$, $n = 10000$ and $l = 5000$.

m	n	l	N_z	K_r	K_z	$k1$	K_{r1}	t	t_1
2	3	5	a	23	14	1	36	0:00:00.00	0:00:00.00
2	3	5	b	23	13	1	36	0:00:00.00	0:00:00.00
2	5	5	a	6	3	0	13	0:00:00.00	0:00:00.00
2	5	5	b	23	14	1	36	0:00:00.00	0:00:00.00
5	10	10	a	p_1	p_2	0	p_{12}	0:00:00.00	0:00:00.00
5	10	10	b	p_3	p_4	1	p_{34}	0:00:05.25	0:00:15.51
10	10	15	b	p_5	p_6	1	C_{35}^{10}	2:51:44.32	7:33:14.05
10	20	20	b	p_7	p_8	1	C_{40}^{10}	08:36:12.94	-
10	20	30	a	10	1	0	31	0:00:00.00	0:00:00.00
30	50	60	a	2	1	0	33	0:00:00.003	0:00:00.04
60	80	120	a	35	1	0	80	0:00:00.28	0:00:01.01
80	130	150	a	26	1	0	90	0:00:00.92	0:00:04.14
100	200	200	a	56	1	0	119	0:00:02.34	0:00:04.42
100	200	300	a	56	1	0	219	0:00:01.39	0:00:05.83
200	200	300	a	40	1	0	123	0:00:07.98	0:01:46.79
200	500	300	a	76	1	0	104	0:00:15.90	0:00:24.49
300	500	1000	a	121	1	0	729	0:05:10.91	0:31:03.15
500	800	2500	a	193	1	0	2139	1:26:34.59	4:27:03.47
600	800	3000	a	516	1	0	2917	3:31:40.75	-
1000	10000	5000	a	74	1	0	4075	0:08:21.18	-
3000	10000	5000	a	216	1	0	1215	09:55:25.28	-

Table 1. Rezults of numerical experiment

6. Conclusion

In this paper, we studied the problem of determining the nonemptiness of the set of strategies of the second player for any strategy of the first player, which arises in game problems with connected variables. A lemma similar to the lemma on the nonemptiness of the set of feasible solutions for a linear programming problem is formulated and proved. An algorithm for solving the problem under consideration has been proposed, which differs from the first part of the [12] algorithm by a significant reduction in the number of mathematical operations for solving the problem under consideration. The algorithm can be used or modified both for solving a game problem with arbitrary situations [7; 12] and for solving a game problem with forbidden situations [1; 2; 13], and also for solving weak problems of linear bilevel programming [3; 9; 10; 15; 16].

References

1. Azizov I. Algorithm for calculating the ϵ - optimal solution of a linear maximin problem with connected variables. *Vesti Academy of Sciences of the BSSR. Ser. fiz.-mat. navuk.*, 1986, no. 1, pp. 14–18. (in Russian)
2. Falk J.E. Linear maxmin problem. *Math. Program.*, 1973. vol. 5, no. 2, pp. 169–188.
3. Fedorov V.V. *Numerical methods of maximin.* Moscow, Nauka Publ., 1979, 280 p. (in Russian)
4. Gabasov R. *Optimization methods.* Minsk, Four quarters Publ., 2011, 472 p. (in Russian)
5. Gabasov R., Kirillova F.M., and Tyatyushkin A.I. *Constructive Optimization Methods, Part 1: Linear Problems.* Minsk, Universitetskoye Publ., 1984, 214 p. (in Russian)
6. Gabasov R., Kirillova F.M. *Linear Programming Methods.* Minsk, BGU Publ., 1977, part 1, 176 p.; part 3, 1980, 368 p.(in Russian)
7. Ivanilov Yu.P. Dual semi-games. *Izvestiya AN SSSR. Technical cybernetics*, 1972, no. 4, pp. 3–9. (in Russian)
8. Lipski W. *Combinatorial Analysis for Software Engineers.* Warszawa, Wydawnictwa Naukowo-Techniczne, 1987.
9. Liu June, Hong Yunfei, Zheng Yue. A New Variant of Penalty Method for Weak Linear Bilevel Programming Problems. *Wuhan University Journal of Natural Sciences*, 2018, vol. 23, no. 4, pp. 328–332. https://doi.org/10.1007/s11859-018-1330-1
10. LIU June, HONG Yunfei, ZHENG Yue. A Branch and Bound-Based Algorithm for the Weak Linear Bilevel Programming Problems. *Wuhan University Journal of Natural Sciences*, 2018, vol. 23, no. 6, pp. 480–486. https://doi.org/10.1007/s11859-018-1352-8
11. Mamatov A.R. High-Order Necessary Optimality Conditions in a Linear Maxmin Problem with Coupled Variables. *Comput. Math. Math. Phys.*, 2010, vol. 50, no. 6, pp. 1017–1022. https://doi.org/10.1134/S0965542510060047
12. Mamatov A.R. An algorithm for solving a two-person game with information transfer. *Comput. Math. Math. Phys.*, 2006, vol. 46, no. 10, pp. 1699–1704. https://doi.org/10.1134/S0965542506100071
13. Mamatov A.R. An Algorithm for Solving a Linear Max-min Problem with Coupled Variables. *Comput. Math. Math. Phys.*, 2005, vol. 45, no. 6, pp. 1006–1010.
14. Mamatov A.R. On theory of the duality linear maximin problems with connected variables. *Siberian J. Num. Math.*, 2007, vol. 10, no. 2, pp. 187–193. (in Russian)
15. Zheng Y., Wan Z., Sun K., Zhang T. An exact penalty method for weak linear bilevel programming problem. *Journal of Applied Mathematics and Computing*, 2013, no. 42, pp. 41–49. https://doi.org/10.1007/s12190-012-0620-6
16. Zheng Y., Fang D., Wan Z. A solution approach to the weak linear bilevel programming problems. *Optimization*, 2016, no. 7, pp. 1437–1449. https://doi.org/10.1080/02331934.2016.1154553

Список источников

1. Азизов И. Алгоритм вычисления ϵ - оптимального решения линейной максиминной задачи со связанными переменными // Весцi АН БССР. Сер. фiз.-мат. навук. 1986. № 1. С. 14–18.
2. Falk J. E. Linear maxmin problem // Math. Program. 1973. Vol. 5, N 2. P. 169–188.

3. Федоров В. В. Численные методы максимина. М. : Наука, 1979. 280 с.
4. Габасов Р. Методы оптимизации. Минск : Четыре четверти, 2011. 472 с.
5. Габасов Р., Кириллова Ф. М., Тятюшкин А. И. Конструктивные методы оптимизации. Ч. 1. Линейные задачи. Минск : Университетское, 1984. 214 с.
6. Габасов Р., Кириллова Ф. М. Методы линейного программирования. Минск : Изд. БГУ, 1977. Ч. 1. 176 с.; 1980. Ч. 3. 368 с.
7. Иванилов Ю. П. Двойственные полуигры // Известия АН СССР. Серия техническая кибернетика. 1972. № 4. С. 3-9.
8. Липский В. Комбинаторика для программистов. М. : Мир, 1988. 200 с.
9. Liu June, Hong Yunfei, Zheng Yue. A New Variant of Penalty Method for Weak Linear Bilevel Programming Problems // Wuhan University Journal of Natural Sciences. 2018. Vol.23, N 4. P. 328–332. https://doi.org/10.1007/s11859-018-1330-1
10. Liu June, Hong Yunfei, Zheng Yue. A Branch and Bound-Based Algorithm for the Weak Linear Bilevel Programming Problems // Wuhan University Journal of Natural Sciences. 2018. Vol. 23, N 6. P. 480–486. https://doi.org/10.1007/s11859-018-1352-8
11. Маматов А. Р. Необходимые условия оптимальности «высокого порядка» в линейной максиминной задаче со связанными переменными // Журнал вычислительной математики и математической физики. 2010. Т. 50, № 6. С. 963–968.
12. Маматов А. Р. Алгоритм решения одной игры двух лиц с передачей информации // Журнал вычислительной математики и математической физики. 2006. Т. 46, № 10. С. 1784–1789.
13. Маматов А. Р. Алгоритм решения линейной максиминной задачи со связанными переменными // Журнал вычислительной математики и математической физики. 2005. Т. 45, № 6. С. 1044–1047.
14. Маматов А. Р. К теории двойственности линейных максиминных задач со связанными переменными // Сибирский журнал вычислительной математики. 2007. Т. 10, №2. С.187–193.
15. An exact penalty method for weak linear bilevel programming problem / Y. Zheng, Z. Wan, K. Sun, T. Zhang // Journal of Applied Mathematics and Computing. 2013. N 3. P. 41–49. https://doi.org/10.1007/s12190-012-0620-6
16. Zheng Y., Fang D., Wan Z. A solution approach to the weak linear bilevel programming problems // Optimization. 2016. N 7. P. 1437–1449. https://doi.org/10.1080/02331934.2016.1154553

Об авторах

Маматов Акмал Равшанович, канд. физ.-мат. наук, ст. науч. сотр., Самаркандский государственный университет им. Ш. Рашидова, Самарканд, 140104, Узбекистан, akmm1964@rambler.ru, https://orcid.org/0000-0002-0610-7015

About the authors

Akmal R. Mamatov, Cand. Sci. (Phys.Math.), Senior Researcher, Samarkand State University named after Sh. Rashidov, Samarkand, 140104, Uzbekistan, akmm1964@rambler.ru, https://orcid.org/0000-0002-0610-7015

Поступила в редакцию / Received 23.11.2022
Поступила после рецензирования / Revised 02.02.2024
Принята к публикации / Accepted 07.02.2024