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Counting Lattice Paths by Using Difference Equations with Non-constant Coefficients

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Abstract. The lattice paths can be counted by the virtue of their step vectors that are aligned to the positive octant. A path can go from one point to an infinite others if there is no restriction applied such that each point only has finitely many predecessors. The linear difference equations with non-constant coefficients will be utilised to incorporate this restriction to study lattice paths that lie on or over a line having a rational slope. The generating functions are obtained and is based on developing a specific method to compute the number of restricted lattice paths.

Keywords: generating function, difference equation, functional equation, lattice path

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Научная статья

Подсчет путей решетки с использованием разностных уравнений с непостоянными коэффициентами

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Аннотация. Траектории решетки могут быть подсчитаны благодаря их шаговым векторам, которые выровнены по положительному октанту. Путь может проходить от одной точки к бесконечному количеству других, если не применяется ограничение, так что каждая точка имеет только конечное число предшественников. Линейные разностные уравнения с непостоянными коэффициентами будут использоваться для учета этого ограничения при изучении траекторий решетки, которые лежат на линии с рациональным наклоном или над ней. Получены генерирующие функции, основанные на разработке конкретного метода вычисления числа ограниченных путей решетки.

Ключевые слова: производящая функция, разностное уравнение, функциональное уравнение, решеточный путь

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1. Introduction

The classical problem in combinatorics is enumeration of lattice paths, which is still considered to be an active field of research. Its interest lies in the fact, that despite being easily understood construction of lattice paths, most of their properties remain unproven. The constant coefficient difference equations in conjunction with the generating functions is an effective tool and also a wide variety of two-dimensional sequences that can lead to generating functions is well-known in the enumerative combinatorial analysis (see [1], [2], [14]), which can be utilised as a standard tool to derive functional equations. Finally, we give a characterization of the nature of generating functions into rational and algebraic functions, which proved to be useful. The derived theory is combined with classical examples of general lattice path counting problems.

Because of the absence of geometric obstacles, the one-dimensional case is studied well (see [7]). In [13], A. Moivre considered the power series

$$f(0) + f(1)w + \ldots + f(k)w^k + \ldots$$

with the coeffcients $f(0), f(1), \ldots$ satisfying the difference equation

$$c_m f(x+m) + c_{m-1} f(x+m-1) + \ldots + c_0 f(x) = 0,\ x = 0, 1, 2 \ldots, \quad (1.1)$$

where $c_m \neq 0$, and $c_j \in \mathbb{C}$ are constants. He proved that this series always represents a rational function (De Moivre's Theorem, [13]).

For example, along with lattice paths problem, one can assume with Bloom's strings, a number of placement of the pieces on the chessboard etc. (see [3]).

2. Known Results and Formulation of the Problem

For $w = (w_1, \dots, w_N)$ where $w_i, i = 1, \dots, N$ are indeterminates, we let $\mathbb{C}[w], \mathbb{C}(w), \mathbb{C}[[w]]$ denote the ring of polynomials, the field of rational functions, and the ring of formal power series in $w_1, \dots, w_N$, where $w^x = w_1^{x_1}, \dots, w_N^{x_N}$. If f is a function on $\mathbb{Z}^N_{\geqslant}$ we will identify it as a function on $\mathbb{Z}^N$ by setting it equal to zero on the complement $\mathbb{Z}^N \backslash \mathbb{Z}^N_{\geqslant}$.

Let $\ell \geqslant 0$. Denote by $\mathfrak{C}$ *a lattice path with length* ℓ, that is a finite sequence $p(0), p(1), \dots, p(L)$ of *points* in $\mathbb{Z}^N$, and its *steps* are the set of lattice vectors $\{0\} \cup \{p(k) - p(k-1) : k = 1, \dots, \ell\}$. Specific *classes* of lattice paths arise by placing conditions on the paths including: the steps are in a specified $S \subset \mathbb{Z}^N$, the points are in a specified $P \subset \mathbb{Z}^N$, fixing the length L, and requiring that the points are distinct (this describes non intersecting paths).

For a class of paths where the points belong to a specified subset $P \subset \mathbb{Z}^N$, we can compute f for the *Restricted lattice class problems.* Clearly $0 \in P$, otherwise there are no paths in P which start from 0. The counting function f has support in P, $f(0) = 1$, and f satisfies the linear homogeneous difference equation when the possible set of steps $S \subset \mathbb{Z}^N$.

$$f(x) = \sum_{y \in S} \chi_P(x)\, f(x - y), \tag{2.1}$$

where χ_P is the *characteristic function* of P. We recall that $\chi_P(x) = 1$ if $x \in P$ and $\chi_P(x) = 0$ if $x \notin P$.

This paper computes f for some general lattice paths. For some general classes $S \subset \mathbb{Z}^N_{\geqslant}$ so f is supported on $\mathbb{R}^N_{\geqslant}$ and therefore f is uniquely specified by its *generating function* $F(w) \in \mathbb{C}[[w]]$.

$$F(w) = \sum_{x \in \mathbb{Z}^N_{\geqslant}} f(x)\, w^x, \tag{2.2}$$

Let $x, m, \gamma \in \mathbb{Z}^N_{\geqslant}$, the characteristic polynomial $P : \mathbb{C}^N \to \mathbb{C}, P(w) = \sum\limits_{0 \leqslant \gamma \leqslant m} c_\gamma w^\gamma$, where $\gamma = (\gamma_1, \dots, \gamma_N)$, c_γ represent constant coefficients, and its characteristic variety

$$V = \{w \in \mathbb{C}^N : P(w) = 0\}. \tag{2.3}$$

Definition 1. Throughout the paper, we assume that the set $\mathfrak{C}$ lies in the positive octant $\mathbb{Z}^N_{\geqslant} = \{(x_1, \dots, x_N) : x_i \in \mathbb{Z}, x_i \geqslant 0, i = 1, \dots, N\}$ of the integer lattice and the inequality $0 \leqslant \gamma \leqslant m$ means that $0 \leqslant \gamma_j \leqslant m_j$ satisfies the condition

$$m = (m_1, \ldots, m_N), m_j = \max\{\gamma_j : j = 1, \ldots, N\} \in \mathfrak{C} : c_m \neq 0 \text{ and } \gamma \in \mathfrak{C}. \tag{2.4}$$

We denote $F_\gamma(w) = \sum\limits_{x \geqslant \gamma} f(x) w^x$, $\Phi_\gamma(w) = \sum\limits_{x \ngeqslant \gamma} \psi(x) w^x$, and $F_\gamma(w) = F(w) - \Phi_\gamma(w)$, where the inequality $x \ngeqslant \gamma$ means, that for at least one $j_0 \in \{1, \ldots, N\}$ the inequality $x_{j_0} < \gamma_{j_0}$ holds.

Let δ_j be a shift operator over j^{th} variable : $\delta_j f(x) = f(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_N)$, then $\delta^\gamma = \delta_1^{\gamma_1} \circ \cdots \circ \delta_N^{\gamma_N}$ and $P(\delta) = \sum\limits_{0 \leqslant \gamma \leqslant m} c_\gamma \delta^\gamma$ be a polynomial difference operator with constant coefficients and $P(\delta) \in \mathbb{C}[w]$. Every delta operator is shift-invariant; it commutes with all shift operators δ^γ where $\delta^\gamma f(x) = f(x + \gamma)$ for every $\gamma \in \mathbb{Z}^N_{\geqslant}$.

Theorem 1. *[5] The generating function $F(w) \in \mathbb{C}[[w]]$ then the identity is represented as*

$$P(w)F(w) - \sum_{0 \leqslant \gamma \leqslant m} c_\gamma w^\gamma \Phi_{m-\gamma}(w) = \sum_{x \geqslant m} P(\delta^{-I}) f(x) w^x \tag{2.5}$$

holds, where $I = (1, \ldots, 1)$.

The identity (2.5) implies that for any function of initial data $\psi(x)$, $x \ngeqslant m$, $x \geqslant 0$ and any function $h(x)$, $x \geqslant m$, the equation $P(\delta^{-I}) f(x) = h(x)$ has a unique solution $f(x)$ satisfying initial data:

$$f(x) = \psi(x),\ x \geqslant 0,\ x \ngeqslant m \text{ (see [4], [10]).} \tag{2.6}$$

If $H(w) = \sum\limits_{x \geqslant m} h(x) w^x$, then identity (2.5) gives

$$P(w)F(w) = \sum_{0 \leqslant \gamma \leqslant m} c_\gamma w^\gamma \Phi_{m-\gamma}(w) + H(w).$$

For a two-dimensional case from the identity (2.5), we denote $F(w_1, w_2) = \sum\limits_{(x_1,x_2) \geqslant (0,0)} f(x_1, x_2) w_1^{x_1} w_2^{x_2}$ is a bivariate generating function, then its corresponding diagonal generating function, $\operatorname{diag}(F) = \sum\limits_{k \geqslant 0} f(k,k) w_1^k w_2^k$, is *algebraic* and $F_{u,v}(w_1, w_2) = \sum\limits_{k=1}^{\infty} f(uk, vk) w_1^{uk} w_2^{vk}$, where $(u, v) \in \mathbb{Z}^2_{\geqslant}$.

Let the function $f(x, y) = \psi(x, y)$, $(x, y) \ngeqslant (u, v)$, $(x, y) \geqslant 0$ satisfy the difference equation

$$P(\delta_1^{-1}, \delta_2^{-1}) f(x, y) = h(x, y),$$

where $h(x,y) = \begin{cases} f(x,y), & \text{if } x = uk,\ y = vk,\ k \geqslant 1 \\ 0, & otherwise \end{cases}$, then

$$F_{u,v}(w_1,w_2) = \sum_{(x,y)\geqslant(u,v)} h(x,y) w_1^x w_2^y$$

and identity (2.5) is

$$F_{u,v}(w_1,w_2) = P(w_1,w_2)F(w_1,w_2) - \sum_{\substack{0\leqslant\gamma_1\leqslant u\\ 0\leqslant\gamma_2\leqslant v}} c_{\gamma_1,\gamma_2} w_1^{\gamma_1} w_2^{\gamma_2} \Phi_{u-\gamma_1,v-\gamma_2}(w_1,w_2).$$

If $w_1 = w_1(t),\ w_2 = w_2(t)$ is a solution to a system $\begin{cases} w_1^u w_2^v = t \\ P(w_1,w_2) = 0 \end{cases}$,
then function $F_{u,v}$ satisfies the formulae

$$F_{u,v}(w_1(t),w_2(t)) = -\sum_{\substack{0\leqslant\gamma_1\leqslant u\\ 0\leqslant\gamma_2\leqslant v}} c_{\gamma_1,\gamma_2} w_1^{\gamma_1}(t) w_2^{\gamma_2}(t) \Phi_{u-\gamma_1,v-\gamma_2}\big(w_1(t),w_2(t)\big).$$

Lattice paths problem has been solved in many works (see [4], [6], [8], [9], [11], [12], [14]). Restricted lattice paths problem has already been solved for the selected classes lattice paths (see [5]).

In the present paper, we will extend those results for some general cases for the Restricted lattice paths problems. In section 2.1, we will give an expression for $f(x,y)$ from which we derive its generating function. By utilising the results in section 2.1, we can assume that the generating functions are expressed in terms of its generating series, the coefficients of the series represents some general paths.

In order to count some general lattice paths, we can consider set of two general steps $\mathfrak{C} = \{(q,p),(p,q)\}$ where $q \in \mathbb{Z}_{\geqslant}, p \in \mathbb{Z}_+$, and $0 \leqslant q < p$, which start from the origin and lie on or over the diagonal $y = x$. We define the number of paths $f(x,y)$ going from $(0,0)$ to (x,y), which satisfies the difference equation

$$f(x,y) - \left(1 - \sum_{d=1}^{p-q} \delta_0(x-y-d)\right) f(x-p,y-q) - f(x-q,y-p) = 0, \quad (2.7)$$

where $\delta_0(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$ is a Kronecker symbol with initial data (from (2.6)):

$$f(x,y) = \psi(x,y), (x,y) \geqslant (0,0), (x,y) \ngeqslant (p,p). \quad (2.8)$$

Let $F_{11}(t)$ be a diagonal power series of $F(w_1,w_2)$:

$$F_{11}(t) = \sum_{k=1}^{\infty} f(k,k) t^k.$$

Corollary 1. *Let* $P(w) = \sum\limits_{0\leqslant\gamma\leqslant m} c_\gamma w^\gamma$ *be a polynomial difference operator with constant coefficients and* $P(w) \in \mathbb{C}[w]$ *then*

$$w_1^{p-q} = \frac{1+\sqrt{1-4t^{p+q}}}{2t^q}.$$

Proof. We can write the polynomial $P(w_1, w_2) = 1 - w_1^q w_2^p - w_1^p w_2^q$ with initial data (2.8) from the polynomial difference operator.

Let $P(w_1, w_2) = 0$, we see that w_1^{p-q} is the root of the following quadratic equation

$$t^q (w_1^{p-q})^2 - w_1^{p-q} + t^p = 0, \tag{2.9}$$

which satisfies $w_1 w_2 = t,\ c_{00} = 1, c_{qp} = c_{pq} = -1$ and we reduce (2.9) to the standard quadratic form to get the root w_1^{p-q}. □

Lemma 1. *The generating function* $H(w_1, w_2)$ *of a solution* $h(x, y)$ *to the difference equation (2.7) with initial data (2.8) is represented as*

$$\sum_{\substack{x\geqslant m_1\\ y\geqslant m_2}} h(x,y) w_1^x w_2^y =$$

$$= \sum_{\substack{x\geqslant m_1\\ y\geqslant m_2}} P(\delta_1^{-I}, \delta_2^{-I}) f(x,y) w_1^x w_2^y = -w_1^p w_2^q \sum_{k\geqslant p-q} f(k,k)(w_1 w_2)^k.$$

Proof. From Theorem 1 we can write the generating function

$$\sum_{\substack{x\geqslant m_1\\ y\geqslant m_2}} P(\delta_1^{-I}, \delta_2^{-I}) f(x,y) w_1^x w_2^y = \sum_{\substack{x\geqslant m_1\\ y\geqslant m_2}} \sum_{\substack{0\leqslant\gamma_1\leqslant m_1\\ 0\leqslant\gamma_2\leqslant m_2}} (c_{\gamma_1,\gamma_2} \delta_1^{-\gamma_1} \delta_2^{-\gamma_2}) f(x,y) w_1^x w_2^y,$$

by taking the initial data (2.8), and from (2.7) we can write

$$\sum_{\substack{x\geqslant p\\ y\geqslant p}} (1 - \delta_1^{-q}\delta_2^{-p} - \delta_1^{-p}\delta_2^{-q}) f(x,y) w_1^x w_2^y =$$

$$= -\sum_{\substack{x\geqslant p\\ y\geqslant p}} (\delta_0(x-y-1) + \ldots + \delta_0(x-y-(p-q))) f(x-p, y-q) w_1^x w_2^y,$$

by applying the Kronecker data $(\delta_0(x))$ notation we get

$$\sum_{\substack{x\geqslant p\\ y\geqslant p}} (1 - \delta_1^{-q}\delta_2^{-p} - \delta_1^{-p}\delta_2^{-q}) f(x,y) w_1^x w_2^y = -w_1^p w_2^q \sum_{k\geqslant p-q} f(k,k)(w_1 w_2)^k. \tag{2.10}$$

□

2.1. From the Difference equation to a Functional equation

Proposition 1. *Let $F(w_1, w_2)$ be the generating function of the solution of (2.7). Then the series $F(w_1, w_2)$ satisfy the following functional equation*

$$(1 - w_1^q w_2^p - w_1^p w_2^q)F(w_1, w_2) - \Phi_{p,p}(w_1, w_2) + w_1^q w_2^p \Phi_{p-q,0}(w_1, z_2) + \\ + w_1^p w_2^q \Phi_{0,p-q}(w_1, w_2) = -w_1^p w_2^q \sum_{k \geqslant p-q} f(k,k)(w_1 w_2)^k. \quad (2.11)$$

Proof. For $N = 2$ and $P(w_1, w_2) = 1 - w_1^q w_2^p - w_1^p w_2^q$, $m = (p, p)$ where $p \in \mathbb{Z}_+$ satisfies (2.4).

Then by Theorem 1, we get

$$(1 - w_1^q w_2^p - w_1^p w_2^q)F(w_1, w_2) - c_{00}\Phi_{p,p}(w_1, w_2) - c_{qp} w_1^q w_2^p \Phi_{p-q,0}(w_1, w_2) - \\ - c_{pq} w_1^p w_2^q \Phi_{0,p-q}(w_1, w_2) = \sum_{\substack{x \geqslant p \\ y \geqslant p}} (1 - \delta_1^{-q}\delta_2^{-p} - \delta_1^{-p}\delta_2^{-q}) f(x, y) w_1^x w_2^y.$$

Using (2.10) from Lemma 1, we get (2.11) where

$$\Phi_{p,p}(w_1, w_2) = \Phi_{p-1,p-1}(w_1, w_2) + \tilde{\Phi}_{0,p-1}(w_1, w_2) + \\ + \tilde{\Phi}_{p-1,0}(w_1, w_2) - f(p-1, p-1)(w_1 w_2)^{p-1},$$

$$\Phi_{0,p-q}(w_1, w_2) = \sum_{i=0}^{p-q-1} \tilde{\Phi}_{0,i}(w_1, w_2) + \sum_{j=0}^{p-q-2} \sum_{n=j}^{p-q-2} f(j, n+1) w_1^j w_2^{n+1},$$

$$\Phi_{p-q,0}(w_1, w_2) = \sum_{i=0}^{p-q-1} \tilde{\Phi}_{i,0}(w_1, w_2) + \sum_{j=0}^{p-q-2} \sum_{n=j}^{p-q-2} f(n+1, j) w_1^{n+1} w_2^j,$$

and $\Phi_{p-1,p-1}(w_1, w_2) = \sum_{i=0}^{p-2} \tilde{\Phi}_{i,0}(w_1, w_2)$, $\tilde{\Phi}_{0,p-1}(w_1, w_2) = \sum_{x=p-1}^{\infty} f(x, p-1) w_1^x w_2^{p-1}$, $\tilde{\Phi}_{p-1,0}(w_1, w_2) = \sum_{y=p-1}^{\infty} f(p-1, y) w_1^{p-1} w_2^y$.

Let $P(w_1, w_2) = 1 - w_1^q w_2^p - w_1^p w_2^q = 0$, as a result we obtain

$$\Phi_{p-1,p-1}(w_1, w_2) + \tilde{\Phi}_{0,p-1}(w_1, w_2) + \\ + \tilde{\Phi}_{p-1,0}(w_1, w_2) - f(p-1, p-1)(w_1 w_2)^{p-1} - \\ - w_1^q w_2^p \left(\sum_{j=0}^{p-q-2} \sum_{n=j}^{p-q-2} f(n+1, j) w_1^{n+1} w_2^j + \sum_{i=0}^{p-q-1} \tilde{\Phi}_{i,0}(w_1, w_2) \right) - \\ - w_1^p w_2^q \left(\sum_{j=0}^{p-q-2} \sum_{n=j}^{p-q-2} f(j, n+1) w_1^j w_2^{n+1} + \sum_{i=0}^{p-q-1} \tilde{\Phi}_{0,i}(w_1, w_2) \right) = \\ = w_1^p w_2^q \sum_{k \geqslant p-q} f(k,k)(w_1 w_2)^k.$$

Since $f(x,y)=0$ below the diagonal, we get

$$\frac{1}{w_1^p w_2^q}\left(\sum_{i=0}^{p-1}\tilde{\Phi}_{i,0}(w_1,w_2) - w_1^q w_2^p\left(\sum_{i=0}^{p-q-1}\tilde{\Phi}_{i,0}(w_1,w_2)\right) - \right.$$
$$\left. - w_1^p w_2^q\left(\sum_{j=0}^{p-q-1}\sum_{n=j}^{p-q-1} f(j,n)w_1^j w_2^n\right)\right) = \sum_{k\geqslant p-q} f(k,k)(w_1w_2)^k. \quad (2.12)$$

For $q=1$ and $p=2$ we have the initial data:

$$f(x,0)=0,\quad f(x,1)=0,\quad x=1,2,3,\dots,\quad f(0,y)=0,$$
$$f(1,y+2)=0,\quad y=1,2,3,\dots,\quad f(y,2y)=1,\quad y=0,1,$$

while the generating function for the initial data equals
$\tilde{\Phi}_{0,0}(w_1,w_2) = \sum\limits_{y=0}^{\infty} f(0,y)w_2^y = 1$, $\tilde{\Phi}_{1,0}(w_1,w_2) = \sum\limits_{y=1}^{\infty} f(1,y)w_1w_2^y = w_1w_2^2$.

Using (2.12) implies

$$-1+\frac{1}{w_1^2w_2} = \sum_{k\geqslant 1} f(k,k)(w_1w_2)^k.$$

The substitution $t=w_1w_2$ implies $w_1 = \dfrac{1+\sqrt{1-4t^3}}{2t}$ (from Corollary 1), we get

$$F_{11}(t) = \sum_{k=1}^{\infty} f(k,k)(t)^k = -1+\frac{1-\sqrt{1-4t^3}}{2t^3} = t^3+2t^6+5t^9+14t^{12}+\cdots. \quad (2.13)$$

□

The coefficients of the series (2.13) represent with

$$f(k,k) = \begin{cases} \frac{3}{k+3}\binom{\frac{2k}{3}}{\frac{k}{3}}, & \text{if } \frac{k}{3}\in\mathbb{Z}_+; (\textit{Catalan numbers}) \\ 0, & \textit{otherwise}, \end{cases} \quad \text{for } k\geqslant 1.$$

In order to count some general lattice paths, we can consider set of three general steps $\mathfrak{C}=\{(q,p),(p,q),(r,r)\}$ where $q\in\mathbb{Z}_{\geqslant}, p,r\in\mathbb{Z}_+$, and $0\leqslant q<p<r$, which start from the origin and lie on or over the diagonal $y=x$. We define the number of paths $f(x,y)$ going from $(0,0)$ to (x,y), which satisfies the difference equation

$$f(x,y) - \left(1-\sum_{d=1}^{p-q}\delta_0(x-y-d)\right) f(x-p,y-q) - f(x-q,y-p) - f(x-r,y-r) = 0, \quad (2.14)$$

with initial data (from (2.6)):

$$f(x, y) = \psi(x, y), (x, y) \geqslant (0, 0), (x, y) \not\geqslant (r, r). \tag{2.15}$$

Corollary 2. *Let $P(w) = \sum\limits_{0 \leqslant \gamma \leqslant m} c_\gamma w^\gamma$ be a polynomial difference operator with constant coefficients and $P(w) \in \mathbb{C}[w]$ then*

$$w_1^{p-q} = \frac{1 - t^r + \sqrt{1 - 2t^r + t^{2r} - 4t^{p+q}}}{2t^q}.$$

Proof. We can write the polynomial $P(w_1, w_2) = 1 - w_1^q w_2^p - w_1^p w_2^q - w_1^r w_2^r$ with initial data (2.15) from the polynomial difference operator.

Let $P(w_1, w_2) = 0$, we see that w_1^{p-q} is the root of the following quadratic equation

$$t^q (w_1^{p-q})^2 - (1 - t^r) w_1^{p-q} + t^p = 0, \tag{2.16}$$

which satisfies $w_1 w_2 = t,\ c_{00} = 1, c_{qp} = c_{pq} = c_{rr} = -1$ and we reduce (2.16) to the standard quadratic form to get the root w_1^{p-q}. □

Lemma 2. *The generating function $H(w_1, w_2)$ of a solution $h(x, y)$ to the difference equation (2.14) with initial data (2.15) is represented as*

$$\sum_{\substack{x \geqslant m_1 \\ y \geqslant m_2}} h(x, y) w_1^x w_2^y =$$

$$= \sum_{\substack{x \geqslant m_1 \\ y \geqslant m_2}} P(\delta_1^{-I}, \delta_2^{-I}) f(x, y) w_1^x w_2^y = -w_1^p w_2^q \sum_{k \geqslant r-q} f(k, k)(w_1 w_2)^k.$$

Proof. As in the proof of Lemma 1 we find the generating function

$$\sum_{\substack{x \geqslant r \\ y \geqslant r}} (1 - \delta_1^{-q} \delta_2^{-p} - \delta_1^{-p} \delta_2^{-q} - \delta_1^{-r} \delta_2^{-r}) f(x, y) w_1^x w_2^y =$$

$$= -w_1^p w_2^q \sum_{k \geqslant r-q} f(k, k)(w_1 w_2)^k. \tag{2.17}$$

□

Proposition 2. *Let $F(w_1, w_2)$ be the generating function of the solution of (2.14). Then the series $F(w_1, w_2)$ satisfy the following functional equation*

$$(1 - w_1^q w_2^p - w_1^p w_2^q - w_1^r w_2^r) F(w_1, w_2) - \Phi_{r,r}(w_1, w_2) + w_1^q w_2^p \Phi_{r-q,r-p}(w_1, w_2) +$$
$$+ w_1^p w_2^q \Phi_{r-p,r-q}(w_1, w_2) = -w_1^p w_2^q \sum_{k \geqslant r-q} f(k, k)(w_1 w_2)^k. \tag{2.18}$$

Proof. $P(w_1,w_2) = 1 - w_1^q w_2^p - w_1^p w_2^q - w_1^r w_2^r$, $m = (r,r)$ where $r \in \mathbb{Z}_+$ satisfies (2.4).

Using Theorem 1 and (2.17) from Lemma 2, we get (2.18)

where $\Phi_{r-q,r-p}(w_1,w_2) = \Phi_{r-p,r-p}(w_1,w_2) + \sum\limits_{j=0}^{p-q-2}\sum\limits_{n=r-p+j}^{r-q-2} f(n+1,r-p+j)w_1^{n+1}w_2^{r-p+j} + \sum\limits_{i=r-p}^{r-q-1}\tilde{\Phi}_{i,0}(w_1,w_2)$,

$\Phi_{r-p,r-q}(w_1,w_2) = \Phi_{r-p,r-p}(w_1,w_2) + \sum\limits_{j=0}^{p-q-2}\sum\limits_{n=r-p+j}^{r-q-2} f(r-p+j,n+1)w_1^{r-p+j}w_2^{n+1} + \sum\limits_{i=r-p}^{r-q-1}\tilde{\Phi}_{0,i}(w_1,w_2)$.

Let $P(w_1,w_2) = 1 - w_1^q w_2^p - w_1^p w_2^q - w_1^r w_2^r = 0$, as a result we obtain

$$\Phi_{r-1,r-1}(w_1,w_2)+\tilde{\Phi}_{0,r-1}(w_1,w_2)+\tilde{\Phi}_{r-1,0}(w_1,w_2)-f(r-1,r-1)(w_1w_2)^{r-1}-$$
$$-w_1^q w_2^p\Bigg(\Phi_{r-p,r-p}(w_1,w_2)+\sum_{j=0}^{p-q-2}\sum_{n=r-p+j}^{r-q-2} f(n+1,r-p+j)w_1^{n+1}w_2^{r-p+j}+$$
$$+\sum_{i=r-p}^{r-q-1}\tilde{\Phi}_{i,0}(w_1,w_2)\Bigg)-w_1^p w_2^q\Bigg(\Phi_{r-p,r-p}(w_1,w_2)+$$
$$+\sum_{j=0}^{p-q-2}\sum_{n=r-p+j}^{r-q-2} f(r-p+j,n+1)w_1^{r-p+j}w_2^{n+1}+\sum_{i=r-p}^{r-q-1}\tilde{\Phi}_{0,i}(w_1,w_2)\Bigg)=$$
$$=w_1^p w_2^q\sum_{k\geqslant r-q} f(k,k)(w_1w_2)^k.$$

Since $f(x,y)=0$ below the diagonal, we get

$$\frac{1}{w_1^p w_2^q}\Bigg(\sum_{i=0}^{r-1}\tilde{\Phi}_{i,0}(w_1,w_2)-w_1^p w_2^q\Bigg(\sum_{i=0}^{r-p-1}\tilde{\Phi}_{i,0}(w_1,w_2)+$$
$$+\sum_{j=0}^{p-q-1}\sum_{n=r-p+j}^{r-q-1} f(r-p+j,n)w_1^{r-p+j}w_2^{n}\Bigg)-w_1^q w_2^p\Bigg(\sum_{i=0}^{r-q-1}\tilde{\Phi}_{i,0}(w_1,w_2)\Bigg)\Bigg)=$$
$$=\sum_{k\geqslant r-q} f(k,k)(w_1w_2)^k. \quad (2.19)$$

For $q=1, p=2$, and $r=3$ we have the initial data:

$$f(x,0)=0,\quad f(x,1)=0,\quad f(x+1,2)=0,\quad x=1,2,3,\ldots,$$
$$f(0,y)=0,\quad f(1,y+2)=0,\quad f(2,y+4)=0,\quad y=1,2,3,\ldots,$$
$$f(y,2y)=1,\quad y=0,1,2,\quad f(y+1,y+2)=y-1,\quad y=1,$$

while the generating function for the initial data equals
$\tilde{\Phi}_{0,0}(w_1,w_2) = \sum_{y=0}^{\infty} f(0,y)w_2^y = 1$, $\tilde{\Phi}_{1,0}(w_1,w_2) = \sum_{y=1}^{\infty} f(1,y)w_1w_2^y = w_1w_2^2$, $\tilde{\Phi}_{2,0}(w_1,w_2) = \sum_{y=2}^{\infty} f(2,y)w_1^2w_2^y = w_1^2w_2^4$.

Using (2.19) implies

$$-1 + \frac{1}{w_1^2w_2} = \sum_{k\geqslant 2} f(k,k)(w_1w_2)^k.$$

The substitution $t = w_1w_2$ implies $w_1 = \dfrac{1-t^3+\sqrt{1-6t^3+t^6}}{2t}$ (from Corollary 3), we get

$$\sum_{k=2}^{\infty} f(k,k)(t)^k = -1+\frac{1-t^3-\sqrt{1-6t^3+t^6}}{2t^3} = 2t^3+6t^6+22t^9+90t^{12}+\cdots. \tag{2.20}$$

□

The coefficients of the series (2.20) represent with

$$f(k,k) = \begin{cases} \frac{3}{k+3}\sum_{i=0}^{\frac{k+3}{3}} \binom{\frac{k+3}{3}}{i}\binom{\frac{2k-3i}{3}}{\frac{k-3i}{3}}, & \text{if } \frac{k-3}{3} \in \mathbb{Z}_{\geqslant}; (\textit{Schröder numbers}) \\ 0, & \textit{otherwise}, \end{cases}$$

for $k \geqslant 1$.

NOTE:

(i) Suppose if we take the set of three general steps $\mathfrak{C} = \{(q,p),(p,q),(r,r)\}$ and where $q \in \mathbb{Z}_{\geqslant}, p,r \in \mathbb{Z}_+$, and $0 \leqslant q < r < p$, then we can write the functional equation

$$\frac{1}{w_1^pw_2^q}\Bigg(\sum_{i=0}^{p-1}\tilde{\Phi}_{i,0}(w_1,w_2) - w_1^qw_2^p\Bigg(\sum_{i=0}^{p-q-1}\tilde{\Phi}_{i,0}(w_1,w_2)\Bigg) - \\ - w_1^pw_2^q\Bigg(\sum_{j=0}^{p-q-1}\sum_{n=j}^{p-q-1} f(j,n)w_1^jw_2^n\Bigg) - w_1^rw_2^r\Bigg(\sum_{i=0}^{p-r-1}\tilde{\Phi}_{i,0}(w_1,w_2)\Bigg)\Bigg) = \\ = \sum_{k\geqslant p-q} f(k,k)(w_1w_2)^k. \tag{2.21}$$

For $q=1, p=3$, and $r=2$ we have the initial data:

$$f(x,0)=0,\quad f(x,1)=0,\quad f(x+2,2)=0,\quad x=1,2,3,\ldots,$$
$$f(x,2)=x-1,\quad x=1,2,\quad f(0,y)=0,\quad f(1,y+3)=0,\quad f(2,y+6)=0,$$
$$y=1,2,3,\ldots,\quad f(y,3y)=1,\quad f(2,y+3)=0,\quad y=0,1,2,$$

while the generating function for the initial data equals

$\tilde{\Phi}_{0,0}(w_1,w_2) = \sum\limits_{y=0}^{\infty} f(0,y)w_2^y = 1, \quad \tilde{\Phi}_{1,0}(w_1,w_2) = \sum\limits_{y=1}^{\infty} f(1,y)w_1 w_2^y = w_1 w_2^3, \quad \tilde{\Phi}_{2,0}(w_1,w_2) = \sum\limits_{y=2}^{\infty} f(2,y)w_1^2 w_2^y = w_1^2 w_2^2 + w_1^2 w_2^6.$

Using (2.21) implies

$$-1 + \frac{1}{w_1^3 w_2} = \sum_{k\geqslant 2} f(k,k)(w_1 w_2)^k.$$

The substitution $t = w_1 w_2$ implies $w_1^2 = \dfrac{1-t^2+\sqrt{1-2t^2-3t^4}}{2t}$ (from Corollary 2), we get

$$\sum_{k=2}^{\infty} f(k,k)(t)^k = -1 + \frac{1-t^2-\sqrt{1-2t^2-3t^4}}{2t^4} = t^2+2t^4+4t^6+9t^8+21t^{10}\cdots . \tag{2.22}$$

The coefficients of the series (2.22) represent with

$$f(k,k) = \begin{cases} \frac{2}{k+2}\sum\limits_{i=0}^{\frac{k+2}{2}} \begin{pmatrix} \frac{k+2}{2} \\ \frac{k-2i}{2} \end{pmatrix} \begin{pmatrix} \frac{k-2i}{2} \\ i \end{pmatrix}, & \text{if } \frac{k-2}{2} \in \mathbb{Z}_{\geqslant}; (\textit{Motzkin numbers}) \\ 0, & \textit{otherwise}, \end{cases}$$

for $k \geqslant 1$.

Lattice paths start from the origin and lie on or over the diagonal $y = x$ using the set of three general steps $\mathfrak{C} = \{(q,p),(p,q),(r,s)\}$ where $q \in \mathbb{Z}_{\geqslant}, p,r,s \in \mathbb{Z}_+$, $0 \leqslant q < r < p < s$ and $s - r = p - q$. We define the number of paths $f(x,y)$ going from $(0,0)$ to (x,y), which satisfies the difference equation

$$f(x,y) - \left(1 - \sum_{d=1}^{p-q} \delta_0(x-y-d)\right) f(x-p,y-q) - f(x-q,y-p) - f(x-r,y-s) = 0, \tag{2.23}$$

with initial data (from (2.6)):

$$f(x,y) = \psi(x,y), (x,y) \geqslant (0,0), (x,y) \ngeqslant (p,s). \tag{2.24}$$

Corollary 3. *Let* $P(w) = \sum\limits_{0\leqslant\gamma\leqslant m} c_\gamma w^\gamma$ *be a polynomial difference operator with constant coefficients and* $P(w) \in \mathbb{C}[w]$ *then*

$$w_1^{p-q} = \frac{1+\sqrt{1-4t^{p+q}-4t^{s+q}}}{2t^q}.$$

Proof. We can write the polynomial $P(w_1,w_2) = 1 - w_1^p w_2^q - w_1^q w_2^p - w_1^r w_2^s$ with initial data (2.24) from the polynomial difference operator.

Let $P(w_1, w_2) = 0$, we see that w_1^{p-q} is the root of the following quadratic equation

$$t^q(w_1^{p-q})^2 - (1 - t^r)w_1^{p-q} + t^p = 0, \tag{2.25}$$

which satisfies $w_1 w_2 = t,\ c_{00} = 1, c_{qp} = c_{pq} = c_{rs} = -1$ and we reduce (2.25) to the standard quadratic form to get the root w_1^{p-q}. □

Lemma 3. *The generating function $H(w_1, w_2)$ of a solution $h(x, y)$ to the difference equation (2.23) with initial data (2.24) is represented as*

$$\sum_{\substack{x \geqslant m_1 \\ y \geqslant m_2}} h(x, y) w_1^x w_2^y =$$

$$= \sum_{\substack{x \geqslant m_1 \\ y \geqslant m_2}} P(\delta_1^{-I}, \delta_2^{-I}) f(x, y) w_1^x w_2^y = -w_1^p w_2^q \sum_{k \geqslant s-q} f(k, k)(w_1 w_2)^k.$$

Proof. As in the proof of Lemma 1 we find the generating function

$$\sum_{\substack{x \geqslant p \\ y \geqslant s}} (1 - \delta_1^{-q}\delta_2^{-p} - \delta_1^{-p}\delta_2^{-q} - \delta_1^{-r}\delta_2^{-s}) f(x, y) w_1^x w_2^y =$$

$$= -w_1^p w_2^q \sum_{k \geqslant s-q} f(k, k)(w_1 w_2)^k. \tag{2.26}$$

□

Proposition 3. *Let $F(w_1, w_2)$ be the generating function of the solution of (2.23). Then the series $F(w_1, w_2)$ satisfy the following functional equation*

$$\begin{aligned}
(1 - w_1^p w_2^q - w_1^q w_2^p - w_1^r w_2^s) F(w_1, w_2) - \\
- \Phi_{p,s}(w_1, w_2) + w_1^p w_2^q \Phi_{0,s-q}(w_1, w_2) + \\
+ w_1^q w_2^p \Phi_{p-q,s-p}(w_1, w_2) + w_1^r w_2^s \Phi_{p-r,0}(w_1, w_2) = \\
= -w_1^p w_2^q \sum_{k \geqslant s-q} f(k, k)(w_1 w_2)^k.
\end{aligned} \tag{2.27}$$

Proof. $P(w_1, w_2) = 1 - w_1^p w_2^q - w_1^q w_2^p - w_1^r w_2^s, m = (p, s)$ where $p, s \in \mathbb{Z}_+$ satisfies (2.4).

Using Theorem 1 and (2.26) from Lemma 4, we get (2.27) where

$$\Phi_{p,s}(w_1, w_2) =$$

$$= \Phi_{p,p}(w_1, w_2) + \sum_{i=p}^{s-1} \tilde{\Phi}_{0,i}(w_1, w_2) + \sum_{j=0}^{p-q-3} \sum_{n=p+j}^{s-2} f(p+j, n+1) w_1^{p+j} w_2^{n+1}, \tag{2.28}$$

$$\Phi_{p-q,s-p}(w_1,w_2)=\tilde{\Phi}_{s-p,s-p}(w_1,w_2)+\sum_{i=s-p}^{p-q-1}\tilde{\Phi}_{i,0}(w_1,w_2).$$

Let $P(w_1,w_2)=1-w_1^pw_2^q-w_1^qw_2^p-w_1^rw_2^s=0$, as a result we obtain

$$\Phi_{p,p}(w_1,w_2)+\sum_{i=p}^{s-1}\tilde{\Phi}_{0,i}(w_1,w_2)+\sum_{j=0}^{p-q-3}\sum_{n=p+j}^{s-2}f(p+j,n+1)w_1^{p+j}w_2^{n+1}-$$
$$-w_1^pw_2^q\left(\sum_{i=0}^{s-q-1}\tilde{\Phi}_{0,i}(w_1,w_2)+\sum_{j=0}^{s-q-2}\sum_{n=j}^{s-q-2}f(j,n+1)w_1^jw_2^{n+1}\right)-$$
$$-w_1^qw_2^p\left(\tilde{\Phi}_{s-p,s-p}(w_1,w_2)+\sum_{i=s-p}^{p-q-1}\tilde{\Phi}_{i,0}(w_1,w_2)\right)-w_1^rw_2^s\left(\sum_{i=0}^{p-r-1}\tilde{\Phi}_{i,0}(w_1,w_2)+\right.$$
$$\left.+\sum_{j=0}^{p-r-2}\sum_{w=j}^{p-r-2}f(w+1,j)w_1^{w+1}w_2^j\right)=w_1^pw_2^q\sum_{k\geqslant s-q}f(k,k)(w_1w_2)^k.$$

Since $f(x,y)=0$ below the diagonal, we get

$$\frac{1}{w_1^pw_2^q}\left(\sum_{i=0}^{p-1}\tilde{\Phi}_{i,0}(w_1,w_2)+\sum_{j=0}^{p-q-2}\sum_{n=p+j}^{s-1}f(p+j,n)w_1^{p+j}w_2^n-\right.$$
$$-w_1^pw_2^q\left(\sum_{j=0}^{s-q-1}\sum_{n=j}^{s-q-1}f(j,n)w_1^jw_2^n\right)-w_1^qw_2^p\left(\sum_{i=0}^{p-q-1}\tilde{\Phi}_{i,0}(w_1,w_2)\right)-$$
$$\left.-w_1^rw_2^s\left(\sum_{i=0}^{p-r-1}\tilde{\Phi}_{i,0}(w_1,w_2)\right)\right)=\sum_{k\geqslant s-q}f(k,k)(w_1w_2)^k. \quad (2.29)$$

For $q=0,p=2,r=1$, and $s=3$ we have the initial data:

$$f(x,0)=0,\quad f(x,1)=0,\quad f(x+2,2)=0,\quad x=1,2,3,\ldots,$$
$$f(0,y)=\frac{1+(-1)^y}{2},\quad y=0,1,2,\ldots,\quad f(1,y)=\frac{(1-(-1)^y)(y-1)}{4},$$
$$y=2,3,4,\ldots,\quad f(y+1,y+1)=y,\quad y=1,$$

while the generating function for the initial data equals
$\tilde{\Phi}_{0,0}(w_1,w_2)=\sum_{y=0}^{\infty}f(0,y)w_2^y=\dfrac{1}{1-w_2^2},\quad \tilde{\Phi}_{1,0}(w_1,w_2)=\sum_{y=1}^{\infty}f(1,y)w_1w_2^y=$
$\dfrac{w_1w_2^3}{(1-w_2^2)^2}.$

Using (2.29) implies

$$-1-w_1^2w_2^2+\frac{1}{w_1^2}=\sum_{k\geqslant 3}f(k,k)(w_1w_2)^k.$$

The substitution $t = w_1 w_2$ implies $w_1^2 = \dfrac{1+\sqrt{1-4t^2-4t^3}}{2}$ (from Corollary 3), we get

$$\sum_{k=3}^{\infty} f(k,k)(t)^k = -1-t^2+\frac{1-\sqrt{1-4t^2-4t^3}}{2(t^2+t^3)} = t^3+2t^4+4t^5+7t^6+15t^7 \cdots . \tag{2.30}$$

□

The coefficients of the series (2.30) coincides with the numbers of general paths ending on the main diagonal $y = x$.

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