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# Soliton Solutions of the Negative Order Modified Korteweg – de Vries Equation

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**Abstract.** In this paper, we study the negative order modified Korteweg-de Vries (nmKdV) equation in the class of rapidly decreasing functions. In particular, we show that the inverse scattering transform technique can be applied to obtain the time dependence of scattering data of the operator Dirac with potential being the solution of the considered problem. We demonstrate the explicit representation of one soliton solution of nmKdV based on the obtained results.

**Keywords:** negative order modified Korteweg – de Vries equation, soliton, inverse scattering transform, scattering data, potential, reflection coefficient

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Научная статья

# Солитонообразные решения модифицированного уравнения Кортевега – де Фриза отрицательного порядка

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**Аннотация**. Исследуется модифицированное уравнение Кортевега – де Фриза (мКдФ) отрицательного порядка в классе быстроубывающих функций. В частности, показано, что с помощью метода обратной задачи рассеяния можно получить временную зависимость данных рассеяния оператора Дирака с потенциалом, являющимся решением рассматриваемой задачи. Продемонстрировано явное представление односолитонного решения мКдФ отрицательного порядка на основе полученных результатов.

**Ключевые слова:** модифицированное уравнение Кортевега – де Фриза отрицательного порядка, солитон, обратная задача теории рассеяния, данные рассеяния, потенциал, коэффициент отражения

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## 1. Introduction

The Korteweg-de Vries equation is the classical example of the nonlinear equation yielding solitary wave solutions and describing waves on shallow water surfaces. The fundamental discovery of C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura [7] presents the inverse scattering transform (IST) for solving KdV equation with rapidly decaying initial data. P. Lax [11] noted the general character of the IST method showing that KdV can be derived as a compatibility condition related to the time evolution of scattering data of the Sturm-Liouville operator. Following such advances the development of IST motivated researchers to study positive-order nonlinear evolution equations and investigate their important properties [1; 10; 15; 19–21].

Another important equation that contributed to the development of the theory of solitary waves is a modified KdV equation which was first investigated by M.Wadati [23]. Numerous works devoted to investigate the positive order mKdV equation and properties of its solutions [3; 5; 6; 9; 26].

Recently, the investigation of negative-order nonlinear equations has become a substantial field of mathematical physics. It is remarkable that the study of negative order KdV equation (nKdV) is an effective tool for investigating the theory of cuspons (cusp soliton) and peakons (peaked solitons). The nKdV equation is related to recursion operators, firstly developed by P.J. Olver [12]. He generalized a recursion formula to describe the process of creating infinitely many symmetries of evolution equations, which can be applicable for the KdV equation. Verosky applied Olver's approach in negative direction for constructing a sequence of equations of increasingly negative orders and derived the representation of the nKdV equation [22]

$$\begin{cases} u_t = \upsilon_x \\ \upsilon_{xxx} + 4u\upsilon_x + 2u_x\upsilon = 0. \end{cases}$$

There is already considerable research related with nKdV equation. Particularly, it was discussed the explicit multisoliton and multikink wave solutons of the nKdV equation using bilinear Backlund transformations [14], derived the Lax representation to show the integrability of considered nKdV equation and obtained its classical solitons, periodic soliton, and kink solutions [13], derived quasiperiodic solutions of the nKdV hierarchy by means of the backward Neumann systems [4]. Besides, it was investigated such important problems such as the initial value problem for the nKdV equation by the Riemann-Gilbert method [16]; integrating the nKdV equation with a self-consistent source in the class of periodic functions [18]; inverse spectral problem for nKdV equation with a special source [17].

Research on the negative order nonlinear equations has expanded the possibilities of investigating other models of the KdV equation. In [24], similarly to Verosky's technique it was obtained the negative order modified KdV (nmKdV) equation

$$u_x u_{xxxt} + 4u^2 u_x u_{xt} + 12uu_x^2 u_t - u_{xx}u_{xxt} - 4u^2 u_{xx} u_t = 0$$

using the relation $Ru_t = u_x$, with the recursion operator $R$ having the form

$$R = -((d^2)_x + 4u^2 + 4u_x d_x^{-1}(u)),$$

where $d_x$ denotes the total derivative with respect to $x$, and $d^{-1}$ its integration operator. Introducing the notation $(u^2)_t = \rho_{xxt}$, it was obtained the following nmKdV equation

$$\begin{cases} \rho_{xx} = u^2 \\ u_{xt} + \alpha u + 2\rho_{xt}u = 0 \end{cases}$$

and constructed the breather solutions of this equation by applying Hirota's bilinear method, where $\alpha$ is a real constant [8].

Unlike the study [8], in this paper, we investigate the following system of equations

$$\begin{cases} \rho_{xx} = -u^2 \\ u_{xt} + \alpha u + 2\rho_{xt}u = 0, \quad x \in \mathbb{R}, \quad t \geq 0 \end{cases} \tag{1.1}$$

by the inverse scattering transform method. The system (1.1) is considered under the initial condition

$$u(x, 0) = u_0(x), x \in \mathbb{R}, \tag{1.2}$$

where the initial complexvalued function has the following properties:

1) $\int_{-\infty}^{+\infty}(1+|x|)\,|u_0(x)|\,dx<\infty$,

2) The operator

$$L(0)=i\begin{pmatrix}\frac{d}{dx} & -u_0\\ u_0 & -\frac{d}{dx}\end{pmatrix}$$

possesses a finite number of eigenvalues, precisely, $2N$ simple eigenvalues

$$\xi_1(0),\xi_2(0),\dots\xi_{2N}(0)\text{ such that } Im\xi_k(0)>0,\xi_{N+k}(0)=-\xi_k(0),$$

$k=1,\dots,N$ [2] and has no spectral singularities.

The function $u=u(x,t)$ is complex valued and sufficiently smooth function of $x$ and $t$, for all $t>0$ satisfying the requirement

$$\int_{-\infty}^{+\infty}(1+|x|)\left(|u(x,t)|+|u_{xt}(x,t)|\right)dx<\infty,$$

$$\begin{cases}\rho(0,t)=0,\;\;\rho_x(x,t)\to 1, & x\to\infty\\ \rho_{xx}(x,t)\to 0,\;\;\;\rho_{xt}(x,t)\to 0, & x\to\pm\infty.\end{cases}\qquad(1.3)$$

The main goal of this study is finding solutions $u(x,t)$, $\rho(x,t)$ of the Cauchy problem (1.1)-(1.3) for the negative order mKdV equation by the inverse scattering technique for the Dirac operator

$$L(t)=i\begin{pmatrix}\frac{d}{dx} & -u\\ u & -\frac{d}{dx}\end{pmatrix}.$$

Note that in [23] it was integrated the positive order mKdV equation by IST.

## 2. Scattering problem

In this part of the work we provide some basic facts on direct and inverse scattering theory for the Dirac system of equation on the real axis

$$\begin{aligned}y_{1x}&=-i\xi y_1+u(x)y_2\\ y_{2x}&=i\xi y_2+u(x)y_1\end{aligned}\qquad(2.1)$$

The scattering theory for the system (2.1) is studied in [25], [2]. In accordance with the condition (1.3), the system of equations (2.1) has the Jost solutions $\phi(x,\xi)$ and $\psi(x,\xi)$. These solutions are unique and the following asymptotics are valid for $Im\xi=0$

$$\left.\begin{array}{l}\phi(x,\xi)\to\begin{pmatrix}1\\0\end{pmatrix}e^{-i\xi x}\\ \bar{\phi}(x,\xi)\to\begin{pmatrix}0\\-1\end{pmatrix}e^{i\xi x}\end{array}\right\},\ x\to-\infty\quad \left.\begin{array}{l}\psi(x,\xi)\to\begin{pmatrix}0\\1\end{pmatrix}e^{i\xi x}\\ \bar{\psi}(x,\xi)\to\begin{pmatrix}1\\0\end{pmatrix}e^{-i\xi x}\end{array}\right\},\ x\to+\infty \tag{2.2}$$

Note, that here and hereafter $\bar{\phi}$ is not complex conjugate to $\phi$. Since the pair of vector-functions $\{\psi,\bar{\psi}\}$ is linearly independent, it holds the relation

$$\phi(x,\xi)=a(\xi)\bar{\psi}(x,\xi)+b(\xi)\psi(x,\xi),\ \ Im\xi=0. \tag{2.3}$$

where $a(\xi)=W\{\phi,\psi\}=\phi_1\psi_2-\phi_2\psi_1$.

The Jost functions $\phi(x,\xi)$ and $\psi(x,\xi)$ admit of analytical continuation into the upper half-plane $Im\xi>0$. Therefore, the function $a(\xi)$ appearing in (2.3) analytically extends to the upper half-plane $Im\xi>0$ and for $|\xi|\to\infty$, $Im\xi\geq 0$ has the asymptotics $a(\xi)=1+\mathrm{O}\left(\frac{1}{|\xi|}\right)$. In addition, $a(\xi)$ may have a finite number of zeros $\xi=\xi_k,\ \ k=1,\dots,N$, lying in the half-plane $Im\xi>0$ and these zeros correspond to the points of the discrete spectrum of the operator $L$.

The operator $L$ can have spectral singularities lying on the continuous spectrum. But in this study, we will assume that the operator $L$ has no spectral singularities and all the eigenvalues are simple. Therefore,

$$\phi_k(x)=c_k\psi(x) \tag{2.4}$$

is valid, where $\phi_k(x)=\phi(x,\xi_k)$, $\psi_k(x)=\psi(x,\xi_k)$ and the quantities $c_k$ are independent of $x$.

The expression

$$r^+(\xi)=\frac{b(\xi)}{a(\xi)} \tag{2.5}$$

defines the reflection coefficient of the scattering problem (2.1).

**Definition 1.** *The collection $\{r^+(\xi),c_k,\xi_k,k=1,\dots,N\}$ is called the set of scattering data.*

The vector function $\psi$ can be expressed in the integral representation

$$\psi=\begin{pmatrix}0\\1\end{pmatrix}e^{i\xi x}+\int_x^\infty K(x,s)e^{i\xi s}ds, \tag{2.6}$$

where $K(x,y)=\begin{pmatrix}K_1(x,y)\\K_2(x,y)\end{pmatrix}$. The kernel $K(x,y)$ does not depend on $\xi$ and has the relation with potential $u(x)$ as follow

$$u(x)=-2K_1(x,x),\int_x^\infty|u(s)|^2\,ds=-2K_2(x,x). \tag{2.7}$$

The components $K_1(x,y),\ K_2(x,y)$ of the kernel for $y>x$ can be found by solving the Gelfand-Levitan-Marchenko system of integral equations

$$\begin{cases} K_2(x,y)+\int_x^\infty K_1(x,s)F(s+y)ds=0 \\ K_1(x,y)+F(x+y)+\int_x^\infty K_2(x,s)F(s+y)ds=0, \end{cases} \tag{2.8}$$

where

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} r(\xi)e^{i\xi x}d\xi - i\sum_{j=1}^{N} c_j e^{i\xi_j x}.$$

The direct scattering problem consists of determining the scattering data in terms of the given potential $u(x)$ of the system (2.1), and, the inverse scattering problem is reconstructing the potential $u(x)$ through the given scattering data.

## 3. Evolution of scattering data

The system (1.1) is equivalent to the following Lax equation

$$L_t+[L,B]=0, \tag{3.1}$$

where $[L,B]=LB-BL$ and

$$L(t)=i\begin{pmatrix} \frac{d}{dx} & -u \\ u & -\frac{d}{dx}, \end{pmatrix}, \tag{3.2}$$

$$B=\frac{i}{2\xi}\begin{pmatrix} -\frac{\alpha}{2}-\rho_{xt} & -u_t \\ u_t & \frac{\alpha}{2}+\rho_{xt} \end{pmatrix}. \tag{3.3}$$

Let $f_0=f_0(x,\xi)$ be any solution of the equation

$$Lf_0=\xi f_0,\ \lambda\in\mathbb{R}. \tag{3.4}$$

Then, it is easy to show, that the function

$$S_0=\dot{f}_0-Bf_0 \tag{3.5}$$

satisfies of the equation (3.4). In fact, taking the derivative with respect to $t$ in (3.4) we have

$$\dot{L}f_0+L\dot{f}_0=\xi\dot{f}_0, \quad \text{for} \quad \xi\in\mathbb{R}.$$

Using (3.1) and (3.4) we deduce that $LS_0=\xi S_0$. Proceeding in a similar manner, it can be shown that the functions

$$S_0^+=\dot{\psi}-B\psi, \quad S_0^-=\dot{\phi}-B\phi \quad \text{and} \quad \bar{S}_0^+=\dot{\bar{\psi}}-B\bar{\psi} \tag{3.6}$$

are also solutions of equation (3.4), where $\psi = \psi(x, \xi, t)$ and $\phi = \phi(x, \xi, t)$ are Jost solutions

Differentiating the integral representation (2.6) for the Jost solution $\psi(x, \xi, t)$ with respect to $t$ and taking into account that, the function $u(x, t)$ (2.7) belongs to the class of rapidly decreasing functions, we find that $\dot{\psi} \to 0$ on $x \to \infty$. Analogously, for the Jost solution $\varphi(x, \xi, t)$ we obtain $\dot{\phi} \to 0$ on $x \to -\infty$

**Remark 1.** By virtue of (2.2), (2.3), (3.3) and the condition (1.3), we have the following asymptotic relations as $x \to \infty$

$$S_0^+ \to -\frac{i}{2\xi}\begin{pmatrix} -\frac{\alpha}{2} & 0 \\ 0 & \frac{\alpha}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} \quad \bar{S}_0^+ \to -\frac{i}{\xi}\begin{pmatrix} -\frac{\alpha}{2} & 0 \\ 0 & \frac{\alpha}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x}$$

as $x \to -\infty$

$$S_0^- \to -\frac{i}{2\xi}\begin{pmatrix} -\frac{\alpha}{2} & 0 \\ 0 & \frac{\alpha}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x}, \ \ Im\xi = 0$$

and by the uniqueness of the Jost solutions we get

$$S_0^+ = -\frac{i\alpha}{4\xi}\psi, \quad \bar{S}_0^+ = \frac{i\alpha}{4\xi}\bar{\psi}, \quad S_0^- = \frac{i\alpha}{4\xi}\phi, \ Im\xi = 0.$$

**Lemma 1.** *For all $\xi \in \mathbb{R}$ the following equality holds*

$$\frac{dr^+(\xi, t)}{dt} = \frac{\alpha i}{2\xi} r^+(\xi, t), \quad Im\xi = 0.$$

*Proof.* We introduce the function

$$S = S_0^- - a(\xi)\bar{S}_0^+ - b(\xi)S_0^+. \tag{3.7}$$

Inserting (3.6) into the expression (3.7) and using (2.3), it follows that

$$S = \dot{a}(\xi)\bar{\psi} + \dot{b}(\xi)\psi. \tag{3.8}$$

Due to Remark 1, we get

$$S = \frac{\alpha i}{2\xi} b(\xi)\psi. \tag{3.9}$$

Comparing the equality (3.8) with (3.9), we obtain

$$\dot{a}(\xi) = 0, \quad \dot{b}(\xi) = \frac{\alpha i}{2\xi} b(\xi).$$

By virtue of (2.5), we arrive at the result, which was required to show. □

**Lemma 2.** *The zeros $\xi = \xi_k, \ k = 1, \ldots, N$ of the function $a(\xi)$ do not depend on $t$.*

*Proof.* Let

$$L\phi_k = \xi_k \phi_k, \tag{3.10}$$

where $\phi_k(x, \xi, t)$ is the eigenfunction corresponding to the eigenvalue $\xi_k$ normalized by condition $\int_{-\infty}^{+\infty} \phi_{k1}\phi_{k2} dx = 1$. The equality (3.10) can be rewritten in the form

$$\begin{cases} \phi'_{k1} - u\phi_{k2} = -i\xi_k\phi_{k1} \\ \phi'_{k2} - u\phi_{k1} = i\xi_k\phi_{k2} \end{cases} \tag{3.11}$$

By the simple computing from the system (3.11), we can derive the following auxiliary relations necessary for further calculations

$$(\phi_{k1}\phi_{k2})' = u(\phi_{k1}^2 + \phi_{k2}^2), \tag{3.12}$$

$$\phi_{k1}^2 - \phi_{k2}^2 = \frac{i}{2\xi_k}\left((\phi_{k1}^2) + (\phi_{k2}^2)\right)' - \frac{2i}{\xi_k} u\phi_{k1}\phi_{k2}. \tag{3.13}$$

Introducing the notation $\mu = \rho_x + \frac{\alpha}{2}t$, the system (1.1) takes the form

$$\begin{cases} \mu_x = -u^2 \\ u_{xt} + 2u\mu_t = 0. \end{cases} \tag{3.14}$$

Differentiating the system (3.11) with respect to $t$ we obtain

$$\begin{cases} \dot{\phi}'_{k1} - u_t\phi_{k2} - u\dot{\phi}_{k2} = -i\dot{\xi}_k\phi_{k1} - i\xi_k\dot{\phi}_{k1} \\ \dot{\phi}'_{k2} - u_t\phi_{k1} - u\dot{\phi}_{k1} = i\dot{\xi}_k\phi_{k2} + i\xi_n\dot{\phi}_{k2} \end{cases} \tag{3.15}$$

Integrating the result of subtraction from the first equation of (3.15) multiplied by $\phi_{k2}$ the second equation of this system multiplied by $\phi_{k1}$ yields

$$\int_{-\infty}^{+\infty} \left(\dot{\phi}'_{k1}\phi_{k2} + u\dot{\phi}_{k1}\phi_{n1} + i\xi_k\dot{\phi}_{k1}\phi_{k2}\right) dx -$$

$$- \int_{-\infty}^{+\infty} \left(\dot{\phi}'_{k2}\phi_{k1} + u\dot{\phi}_{k2}\phi_{k2} - i\xi_k\dot{\phi}_{k2}\phi_{k1}\right) dx -$$

$$- \int_{-\infty}^{+\infty} u_t(\phi_{k2}^2 - \phi_{k1}^2) dx = -2i\dot{\xi}_k \int_{-\infty}^{+\infty} (\phi_{k1}\phi_{k2})\, dx.$$

Integrating by parts the first two integrals in the last equation and using the normalization condition it follows that

$$\int_{-\infty}^{+\infty} u_t(\phi_{k2}^2 - \phi_{k1}^2) dx = 2i\dot{\xi}_k. \tag{3.16}$$

Substituting (3.13) to the (3.16) and using the first equation of (3.14), we get

$$\int_{-\infty}^{+\infty} u_t(\phi_{k2}^2 - \phi_{k1}^2) dx =$$

$$\frac{i}{2\xi_k}\int_{-\infty}^{+\infty} u_t\left((\phi_{k1}^2)+(\phi_{k2}^2)\right)' dx + \frac{i}{\xi_k}\int_{-\infty}^{+\infty} \mu_{xt}\phi_{k1}\phi_{k2}dx.$$

Integrating by parts the each integral in the right-hand side of the last expression we have

$$\int_{-\infty}^{+\infty} u_t(\phi_{k2}^2-\phi_{k1}^2)dx =$$

$$-\frac{i}{2\xi_k}\int_{-\infty}^{+\infty} u_{xt}\left((\phi_{k1}^2)+(\phi_{k2}^2)\right) dx - \frac{i}{\xi_k}\int_{-\infty}^{+\infty} \mu_t(\phi_{k1}\phi_{k2})'dx.$$

Using (3.12) and the second equation of (3.14), we deduce that $\frac{d\xi_k}{dt}=0$. Lemma is proved.

□

**Lemma 3.** *For the function $c_k(t)$ it holds the following equation*

$$\frac{dc_k(t)}{dt} = \frac{\alpha i}{2\xi_k}c_k(t), \quad k=1,\dots,N.$$

*Proof.* Similarly to the process for continuous spectrum for the discrete spectrum we construct the functions

$$S_k^- = \dot{\phi}_k - B\phi_k, \quad S_k^+ = \dot{\psi}_k - B\psi_k \tag{3.17}$$

and through these, we introduce the following function

$$S_k = S_k^- - c_k S_k^+. \tag{3.18}$$

Differentiating the equality (2.4) by $t$ and in accordance with the Lemma 2 we have

$$\frac{d\phi_k}{dt} = \frac{dc_k}{dt}\psi_k + c_k\frac{d\psi_k}{dt}. \tag{3.19}$$

Using (3.19) and (3.17), the expression (3.18) turns into

$$S_k = \frac{dc_k}{dt}\psi_k. \tag{3.20}$$

Due to the analycity of Jost functions and Remark 1, we have $S_k^- = \frac{\alpha i}{4\xi_k}\phi_k$ and $S_k^+ = -\frac{\alpha i}{4\xi_k}\psi_k$. Therefore, according to (2.4) we find that

$$S_k = \frac{\alpha i}{2\xi_k}\psi_k c_k. \tag{3.21}$$

Comparing the expression (3.20) with (3.21), we find the evolution on $t$ of the norming constant. □

Thus we have proved the following theorem which is the main result of this study.

**Theorem 1.** *If the functions $u(x,t)$, $\rho(x,t)$ are solutions to the problem* (1.1) – (1.3)*, then the scattering data of the system* (2.1) *evolve in time according to the following differential equations,*

$$\frac{dr^{+}(\xi,t)}{dt}=\frac{\alpha i}{2\xi}r^{+}(\xi,t), Im\xi=0,$$

$$\frac{d\xi_k}{dt}=0,\quad k=1,2,...,N,$$

$$\frac{dc_k(t)}{dt}=\frac{\alpha i}{2\xi_k}c_k(t).$$

The above relations completely determine the evolution of scattering data for the operator $L(t)$ which allows using the inverse scattering transform method to solve the Cauchy problem (1.1)–(1.3).

## 4. Construction of a soliton solution for the nmKdV equation

In this section, we apply the result of the Theorem for constructing a one-soliton solution of the system (1.1). For this purpose, we assume that the system is considered under the initial condition

$$u_0(x)=-\frac{i}{ch2x}. \tag{4.1}$$

Solving the direct problem for the operator $L(0)$ with potential (3.19) for $N=1$ we have

$$r(x,0)=0,\quad \xi_1(0)=i,\quad c_1(0)=2. \tag{4.2}$$

Applying the Theorem1 for the case (3.20), we obtain the following values of scattering data

$$r(x,t)=0, \xi_1(t)=\xi_1(0)=i, c_1(t)=2e^{\frac{\alpha}{2}t}, \tag{4.3}$$

Solving the Gelfand-Levitan-Marchenko system of equations (2.8) with the obtained results yields $K_1(x,y)=\frac{2ie^{-x-y+\frac{\alpha}{2}t}}{1+e^{-4x+\alpha t}}$. Then by the formula (2.7) we recover the potential

$$u(x,t)=-2i\sec h\left(2x-\frac{\alpha}{2}t\right).$$

Consequently, due to the conditions (1.3) we find

$$\rho(x,t)=\ln ch\left(2x-\frac{\alpha}{2}t\right)-x-\ln ch\left(\frac{\alpha}{2}t\right).$$

Figure 1 shows the soliton solutions of the negative order modified Korteweg – de Vries equation.

## 5. Conclusion

In this paper, we have derived the time dependence of scattering data of the Dirac operator. The obtained results specify completely the evolution of the scattering data of the operator $L(t)$ which allows applying the IST method to find the solution to the problem (1.1) - (1.3).

Table 1

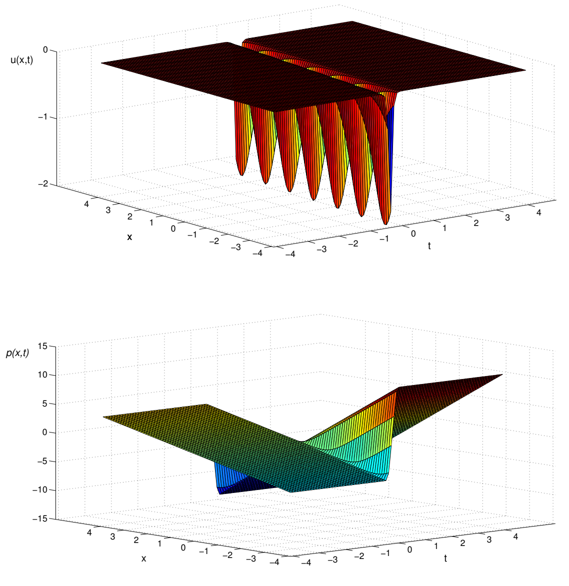


Figure 1. The solution of the negative order modified Korteweg – de Vries equation corresponding to the parameter $\alpha = 50$.

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