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# Classical and Mild Solution of the First Mixed Problem for the Telegraph Equation with a Nonlinear Potential

**Viktor I. Korzyuk**[1,2], **Jan V. Rudzko**[2]✉

[1] Belarusian State University, Minsk, Belarus
[2] Institute of Mathematics of the National Academy of Sciences of Belarus, Minsk, Belarus
✉ janycz@yahoo.com

**Abstract.** We study the first mixed problem for the telegraph equation with a nonlinear potential in the first quadrant. We pose the Cauchy conditions on the lower base of the domain and the Dirichlet condition on the lateral boundary. By the method of characteristics, we obtain an expression for the solution of the problem in an implicit analytical form as a solution of some integral equations. To solve these equations, we use the method of sequential approximations. The existence and uniqueness of the classical solution under specific smoothness and matching conditions for given functions are proved. Under inhomogeneous matching conditions, we consider a problem with conjugation conditions. When the given data is not smooth enough, we construct a mild solution.

**Keywords:** nonlinear wave equation, classical solution, mixed problem, matching conditions, generalized solution

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Научная статья

# Классическое и слабое решение первой смешанной задачи для телеграфного уравнения с нелинейным потенциалом

**В. И. Корзюк**[1,2], **Я. В. Рудько**[2]✉

[1] Белорусский государственный университет, Минск, Беларусь
[2] Институт математики Национальной академии наук Беларуси, Минск, Беларусь
✉ janycz@yahoo.com

**Аннотация**. Исследуется первая смешанная задача для телеграфного уравнения с нелинейным потенциалом в первом квадранте плоскости. На нижнем основании задаются условия Коши, а на боковой границе – условие Дирихле. Методом характеристик строится выражение решения задачи в неявном аналитическом виде как решение некоторых интегральных уравнений. Для получения решений этих интегральных уравнений используется метод последовательных приближений. Доказывается существование и единственность классического решения при определенных условиях гладкости и условиях согласования заданных функций. При неоднородных условиях согласования рассматривается задача с условиями сопряжения. В случае недостаточно гладких данных строится слабое решение.

**Ключевые слова:** нелинейное волновое уравнение, классическое решение, смешанная задача, условия согласования, обобщенное решение

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## 1. Introduction

This article is a continuation of the work [8]. In this article, we will show that we can take the condition of a Lipschitz–Carathéodory type for nonlinearity instead of the Lipschitz condition, discuss in more detail the problem with conjugation conditions and construct a mild solution and prove its existence and uniqueness.

The paper is organized in the following way. In Sec. 2, the problem statement is formulated. In Sec. 3, we reduce the problem to solving the integral equation and prove its solvability, uniqueness, and well-posedness. In Sec. 4, we construct a piecewise smooth solution. In Sec. 5, we build a classical solution. In Sec. 6, we consider the problem with conjugation conditions on the characteristic. In Sec. 7, we discuss a mild solution. Section 8 presents the conclusions of the work done.

The reduction of the differential formulation of the problem to the integral one is done by the method of characteristics. We use the method of successive approximations to solve integral equations. A multidimensional generalization of the Grönwall lemma is used to prove the well-posedness of the problem.

## 2. Statement of the problem

In the domain $Q = (0, \infty) \times (0, \infty)$ of two independent variables $(t, x) \in Q \subset \mathbb{R}^2$, consider the one-dimensional nonlinear equation

$$\Box u(t,x) - f(t,x,u(t,x)) = F(t,x), \tag{2.1}$$

where $\Box = \partial_t^2 - a^2\partial_x^2$ is the d'Alembert operator ($a > 0$ for definiteness), $F$ is a function given on the set $\overline{Q}$, and $f$ is a function given on the set $[0,\infty) \times [0,\infty) \times \mathbb{R}$ and satisfying the condition of the Lipschitz–Carathéodory type in the third variable; i.e. there exists a function $k$ of the class $L_2^{\mathrm{loc}}(\overline{Q})$ such that $|f(t,x,z_1) - f(t,x,z_2)| \leq k(t,x)|z_1 - z_2|$. Equation (2.1) is equipped with the initial condition

$$u(0,x) = \varphi(x), \partial_t u(0,x) = \psi(x), x \in [0,\infty), \tag{2.2}$$

and the boundary condition

$$u(t,0) = \mu(t), t \in [0,\infty), \tag{2.3}$$

where $\varphi$, $\psi$ and $\mu$ are functions given on the half-line $[0,\infty)$.

Equations of the form (2.1) arise in various areas of physics, mathematics, and engineering, e. g. superconductivity, dislocations in crystals, waves in ferromagnetic materials, laser pulses in two-phase media, propagation of spin waves in anisotropic spin liquids [14].

Such mixed problems with Dirichlet conditions in unbounded domains have been discussed previously by various authors, e. g. [2–4; 12]. However, in these papers, as a rule, nonlinearities of power type and weak solutions were considered.

## 3. Integral equation

We divide the domain $Q$ by the characteristic $x - at = 0$ into two subdomains $Q^{(j)} = \{(t,x) \mid (-1)^j(at - x) > 0\}$, $j = 1, 2$. In the closure $\overline{Q^{(j)}}$ of each of the subdomains $Q^{(j)}$, we consider the integral equation

$$u^{(j)}(t,x) = g^{(1,j)}(x - at) + g^{(2)}(x + at) - \\ - \frac{1}{4a^2}\int_0^{x-at} dy \int_{(-1)^j(at-x)}^{x+at} \left[ F\left(\frac{z-y}{2a}, \frac{z+y}{2}\right) + \right. \\ \left. + f\left(\frac{z-y}{2a}, \frac{z+y}{2}, u^{(j)}\left(\frac{z-y}{2a}, \frac{z+y}{2}\right)\right)\right] dz, (t,x) \in \overline{Q^{(j)}}, j = 1, 2, \tag{3.1}$$

where $g^{(1,1)}$, $g^{(2)}$, and $g^{(1,2)}$ – are some functions, the first two of them given on the nonnegative half-line and the last one, on the nonpositive half-line.

On the closure $\overline{Q}$ of the domain $Q$, we define a function $u$ as the one coinciding with the solution $u^{(j)}$ of the integral equation (3.1)

$$u(t,x) = u^{(j)}(t,x), \quad (t,x) \in \overline{Q^{(j)}}, \quad j = 1,2, \tag{3.2}$$

on the closure $\overline{Q^{(j)}}$ of the domain $Q^{(j)}$.

**Lemma 1.** *Let the conditions $f \in C^1(\overline{Q} \times \mathbb{R})$ and $F \in C^1(\overline{Q})$ be satisfied. The function $u^{(1)}$ belongs to the class $C^2(\overline{Q^{(1)}})$ and satisfies Eq. (2.1) in $\overline{Q^{(1)}}$ if and only if it is a continuous solution of Eq. (3.1) for $j = 1$ in which the functions $g^{(1,1)}$ and $g^{(2)}$ are in the class $C^2([0,\infty))$.*

**Lemma 2.** *Let the conditions $f \in C^1(\overline{Q} \times \mathbb{R})$ and $F \in C^1(\overline{Q})$ be satisfied. The function $u^{(2)}$ belongs to the class $C^2(\overline{Q^{(2)}})$ and satisfies Eq. (2.1) in $\overline{Q^{(2)}}$ if and only if it is a continuous solution of Eq. (3.1) for $j = 2$ in which the functions $g^{(1,2)}$ and $g^{(2)}$ belong to the classes $C^2((-\infty,0])$ and $C^2([0,\infty))$, respectively.*

**Theorem 1.** *Let the conditions $f \in C^1(\overline{Q}\times\mathbb{R})$ and $F \in C^1(\overline{Q})$ be satisfied. The function $u$ belongs to the class $C^2(\overline{Q})$ and satisfies Eq. (2.1) if and only if for each $j = 1,2$ it is a continuous solution of Eq. (3.1) in which the functions $g^{(1,1)}$, $g^{(1,2)}$, and $g^{(2)}$ are in the classes $C^2([0,\infty))$, $C^2((-\infty,0])$, and $C^2([0,\infty))$, respectively, and the following matching conditions are satisfied:*

$$g^{(1,1)}(0) - g^{(1,2)}(0) = 0, \tag{3.3}$$

$$Dg^{(1,1)}(0) - Dg^{(1,2)}(0) = 0, \tag{3.4}$$

$$D^2 g^{(1,1)}(0) - D^2 g^{(1,2)}(0) + \frac{1}{a^2}\big(F(0,0) + f(0,0,g^{(1,1)}(0) + g^{(2)}(0))\big) = 0. \tag{3.5}$$

*Proof.* The proofs of Lemmas 1, 2 and Theorem 1 are presented in the work [8]. □

**Theorem 2.** *Let $F \in L_1^{\text{loc}}(\overline{Q})$ and $f \in C(\overline{Q} \times \mathbb{R})$, let the function $f$ satisfy the condition of the Lipschitz–Carathéodory type with respect to the third variable, i.e., there is a function $k \in L_2^{\text{loc}}(\overline{Q})$ such that $|f(t,x,z_1) - f(t,x,z_2)| \le k(t,x)|z_1 - z_2|$, and let the functions $g^{(1,1)}$, $g^{(1,2)}$, and $g^{(2)}$ be continuous. Then there exist unique solutions of Eqs. (3.1), and these solutions continuously depend on the initial data.*

*Proof.* The proof of the theorem will be carried out by the scheme set forth in [15] (in complete form) and in [1;5;10] (briefly). To be definite, consider

Eq. (3.1) for $j = 1$. It will be solved by the successive approximation method. Set $G(t,x) = g^{(1,j)}(x-at) + g^{(2)}(x+at)$. Take the initial approximation

$$u^{(1,0)}(t,x) = G(t,x) - \frac{1}{4a^2} \int\limits_0^{x-at} dy \int\limits_{x-at}^{x+at} F\left(\frac{z-y}{2a}, \frac{z+y}{2}\right) dz.$$

Then every subsequent approximation will be calculated by the formula

$$u^{(1,m)}(t,x) = G(t,x) - \frac{1}{4a^2} \int\limits_0^{x-at} dy \int\limits_{x-at}^{x+at} \left[F\left(\frac{z-y}{2a}, \frac{z+y}{2}\right) + \right.$$
$$\left. + f\left(\frac{z-y}{2a}, \frac{z+y}{2}, u^{(1,m-1)}\left(\frac{z-y}{2a}, \frac{z+y}{2}\right)\right)\right] dz, (t,x) \in \overline{Q^{(1)}}. \tag{3.6}$$

Let us establish estimates for the successive approximations. Let $\widetilde{x} > 0$, $\mathcal{A} = \overline{Q^{(1)}} \cap ([0, \widetilde{x}/a] \times [0, \widetilde{x}])$, $M_G = \max\limits_{(t,x)\in\mathcal{A}} |G(t,x)|$,
$K = \sup\limits_{(t,x)\in\mathcal{A}} \sqrt{\int\limits_0^{x-at} dy \int\limits_{x-at}^{x+at} \left|k\left(\frac{z-y}{2a}, \frac{z+y}{2}\right)\right|^2 dz}$.

Then $\left|(u^{(1,1)} - u^{(1,0)})(t,x)\right| \leqslant \mathcal{M}$,

$$\left|(u^{(1,2)} - u^{(1,1)})(t,x)\right| \leqslant$$
$$\leqslant \left|\frac{1}{4a^2} \int\limits_0^{x-at} dy \int\limits_{x-at}^{x+at} f\left(\frac{z-y}{2a}, \frac{z+y}{2}, u^{(1,1)}\left(\frac{z-y}{2a}, \frac{z+y}{2}\right)\right) dz - \right.$$
$$\left. - f\left(\frac{z-y}{2a}, \frac{z+y}{2}, u^{(1,0)}\left(\frac{z-y}{2a}, \frac{z+y}{2}\right)\right) dz\right| \leqslant$$
$$\leqslant \frac{1}{4a^2}\left| \int\limits_0^{x-at} dy \int\limits_{x-at}^{x+at} k\left(\frac{z-y}{2a}, \frac{z+y}{2}\right) \left|u^{(1,1)} - u^{(1,0)}\right| \left(\frac{z-y}{2a}, \frac{z+y}{2}\right) dz\right| \leqslant$$
$$\leqslant \frac{1}{4a^2}\left|\sqrt{\int\limits_0^{x-at} dy \int\limits_{x-at}^{x+at} \left|k\left(\frac{z-y}{2a}, \frac{z+y}{2}\right)\right|^2 dz} \sqrt{\int\limits_0^{x-at} dy \int\limits_{x-at}^{x+at} \mathcal{M}^2\, dz}\right| \leqslant$$
$$\leqslant \frac{K\mathcal{M}\sqrt{2}\sqrt{at(x-at)}}{4a^2}, (t,x) \in \mathcal{A}.$$

In what follows, by induction in which the last inequality is chosen as the base case, one can readily prove the estimate

$$\left|(u^{(1,i+1)} - u^{(1,i)})(t,x)\right| \leq \frac{K^i \mathcal{M}\sqrt{\widetilde{x}t(x-at)^{i-1}(x+at)^{i-1}}}{2\cdot 4^{i-1}a^{2i-1}\sqrt{2\cdot(1)_{i-1}(2)_{i-1}a}}, (t,x)\in\mathcal{A}, \tag{3.7}$$

where we have used the notation $(x)_n = \prod\limits_{k=1}^{n}(x+k-1)$ for the Pochhammer symbol.

Note that $u^{(1,m)} = u^{(1,0)} + \sum\limits_{j=0}^{m-1}(u^{(1,j+1)} - u^{(1,j)})$. The estimate (3.7) implies the absolute and uniform convergence of the series $u^{(1,\infty)} = u^{(1,0)} + \sum\limits_{j=0}^{\infty}(u^{(1,j+1)} - u^{(1,j)})$ on the set $\mathcal{A}$, since its terms are majorized in magnitude by the terms of the uniformly converging series[1]

$$\mathcal{M} + M_G + \sum_{i=1}^{\infty}\frac{K^i\mathcal{M}\sqrt{\widetilde{x}t(x-at)^{i-1}(x+at)^{i-1}}}{2\cdot 4^{i-1}a^{2i-1}\sqrt{2\cdot(1)_{i-1}(2)_{i-1}a}} \leqslant$$
$$\leqslant M\left(1+\frac{K\sqrt{\widetilde{x}t}}{2a\sqrt{2a}}\exp\left(\frac{K\sqrt{(x-at)(x+at)}}{4a^2}\right)\right),$$

where $M = \mathcal{M} + M_G$. Thus, the successive approximations by the continuous functions $u^{(1,m)}$ uniformly tend on the set $\mathcal{A}$ to a function $u^{(1)} : \mathbb{R}^2 \supset \overline{Q^{(1)}} \supset \mathcal{A} \ni (t,x) \to u^{(1)}(t,x) \in \mathbb{R}$ continuous in $\mathcal{A}$, and, by virtue of of arbitrariness of $\widetilde{x}$, to a function $u^{(1)} : \mathbb{R}^2 \supset \overline{Q^{(1)}} \ni (t,x) \to u^{(1)}(t,x) \in \mathbb{R}$, continuous in $\overline{Q^{(1)}}$. Passing to the limit as $m \longrightarrow \infty$ in (3.6), we conclude that the function $u^{(1)}$ is a solution of Eq. (3.1) for $j=1$ on the set $\overline{Q^{(1)}}$.

Let us prove the uniqueness of solution of Eq. (3.1) for $j=1$ by contradiction. Let Eq. (3.1) for $j=1$ have two solutions $u^{(1)}$ and $\widetilde{u}^{(1)}$. Denote $U = u^{(1)} - \widetilde{u}^{(1)}$. Then

$$U(t,x) = -\frac{1}{4a^2}\int\limits_0^{x-at}dy\int\limits_{x-at}^{x+at} f\left(\frac{z-y}{2a},\frac{z+y}{2},u^{(1)}\left(\frac{z-y}{2a},\frac{z+y}{2}\right)\right)dz +$$
$$+\frac{1}{4a^2}\int\limits_0^{x-at}dy\int\limits_{x-at}^{x+at} f\left(\frac{z-y}{2a},\frac{z+y}{2},\widetilde{u}^{(1)}\left(\frac{z-y}{2a},\frac{z+y}{2}\right)\right)dz, (t,x)\in\overline{Q^{(1)}}. \tag{3.8}$$

The function $U$ is continuous, and hence $|U(t,x)| \leq M_U$ under the condition $(t,x)\in\mathcal{A}$, where $M_U$ is some constant. It follows from (3.8) with

[1] We can give more precise estimates, see https://math.stackexchange.com/questions/4396003/closed-form-of-sum-i-0-infty-xi-ii1-1-2

allowance for the condition of Lipschitz–Carathéodory type and Cauchy–Bunyakovsky–Schwarz inequality that

$$|U(t,x)| \leq \frac{1}{4a^2}\sqrt{K^2 \int\limits_0^{x-at} dy \int\limits_{x-at}^{x+at} M_U^2 dz} \leq \frac{KcM_U\sqrt{at\widetilde{x}}}{2\sqrt{2}a^2}, (t,x) \in \mathcal{A}.$$

By induction, we arrive at the estimate

$$|U(t,x)| \leq \frac{K^{i+1}M_U\widetilde{x}^{1+i}}{2\cdot 4^i a^{2i+2}\sqrt{2\cdot(1)_i(2)_i}}$$

for each positive integer $i$ and any pair $(t,x)$ in $\mathcal{A}$. It follows that $U\equiv 0$ on the set $\mathcal{A}$ and, by virtue of the arbitrariness of $\widetilde{x}$, that $U \equiv 0$ on the set $\overline{Q^{(j)}}$. Thus, we have proved the existence of a unique continuous solution of Eq. (3.1) for $j=1$.

To prove the continuous dependence of the solution on the initial data, along with Eq. (3.1) for $j=1$ we consider the perturbed equation

$$(u^{(1)}+\Delta u)(t,x) = (G+\Delta G)(t,x) - \\ - \frac{1}{4a^2}\int\limits_0^{x-at} dy \int\limits_{x-at}^{x+at}\left[F\left(\frac{z-y}{2a},\frac{z+y}{2}\right)+\right. \\ \left. + f\left(\frac{z-y}{2a},\frac{z+y}{2},(u^{(1)}+\Delta u)\left(\frac{z-y}{2a},\frac{z+y}{2}\right)\right)\right]dz, (t,x)\in\overline{Q^{(1)}}, \quad (3.9)$$

and the difference of the perturbed (3.9) and unperturbed (3.1) equations,

$$\Delta u(t,x) = \Delta G(t,x) - \\ - \frac{1}{4a^2}\int\limits_0^{x-at} dy \int\limits_{x-at}^{x+at}\left[f\left(\frac{z-y}{2a},\frac{z+y}{2},(u^{(1)}+\Delta u)\left(\frac{z-y}{2a},\frac{z+y}{2}\right)\right)-\right. \\ \left. - f\left(\frac{z-y}{2a},\frac{z+y}{2},u^{(1)}\left(\frac{z-y}{2a},\frac{z+y}{2}\right)\right)\right]dz, (t,x)\in\overline{Q^{(1)}}. \quad (3.10)$$

For Eq. (3.10) for the disturbance $\Delta u$, one has the following estimate of the disturbance modulus:

$$|\Delta u(t,x)| \leqslant M_{\Delta G} + \\ + \frac{1}{4a^2}\int\limits_0^{x-at} dy \int\limits_{x-at}^{x+at} k\left(\frac{z-y}{2a},\frac{z+y}{2}\right)\left|\Delta u\left(\frac{z-y}{2a},\frac{z+y}{2}\right)\right| dz,$$

where $M_{\Delta G} = \max\limits_{(t,x)\in\mathcal{A}} |\Delta G(t,x)|$. Applying the multidimensional Grönwall lemma [13] to the previous inequality, we obtain $|\Delta u(t,x)| \leqslant C^{(1)} M_{\Delta G}$, where $C^{(1)}$ is some positive constant depending only on the set $\mathcal{A}$, the function $k$, and the number $a$. The resulting inequality implies that whatever a small perturbation $\Delta G$, $M_{\Delta G} = \varepsilon$, is taken, the perturbation of the solution obeys the inequality $|\Delta u(t,x)| = \delta \leqslant \varepsilon C^{(1)}$ on the set $\mathcal{A}$. By virtue of the arbitrariness of $\widetilde{x}$, we conclude that the solution of 3.1) for $j = 1$ continuously depends on the initial data.

The existence of a unique continuous solution of Eq. (3.1) for $j = 2$, which continuously depends on the initial data, can be proved in a similar way. The proof of the theorem is complete. □

This theorem allows us to strengthen the result of the work [8] and generate new solutions to Eq. (2.1).

**Remark 1.** In Theorem 1, we can take three following conditions instead of $f \in C(\overline{Q} \times \mathbb{R})$, namely:

1) The function $f_1 \colon \overline{Q} \ni (t,x) \mapsto f(t,x,z) \in \mathbb{R}$ is measurable for any fixed $z \in \mathbb{R}$;

2) The function $f_2 \colon \mathbb{R} \ni z \mapsto f(t,x,z) \in \mathbb{R}$ is continuous on the set $\mathbb{R}$ for almost any fixed point $(t,x) \in \overline{Q}$;

3) The function $f$ satisfies the grow condition $|f(t,x,z)| \leqslant \alpha(t,x) + \beta|z|$, where $\alpha \in L_1^{\text{loc}}(\overline{Q})$, $\beta \in \mathbb{R}$.

*Proof.* It is necessary to show that for any continuous function $u^{(j)}$ the right side of the equation (3.1) is also a continuous function. Note that if we fix a function $u^{(j)}$ and a compact set $\mathcal{K} \subset \overline{Q}$, then under the conditions specified in this remark, the expression $\mathcal{K} \ni (t,x) \mapsto f\left(t,x,u^{(j)}(t,x)\right)$ defines a function of the class $L_1(\mathcal{K})$ [16]. And, by virtue of the arbitrariness of $\mathcal{K}$, the formula $\overline{Q} \ni (t,x) \mapsto f\left(t,x,u^{(j)}(t,x)\right)$ defines a function of the class $L_1^{\text{loc}}(\overline{Q})$. Then, using the absolute continuity of the Lebesgue integral, we conclude that the right side of the equation (3.1) is a continuous function too. □

## 4. Constructing the solution of the mixed problem

Determining the functions $g^{(1,1)}$ and $g^{(2)}$ from the Cauchy conditions (2.2) and the function $g^{(1,2)}$ from the boundary condition (2.3), we obtain [8]

$$u^{(1)}(t,x) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} \psi(z)\,dz +$$

$$+\frac{1}{4a^2}\int\limits_{x-at}^{x+at}dz\int\limits_{x-at}^{z}\left[F\left(\frac{z-y}{2a},\frac{z+y}{2}\right)+\right.$$
$$\left.+f\left(\frac{z-y}{2a},\frac{z+y}{2},u^{(1)}\left(\frac{z-y}{2a},\frac{z+y}{2}\right)\right)\right]dy,\quad (t,x)\in\overline{Q^{(1)}},$$

$$u^{(2)}(t,x)=\mu\left(t-\frac{x}{a}\right)+\frac{\varphi(x+at)-\varphi(at-x)}{2}+\frac{1}{2a}\int\limits_{at-x}^{x+at}\psi(z)dz-$$
$$-\frac{1}{4a^2}\int\limits_{0}^{x-at}dy\int\limits_{at-x}^{x+at}\left[F\left(\frac{z-y}{2a},\frac{z+y}{2}\right)+\right.$$
$$\left.+\ f\left(\frac{z-y}{2a},\frac{z+y}{2},u^{(2)}\left(\frac{z-y}{2a},\frac{z+y}{2}\right)\right)\right]dz+$$
$$+\ \frac{1}{4a^2}\int\limits_{at-x}^{x+at}dz\int\limits_{0}^{z}\left[F\left(\frac{z-y}{2a},\frac{z+y}{2}\right)+\right.$$
$$\left.+\ f\left(\frac{z-y}{2a},\frac{z+y}{2},u^{(1)}\left(\frac{z-y}{2a},\frac{z+y}{2}\right)\right)\right]dy,\quad (t,x)\in\overline{Q^{(2)}}.\quad (4.1)$$

We note that the equation for defining the function $u^{(2)}$ can be derived by the curvilinear parallelogram identity [9].

**Lemma 3.** *Let the conditions $f\in C^1(\overline{Q}\times\mathbb{R})$, $F\in C^1(\overline{Q})$, $\varphi\in C^2([0,\infty))$, $\psi\in C^1([0,\infty))$, and $\mu\in C^2([0,\infty))$ hold, and let the function $f$ satisfy the condition of Lipschitz–Carathéodory type with respect to the third variable. Then there exists solutions $u^{(j)}$ $(j=1,2)$ of Eqs. (4.1); they are unique in the class $C^2(\overline{Q^{(j)}})$ and continuously depend on the functions $\varphi$, $\psi$, and $\mu$.*

*Proof.* This lemma follows from Theorems 1 and 2. □

Thus, we have constructed a piecewise smooth solution of problem (2.1) – (2.3) determined by formulas (4.1) and (3.2).

## 5. Classical solution

For the function $u$ to belong to the class $C^2(\overline{Q})$, in addition to the requirements of smoothness for the functions $f$, $F$, $\varphi$, $\psi$, and $\mu$, it is necessary and sufficient that the equalities (3.3) – (3.5), be satisfied, according to Theorem 1. Calculating the quantities occurring in the expressions (3.3) – (3.5), we obtain the following matching conditions:

$$\mu(0) = \varphi(0), \tag{5.1}$$

$$\mu'(0) = \psi(0), \tag{5.2}$$

$$\mu''(0) = \frac{1}{2}\Big(f(0,0,\mu(0)) + f(0,0,\varphi(0))\Big) + F(0,0) + a^2\varphi''(0). \tag{5.3}$$

**Theorem 3.** *Let the conditions $f \in C^1(\overline{Q}\times\mathbb{R})$, $F\in C^1(\overline{Q})$, $\varphi\in C^2([0,\infty))$, $\psi \in C^1([0,\infty))$, and $\mu \in C^2([0,\infty))$ be satisfied, and let the function $f$ satisfy the condition of Lipschitz–Carathéodory type with respect to the third variable. The first mixed problem (2.1) – (2.3) has a unique solution $u$ in the class $C^2(\overline{Q})$ if and only if conditions (5.1) – (5.3) are satisfied. This solution is determined by formulas (3.2) and (4.1).*

*Proof.* It follows from Theorem 1, Lemma 3, and the above argument. □

## 6. Inhomogeneous matching conditions

Let's consider the problem (2.1) – (2.3) in the case when the matching conditions (5.1) – (5.3) partially or completely not fulfilled as it was done in [5–7; 10; 11].

According to Theorem 1, the presence of inhomogeneous matching conditions breaks the continuity of the function $u$ or its derivatives, or all together. This conclusion can be formulated as the following statement.

**State 1.** *If the homogeneous matching conditions (5.1) – (5.3) are not satisfied for the given functions $\mu$, $\varphi$, $\psi$, $f$, and $F$, then no matter how smooth are the functions $f$, $F$, $\mu$, $\varphi$, and $\psi$, the problem (2.1) – (2.3) does not have a classical solution defined on $\overline{Q}$.*

*Proof.* It follows from Theorem 1. □

Let the given functions of the equation (2.1), the initial conditions (2.2), and the boundary condition (2.3) are sufficiently smooth as in Theorem 3: $\varphi \in C^2([0,\infty))$, $\psi \in C^1([0,\infty))$, $\mu \in C^2([0,\infty))$, $f \in C^1(\overline{Q} \times \mathbb{R})$, and $F \in C^1(\overline{Q})$. Since the matching conditions (5.1) – (5.3), generally speaking, are not satisfied, we obtain discontinuities of the function $u$ and its derivatives according to the following expressions

$$[(u)^+ - (u)^-](t, x = at) = \varphi(0) - \mu(0),$$
$$[(\partial_t u)^+ - (\partial_t u)^-](t, x = at) = -a[(\partial_x u)^+ - (\partial_x u)^-](t, x = at) = \psi(0) -$$
$$- \mu'(0) + \frac{1}{4a} \int\limits_0^{2at} \left[ f\left(\frac{z}{2a}, \frac{z}{2}, (u)^+\left(\frac{z}{2a}, \frac{z}{2}\right)\right) - f\left(\frac{z}{2a}, \frac{z}{2}, (u)^-\left(\frac{z}{2a}, \frac{z}{2}\right)\right)\right] dz,$$

$$[(\partial_t^2 u)^+ - (\partial_t^2 u)^-](t, x = at) =$$
$$= F(0,0) + \frac{1}{2}\big(f(0,0,(u)^+(0,0)) + f(0,0,(u)^-(0,0))\big) +$$
$$+ \frac{1}{2}\big(f(t, at, (u)^+(t, at)) - f(t, at, (u)^-(t, at))\big) - \mu''(0) + a^2\varphi''(0) + \frac{1}{8a}\times$$
$$\times \Bigg( \int\limits_0^{2at} \Bigg[ \left((\partial_t u)^+\left(\frac{z}{2a}, \frac{z}{2}\right) - a(\partial_x u)^+\left(\frac{z}{2a}, \frac{z}{2}\right)\right) \partial_y f\left(\frac{z}{2a}, \frac{z}{2}, y =\right.$$
$$\left. = (u)^+\left(\frac{z}{2a}, \frac{z}{2}\right)\right) - a\partial_x f\left(\frac{z}{2a}, \frac{z}{2}, (u)^+\left(\frac{z}{2a}, \frac{z}{2}\right)\right) +$$
$$\partial_t f\left(\frac{z}{2a}, \frac{z}{2}, (u)^+\left(\frac{z}{2a}, \frac{z}{2}\right)\right)\Bigg] dz -$$
$$- \int\limits_0^{2at} \Bigg[ \left((\partial_t u)^-\left(\frac{z}{2a}, \frac{z}{2}\right) - a(\partial_x u)^-\left(\frac{z}{2a}, \frac{z}{2}\right)\right) \partial_y f\left(\frac{z}{2a}, \frac{z}{2}, y =\right.$$
$$\left. = (u)^-\left(\frac{z}{2a}, \frac{z}{2}\right)\right) -$$
$$- a\partial_x f\left(\frac{z}{2a}, \frac{z}{2}, (u)^-\left(\frac{z}{2a}, \frac{z}{2}\right)\right) + \partial_t f\left(\frac{z}{2a}, \frac{z}{2}, (u)^-\left(\frac{z}{2a}, \frac{z}{2}\right)\right)\Bigg] dz \Bigg) =$$
$$= f(t, x, (u)^+(t, x)) - f(t, x, (u)^-(t, x)) + a^2[(\partial_x^2 u)^+ - (\partial_x^2 u)^-] =$$
$$= \frac{1}{2}\big(f(t, x, (u)^+(t, x)) - f(t, x, (u)^-(t, x))\big) - a[(\partial_t\partial_x u)^+ - (\partial_t\partial_x u)^-]. \tag{6.1}$$

Here by $()^{\pm}$ – we have denoted the limit values of the function $u$ and its partial derivatives calculated on different sides of the characteristic $x - at = 0$; i.e., $(\partial_t^p u)^{\pm}(t, x = at) = \lim\limits_{\delta\to 0+} \partial_t^p u(t, at \pm \delta)$. Let us denote $\widetilde{Q} = \overline{Q} \backslash \{(t, x) \mid x - at = 0\}$.

**Theorem 4.** *Let the conditions $f \in C^1(\overline{Q} \times \mathbb{R})$, $F \in C^1(\overline{Q})$, $\varphi \in C^2([0, \infty))$, $\psi \in C^1([0, \infty))$, and $\mu \in C^2([0, \infty))$ be satisfied, and let the function $f$ satisfy the condition of Lipschitz–Carathéodory type with respect to the third variable. The first mixed problem (2.1) – (2.3) has a unique solution $u$ in* *the class $C^2(\widetilde{Q})$ if and only if conditions (6.1) are satisfied. This solution is determined by formulas (3.2) and (4.1).*

*Proof.* It follows from the above arguments. □

**Theorem 5.** *Let the conditions $f \in C^1(\overline{Q} \times \mathbb{R})$, $F \in C^1(\overline{Q})$, $\varphi \in C^2([0,\infty))$, $\psi \in C^1([0,\infty))$, and $\mu \in C^2([0,\infty))$ be satisfied, and let the function $f$ satisfy the condition of Lipschitz–Carathéodory type with respect to the third variable. The first mixed problem (2.1) – (2.3) has a unique solution $u$ in the class $C^2(\widetilde{Q}) \cap C(\overline{Q})$ if and only if conditions (6.1) and (5.1) are satisfied. This solution is determined by formulas (3.2) and (4.1).*

*Proof.* It follows from Theorems 1 – 3 and the above arguments. Indeed, if $\varphi(0) = \mu(0)$, then the solution $u$ is continuous on the set $\{(t,x) \mid x - at = 0\}$ by virtue of (6.1). Therefore, in addition to $u \in C^2(\widetilde{Q})$, the solution $u$ is a continuous function on the closure $\overline{Q}$. □

**Theorem 6.** *Let the conditions $f \in C^1(\overline{Q} \times \mathbb{R})$, $F \in C^1(\overline{Q})$, $\varphi \in C^2([0,\infty))$, $\psi \in C^1([0,\infty))$, and $\mu \in C^2([0,\infty))$ be satisfied, and let the function $f$ satisfy the condition of Lipschitz–Carathéodory type with respect to the third variable. The first mixed problem (2.1) – (2.3) has a unique solution $u$ in the class $C^2(\widetilde{Q}) \cap C^1(\overline{Q})$ if and only if conditions (6.1), (5.1), and (5.2) are satisfied. This solution is determined by formulas (3.2) and (4.1).*

*Proof.* It easily follows from Theorems 1 – 5 and the formulas (6.1), since in this case the function $u$ is continuous on the set $\{(t,x) \mid x - at = 0\}$, but by virtue of the condition (6.1), (5.1), and (5.2) the solution $u$ has continuous derivatives of the first order. □

**Remark 2.** If the given functions of problem (2.1) – (2.3) do not satisfy the homogeneous matching conditions (5.1) – (5.3), then the solution of problem (2.1) – (2.3) is reduced to solving the corresponding matching problem in which the matching conditions are given on the characteristic $x - at = 0$.

The conditions (6.1) can be taken for the matching conditions. Now problem (2.1) – (2.3) can be stated using the matching conditions (6.1) as follows.

**Problem (2.1) – (2.3) with matching conditions on characteristics.** *Find a classical solution of Eq. (2.1) with the Cauchy conditions (2.2), the boundary conditions (2.3), and the matching conditions (6.1).*

Note that such a statement of the problem in question with matching conditions is more acceptable for its numerical implementation.

## 7. Mild solution

Let's consider the problem (2.1) – (2.3) in the case when the functions $\mu$, $\varphi$, $\psi$, $\gamma$, $f$, and $F$ are not enough smooth.

**Definition 1.** *The function $u$ is a mild solution of the problem (2.1) – (2.3), if it is representable in the form (3.2) and (4.1).*

**Remark 3.** Obviously, any classical solution of the problem (2.1) – (2.3) is a mild solution of this problem too. In its turn, if a mild solution of problem (2.1) – (2.3) belongs to the class $C^2(\overline{Q})$, then it will be a classical solution of that problem.

**Theorem 7.** *Let the conditions $f \in C(\overline{Q} \times \mathbb{R})$, $F \in L_1^{\mathrm{loc}}(\overline{Q})$, $\varphi \in C([0,\infty))$, $\psi \in L_1^{\mathrm{loc}}([0,\infty))$, and $\mu \in C([0,\infty))$ be satisfied, and let the function $f$ satisfy the condition of Lipschitz–Carathéodory type with respect to the third variable. The first mixed problem (2.1) – (2.3) has a unique mild solution $u$ in the class $C(\widetilde{Q})$.*

*Proof.* The solvability of the Eqs. (4.1) and continuity of their solutions follows from Theorem 1. □

**Remark 4.** As in Theorem 2, in Theorem 7 we can take three conditions specified in Remark 1 instead of $f \in C(\overline{Q} \times \mathbb{R})$ too.

The conjugation conditions (6.1), generally speaking, are not satisfied for the mild solution. But it is possible to guarantee the fulfillment of the condition $[(u)^+ - (u)^-](t, x = at) = \varphi(0) - \mu(0)$ according to the representations (4.1).

## 8. Conclusion

The paper shows that a unique classical solution of problem (2.1) – (2.3) exists if and only if the smoothness conditions and the matching conditions are satisfied. But when matching conditions are not satisfied, we have considered a problem with conjugation conditions and built its classical solution. And in the case of insufficiently smooth given functions, we have constructed a unique mild solution.

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## Об авторах

**Корзюк Виктор Иванович**, д-р физ.-мат. наук, проф., Белорусский государственный университет, Беларусь, 220030, г. Минск, korzyuk@bsu.by

**Рудько Ян Вячеславович**, аспирант, магистр наук (математика и компьютерные науки), Институт математики Национальной академии наук Беларуси, Беларусь, 220072, г. Минск, janycz@yahoo.com, https://orcid.org/0000-0002-1482-9106

## About the authors

**Viktor I. Korzyuk**, Dr. Sci. (Phys.–Math.), Prof., Belarusian State University, Minsk, 220030, Belarus, korzyuk@bsu.by

**Jan V. Rudzko**, Postgraduate Student, Master's Degree (Mathematics and Computer Sciences), Institute of Mathematics of the National Academy of Sciences of Belarus, Minsk, 220072, Belarus, janycz@yahoo.com, https://orcid.org/0000-0002-1482-9106

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