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Optimal Control of Manipulator

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Abstract. When solving the problem of optimal performance for manipulative robots, the scientific team headed by F. L. Chernousko actively uses the Pontryagin maximum principle. The application of the maximum principle is complicated by the nonlinearities of controlled systems of manipulation robots. Therefore, when using it, the original mathematical model is replaced with a simpler one. These substitutions made it possible to analytically solve the problems of finding the switching points of relay controls for individual models of manipulation robots. In this paper, when finding the switching moments of relay controls for a manipulating robot, the original nonlinear controlled system is used. The problem is reduced to the problem of the existence of a solution to the boundary value problem for a controlled nonlinear system in the selected class of permissible controls that guarantee the arrival of the manipulator in the final position with zero speeds.

Keywords: optimal control, Pontryagin's maximum principle, manipulator

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Научная статья

Оптимальное управление манипулятором

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Аннотация. При решении задачи оптимального быстродействия для манипуляционных роботов в научном коллективе, возглавляемом Ф. Л. Черноусько, активно используется принцип максимума Понтрягина. Применение принципа максимума усложняется нелинейностями управляемых систем манипуляционных роботов. Поэтому при его использовании исходную математическую модель заменяют на более простую. Эти замены позволили аналитически решить задачи нахождения моментов переключения релейных управлений для отдельных моделей манипуляционных роботов. В настоящей работе при нахождении моментов переключения релейных управлений для манипуляционного робота используется исходная нелинейная управляемая система. Поставленная задача сводится к проблеме существования решения краевой задачи для управляемой нелинейной системы в выбранном классе допустимых управлений, гарантирующих приход манипулятора в конечное положение с нулевыми скоростями.

Ключевые слова: оптимальное управление, принцип максимума Понтрягина, манипулятор

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1. Introduction

The plane movement of an anthropomorphic manipulator is considered. The manipulator consists of a base 1, a rack 2, a vertically oriented shaft 3, a guide 4 rigidly connected to it, and a hand 5 with a grip. All elements of the manipulator are considered absolutely solid bodies. The arm can move progressively along the horizontal guide, while its center of mass moves along the axis of the guide 4, perpendicular to the axis of the shaft. We neglect the size of the grip and model its material point.

The plane movements of the robot are controlled by the moment M applied to the shaft axis and the horizontal force F applied to the arm. The force F and the moment M are created by two independent drives. The plane movements of the manipulator are described by generalized co-

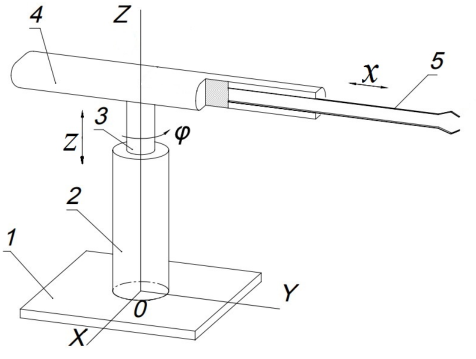

Figure 1. Manipulation Robot

ordinates: x is the coordinate of the center of mass of the arm, and φ is the angle of rotation of the arm.

Lagrange equations of the second kind have the form

$$\left(J_1 + J_2 + m_2 x^2\right) \ddot{\varphi} + 2 m_2 x \dot{x} \dot{\varphi} = M(t), \tag{1.1}$$

$$m_2 \left(\ddot{x} - x \dot{\varphi}^2\right) = F(t), \tag{1.2}$$

where m_2 is the mass of the manipulator arm, J_1 is the total moment of the shaft and the guiding arm relative to the axis OZ, J_2 is the moment of inertia of the arm relative to the vertical axis passing through the center of mass of the arm.

It is required to find the control $M(t)$ and $F(t)$ that translates the system (1.1), (1.2) from the initial position

$$\varphi(0) = 0, \quad \dot{\varphi}(0) = 0, \quad x(0) = x_0 > 0, \quad \dot{x}(0) = 0 \tag{1.3}$$

to the assigned final position

$$\varphi(T) = \varphi_T, \quad \dot{\varphi}(T) = 0, \quad x(T) = x_T > 0, \quad \dot{x}(T) = 0. \tag{1.4}$$

The following restrictions are imposed on the resources of the controls:

$$|M(t)| \leq M_0, \quad |F(t)| \leq F_0, \quad t \in (0, T].$$

The nonlinearity of the mathematical model of the manipulator complicates the application of the Pontryagin [11] maximum principle when solving the time-optimal control problem. In [3], a mathematical model is considered in which the equation (1.2) is replaced by a linear equation. For this model, an analytical method is proposed for finding relay controls that solve the problem of transferring the manipulator from the initial equilibrium position to the final position. These controls do not guarantee the arrival of the manipulator in the final position with zero speeds.

Several options for constructing such relay controls have been proposed. In this paper, we study the formulation of the optimal speed problem for the manipulator, in which it is guaranteed to arrive at the final position with zero speeds. In this formulation, no additional geometric restrictions are imposed on the movement of the manipulator. The problem of optimal performance for a mechanical object with geometric constraints was considered in [9]. If the manipulator comes to the final position with non-zero speeds, then additional impulse controls can be used to slow it down. In the [7; 8] it is shown that the transition of the manipulator from the initial to the final equilibrium position can be realized using only impulse controls. In [4], the question of the existence of optimal impulse controls for a two-link manipulator was investigated.

2. Formulation of the problem of optimal manipulator control

The problem of speed is considered, that is, the optimal controls $M = M^0$ and $F = F^0$ should provide the shortest time T^0 for the transition of the manipulator from the initial position to the final one. According to the Pontryagin maximum principle, optimal controls are relay [10]. To implement them, we will clarify the formulation of the management task. We will assume that the transition times of the manipulator from the initial position to the final position can be different in various generalized coordinates, namely T_φ and T_x, $T^0 = max(T_\varphi, T_x)$. In this case, the solutions of the managed system (1.1), (1.2) must meet the following conditions

$$\varphi(t) = \varphi_T, \quad t \in [T_\varphi, T^0], \text{ when } T_\varphi < T^0,$$

or

$$x(t) = x_T, \quad t \in [T_x, T^0], \text{ when } T_x < T^0.$$

The latter conditions can be provided by choosing controls for which equality takes place:

$$M^0(t) = 0, \quad t \in [T_\varphi, T^0], \text{ when } T_\varphi < T^0,$$

or

$$F^0(t) = -m_2 x_T \dot{\varphi}^2, \quad t \in [T_x, T^0], \text{ when } T_x < T^0.$$

Indeed, with $T_\varphi < T^0$, the function $\varphi(t), t \in [T_\varphi, T^0]$, according to (1.1), is the solution of the differential equation

$$\frac{d}{dt}\left((J_1 + J_2 + m_2 x^2)\dot{\varphi}\right) = 0$$

with initial conditions $\varphi(T_\varphi) = \varphi_T$, $\dot{\varphi}(T_\varphi) = 0$. For $T_x < T^0$, the function $x(t)$, $t \in [T_x, T^0]$, according to (1.2), is the solution of the differential equation

$$m_2\ddot{x} - m_2(x - x_T)\dot{\varphi}^2(t) = 0$$

with initial conditions $x(T_x) = x_T$, $\dot{x}(T_x) = 0$.

Let's limit ourselves to studying controls with one switching point t_φ at the generalized coordinate φ and with one switching point at the generalized coordinate x.

The set of controls under consideration is described by formulas

$$M^0(t) = \begin{cases} M_0, & 0 \le t < t_\varphi, \\ -M_0 & t_\varphi \le t < T_\varphi, \\ 0, & T_\varphi \le t \le T^0, \end{cases} \tag{2.1}$$

$$F^0(t) = \begin{cases} F_0, & 0 \le t < t_x, \\ -F_0 & t_x \le t < T_x, \\ -m_2 x_T \dot{\varphi}^2(t), & T_x \le t \le T^0. \end{cases} \tag{2.2}$$

According to the maximum principle [10], optimal control of the performance task for the manipulator belong to the set (2.1), (2.2), while for the parameters t_φ, T_φ, t_x, T_x is required existence of a solution to the boundary value problem (1.1), (1.2), (1.3), (1.4) with controls defined by formulas (2.1), (2.2).

The purpose of this work is to find unknown parameters of controls t_φ, T_φ, t_x, T_x. Other methods for finding switching moments in time-optimal control problems were proposed in [1; 2; 5; 6] for mechanical objects other than the manipulator considered in this paper.

3. A model of a manipulator with a slow rotation of the arm

In [3], a simplified model of the manipulator is proposed, describing approximately its movements with a slow rotation of the arm. The corresponding system of differential equations has the form

$$\frac{d}{dt}\left((J_1 + J_2 + m_2 x^2)\dot{\varphi}\right) = M(t), \tag{3.1}$$

$$m_2 \ddot{x} = F(t). \tag{3.2}$$

For this model, you need to change a lot of controls (2.2), assuming

$$F^0(t) = \begin{cases} F_0, & 0 \le t < t_x, \\ -F_0 & t_x \le t < T_x, \\ 0, & T_x \le t \le T^0. \end{cases} \tag{3.3}$$

Optimal controls of the speed problem for a manipulator with a slow rotation of the arm are found by examining the solvability condition of the boundary value problem (3.1), (3.2), (1.3), (1.4) with controls defined by formulas (2.1), (3.3).

Theorem 1. *Optimal control parameters (2.1), (3.3) the speed-in-action problem of system (3.1), (3.2) are determined by formulas*

$$t_\varphi = T_\varphi/2, \quad T_x = 2\sqrt{\frac{m_2(x_T - x_0)}{F_0}} = 2\,t_x,$$

where T_φ *is the root of the following equation*

$$\int_0^{\frac{T_\varphi}{2}} \frac{s\,ds}{J_1 + J_2 + m_2\,x^2(s)} + \int_{\frac{T_\varphi}{2}}^{T_\varphi} \frac{(T_\varphi - s)\,ds}{J_1 + J_2 + m_2\,x^2(s)} = \frac{\varphi_T}{M_0},$$

and $x(t)$ *changes according to the following formula*

$$x(t) = \begin{cases} x_0 + \frac{F_0 t^2}{2\,m_2}, & 0 \le t < T_x/2, \\ x_0 - \frac{F_0\,T_x^2}{4\,m_2} + \frac{F_0\,T_x\,t}{m_2} - \frac{F_0\,t^2}{2\,m_2}, & T_x/2 \le t < T_x, \\ x_T, & T_x \le t \le T^0. \end{cases} \tag{3.4}$$

Proof. We find the conditions for the existence of a solution to the boundary value problem for the equation (3.2) with controls (3.3) and boundary conditions

$$x(0) = x_0, \quad \dot{x}(0) = 0, \quad x(T_x) = x_T, \quad \dot{x}(T_x) = 0.$$

We have

$$x(t) = \begin{cases} x_0 + \frac{F_0\,t^2}{2\,m_2}, & 0 \le t < t_x, \\ x(t_x) + \dot{x}(t_x)(t - t_x) - \frac{F_0\,(t-t_x)^2}{2\,m_2} & t_x \le t < T_x, \\ x_T, & T_x \le t \le T^0. \end{cases}$$

From the boundary condition $\dot{x}(T_x) = 0$ we find $T_x = 2\,t_x$, and from the boundary condition $x(T_x) = x_T$ we have $T_x = 2\sqrt{\frac{m_2\,(x_T - x_0)}{F_0}}$. As a result, the function $x(t)$ is described by the formula (3.4).

We proceed to finding the conditions for the existence of a solution of the boundary value problem for the equation (3.1) with control (2.1) and boundary conditions

$$\varphi(0) = 0, \quad \dot{\varphi}(0) = 0, \quad \varphi(T_\varphi) = \varphi_T, \quad \dot{\varphi}(T_\varphi) = 0.$$

Taking into account the boundary condition $\dot{\varphi}(0) = 0$, we have

$$\dot{\varphi}(t) = \begin{cases} \frac{M_0\,t}{J_1 + J_2 + m_2\,x^2(t)}, & 0 \le t < t_\varphi, \\ \frac{M_0\,(2\,t_\varphi - t)}{J_1 + J_2 + m_2\,x^2(t)}, & t_\varphi \le t < T_\varphi, \\ \frac{M_0\,(2\,t_\varphi - T_\varphi)}{J_1 + J_2 + m_2\,x^2(t)}, & T_\varphi \le t \le T^0. \end{cases}$$

Using the condition $\dot\varphi(T_\varphi) = 0$, we find $T_\varphi = 2\,t_\varphi$. Taking into account this equality and the boundary condition $\varphi(T_\varphi) = \varphi_T$, we get the function

$$\varphi(t) = \begin{cases} \varphi_T + \int_{T_\varphi}^{t} \frac{M_0\,\alpha(s,T_\varphi)\,d\,s}{J_1+J_2+m_2\,x^2(s)}, & 0 \le t < T_\varphi, \\ \varphi_T, & T_\varphi \le t \le T^0, \end{cases}$$

where

$$\alpha(t,T_\varphi) = \begin{cases} t, & 0 \le t < T_\varphi/2, \\ T_\varphi - t, & T_\varphi/2 \le t \le T_\varphi. \end{cases}$$

Using the boundary condition $\varphi(0) = 0$, we have the equality

$$\Phi(T_\varphi) = \int_0^{T_\varphi} \frac{\alpha(s,T_\varphi)\,ds}{J_1 + J_2 + m_2\,x^2(s)} = \frac{\varphi_T}{M_0}. \tag{3.5}$$

Derivative of the $\Phi(T_\varphi)$ function

$$\Phi'(T_\varphi) = \int_{\frac{T_\varphi}{2}}^{T_\varphi} \frac{ds}{J_1 + J_2 + m_2\,x^2(s)} > 0, \quad 0 < T_\varphi < +\infty.$$

Here, in the definition of the function Φ, the following continuation of the function $x(t) = x_T$ is used for $T_x < t < +\infty$.

We have $\Phi(0) = 0$, $\lim_{T_\varphi \to +\infty} \Phi(T_\varphi) = +\infty$. Therefore, the equation (3.5) has a unique solution. □

The integral included in the definition of the function $\Phi(T_\varphi)$ can be calculated analytically. For $T_\varphi \le T_x/2$, the analytical representation of this function has the form

$$\Phi(T_\varphi) = \frac{1}{F_0\sqrt{J}}\,\mathrm{arctg}\,\frac{J\Big(J\big(x_0 - x(T_\varphi)\big)/2 + x(T_\varphi)\big(x^2(T_\varphi) + 3\,x_0^2\big)/4\Big)}{J^2 + J\big(x^2(T_\varphi/2) + x(T_\varphi)\,x_0\big) + x(T_\varphi)\,x^2(T_\varphi/2)\,x_0} + \\ + \frac{T_\varphi}{4\,a\sqrt{m_2\,b}} \times \\ \times \ln \frac{\big(x(T_\varphi) + x_0\big)^2 + a\,\big(x(T_\varphi) - x_0\big) + 4\,J - T_\varphi\sqrt{\frac{b}{m_2}}\big(x(T_\varphi) - x_0 - 2\,a\big)}{\big(x(T_\varphi) + x_0\big)^2 + a\,\big(x(T_\varphi) - x_0\big) + 4\,J + T_\varphi\sqrt{\frac{b}{m_2}}\big(x(T_\varphi) - x_0 - 2\,a\big)} + \\ + \frac{T_\varphi}{2\,a\sqrt{m_2\,b}}\,\mathrm{arctg}\,\frac{4\,F_0\,T_\varphi\Big(2\,m_2\big(x(T_\varphi) - x_0\big) + \sqrt{m_2\,b} + b\Big)}{4\Big(2\,m_2\big(x(T_\varphi) - x_0\big) + \sqrt{m_2\,b} + b\Big)^2 - F_0^2\,T_\varphi^2\big(1 + \frac{9\,b}{m_2}\big)},$$

where $J = (J_1 + J_2)/2$, $a = \sqrt{J + x_0^2}$, $b = F_0(a - x_0)$, and $x(T_\varphi)$, $x(T_\varphi/2)$ are defined by formulas

$$x(T_\varphi) = x_0 + \frac{F_0\,T_\varphi^2}{2\,m_2}, \quad x(T_\varphi/2) = x_0 + \frac{F_0\,T_\varphi^2}{8\,m_2}.$$

4. Determination of optimal control parameters for the manipulator without restriction on the nature of movements

We proceed to finding the solvability conditions of the boundary value problem (1.1), (1.2), (1.3), (1.4) with controls defined by formulas (2.1), (2.2).

Theorem 2. *If the following condition is met*

$$x(T_\varphi, T_\varphi) + \frac{m_2\, \dot{x}^2(T_\varphi, T_\varphi)}{2\, F_0} \leq x_T, \tag{4.1}$$

the parameters of optimal controls (2.1), (2.2) are determined by the formulas $t_\varphi = T_\varphi/2$,

$$t_x = T_\varphi + \frac{m_2}{F_0}\left(\sqrt{\frac{\dot{x}^2(T_\varphi, T_\varphi)}{2} + \frac{F_0}{m_2}\big(x_T - x(T_\varphi, T_\varphi)\big)} - \dot{x}(T_\varphi, T_\varphi)\right), \tag{4.2}$$

$$T_x = T_\varphi + \frac{m_2}{F_0}\left(2\sqrt{\frac{\dot{x}^2(T_\varphi, T_\varphi)}{2} + \frac{F_0}{m_2}\big(x_T - x(T_\varphi, T_\varphi)\big)} - \dot{x}(T_\varphi, T_\varphi)\right). \tag{4.3}$$

The parameter T_φ *is the solution of the equation*

$$\Phi(T_\varphi) = \int_0^{T_\varphi} \frac{\alpha(s, T_\varphi)\, ds}{J_1 + J_2 + m_2\, x^2(s, T_\varphi)} = \frac{\varphi_T}{M_0}, \tag{4.4}$$

and the function $x(t, T_\varphi)$, $0 \leq t \leq T_\varphi$ *— solution of the differential equation*

$$\ddot{x} - \frac{x\, M_0^2\, \alpha^2(t, T_\varphi)}{\big(J_1 + J_2 + m_2 x^2(t)\big)^2} = \frac{F_0}{m_2} \tag{4.5}$$

with initial conditions $x(0) = x_0$, $\dot{x}(0) = 0$.

Proof. Let $T_\varphi \leq t_x < T_x$. We find the solvability conditions of the boundary value problem (1.1), (1.2), (1.3), (1.4) with controls defined by the formulas (2.1), (2.2).

For the equation (1.2) on the section $[T_\varphi, T_x]$ we have the boundary value problem

$$\ddot{x} = \begin{cases} \frac{F_0}{m_2}, & T_\varphi \leq t \leq t_x, \\ -\frac{F_0}{m_2}, & t_x < t \leq T_x, \end{cases} \tag{4.6}$$

$$x(T_\varphi) = x(T_\varphi, T_\varphi), \quad \dot{x}(T_\varphi) = \dot{x}(T_\varphi, T_\varphi), \tag{4.7}$$

$$x(T_x) = x_T, \quad \dot{x}(T_x) = 0, \tag{4.8}$$

where $x(\cdot, T_\varphi)$ is the solution of the equation (4.5). For the equation (4.6), the solution of the Cauchy problem with initial conditions (4.8) is determined by the formulas

$$x(t) = x_T - \frac{F_0(T_x - t)^2}{2\,m_2}, \quad t_x < t \le T_x, \tag{4.9}$$

$$x(t) = x_T - \frac{F_0(T_x - t_x)^2}{2\,m_2} + \frac{F_0(T_x - t_x)(t - t_x)}{m_2} + \frac{F_0(t - t_x)^2}{2\,m_2}, \quad T_\varphi \le t \le t_x.$$

Given the boundary conditions (4.7), we obtain the equations

$$\dot{x}(T_\varphi, T_\varphi) = \frac{F_0(T_x - t_x)}{m_2} + \frac{F_0(T_\varphi - t_x)}{m_2},$$

$$x(T_\varphi, T_\varphi) = x_T - \frac{F_0(T_x - t_x)^2}{2\,m_2} + \frac{F_0(T_x - t_x)(T_\varphi - t_x)}{m_2} + \frac{F_0(T_\varphi - t_x)^2}{2\,m_2}.$$

From the first equation we find

$$t_x = \frac{1}{2}\left(T_x + T_\varphi - \frac{m_2\,\dot{x}(T_\varphi, T_\varphi)}{F_0}\right). \tag{4.10}$$

Taking into account this equality, the second equation is transformed to the following form

$$(T_x - T_\varphi)^2 + \frac{2\,m_2}{F_0}\dot{x}(T_\varphi, T_\varphi)(T_x - T_\varphi) + \frac{4\,m_2}{F_0}\big(x(T_\varphi, T_\varphi) - x_T\big) - \frac{m_2^2}{F_0^2}\dot{x}^2(T_\varphi, T_\varphi) = 0.$$

When the condition (4.1) is met, the last equation has a unique solution satisfying the requirement $T_x > T_\varphi$ and defined by the formula (4.3). From the equality (4.10) with (4.3), we obtain the formula (4.2). When the condition (4.1) is met, the condition $t_x > T_\varphi$ is satisfied, as well as $T^0 = T_x$.

Consider a boundary value problem for the equation (1.1) with boundary conditions

$$\varphi(0) = 0, \quad \dot{\varphi}(0) = 0, \quad \varphi(T_\varphi) = \varphi_T, \quad \dot{\varphi}(T_\varphi) = 0, \tag{4.11}$$

and control defined by the formula (2.1). For the equation (1.1) there is the first integral

$$\dot{\varphi} = \frac{M_0\alpha(t, T_\varphi)}{J_1 + J_2 + m_2 x^2}, \tag{4.12}$$

where

$$\alpha(t, T_\varphi) = \begin{cases} t, & 0 \le t < t_\varphi, \\ 2\,t_\varphi - t, & t_\varphi \le t \le T_\varphi. \end{cases}$$

Taking into account the boundary condition $\dot\varphi(T_\varphi)=0$, we have $T_\varphi = 2\,t_\varphi$.

On the section $[0,T_\varphi]$, the equation (1.2) is transformed to the form (4.5). Its solution is $x(t,T_\varphi)$, $0\le t\le T_\varphi$, with initial conditions $x(0)=x_0$, $\dot x(0)=0$, as well as the solution (4.9) of the equation (4.6), with parameters set by the formulas (4.2), (4.3), determines the solution of the equation (1.2) on the section $[0,T_x]$ with boundary conditions

$$x(0)=x_0,\quad \dot x(0)=0,\quad x(T_x)=x_T,\quad \dot x(T_x)=0.$$

Using (4.12), we find that the solution of the equation (1.1) on the section $[0,T_\varphi]$ with boundary conditions (4.11) is determined by the formula

$$\varphi(t)=\int_0^t \frac{M_0\,\alpha(s,T_\varphi)\,ds}{J_1+J_2+m_2\,x^2(s,T_\varphi)},\quad t\in[0,T_\varphi],$$

where the parameter T_φ is the solution of equation (4.4). □

When finding the parameter T_φ, another representation of the function $\Phi(T_\varphi)$ can be used, using the properties of the solution $x(t,T_\varphi)$, $0\le t\le T_\varphi$, of the differential equation (4.5). We have

$$\frac{d\,\dot x^2(t,T_\varphi)}{d\,t}+\frac{M_0^2\,\alpha^2(t,T_\varphi)}{m_2}\frac{d}{d\,t}\frac{1}{y(t,T_\varphi)}=\frac{2\,F_0}{m_2}\frac{d\,x(t,T_\varphi)}{d\,t}, \tag{4.13}$$

where $y(t,T_\varphi)=J_1+J_2+m_2x^2(t,T_\varphi)$, $0\le t\le T_\varphi$. Using the definition of the function α and the independence of the solution of the equation (4.5) from the parameter T_φ on the section $[0,T_\varphi/2]$ $(x(t,T_\varphi)=x(t,0)=x(t), y(t,T_\varphi)=y(t))$ we get

$$\dot x^2(T_\varphi/2)+\frac{M_0^2}{m_2}\int_0^{T_\varphi} s^2\frac{d}{ds}\frac{1}{y(s)}\,ds=\frac{2\,F_0}{m_2}\Big(x(T_\varphi/2)-x_0\Big),$$

$$\dot x^2(T_\varphi,T_\varphi)-\dot x^2(T_\varphi/2)+\frac{M_0^2}{m_2}\int_{\frac{T_\varphi}{2}}^{T_\varphi}(T_\varphi-s)^2\frac{d}{ds}\frac{1}{y(s,T_\varphi)}\,ds= \\ =\frac{2\,F_0}{m_2}\Big(x(T_\varphi,T_\varphi)-x(T_\varphi/2)\Big).$$

Partial integration formulas are used to transform integrals

$$\int_0^{\frac{T_\varphi}{2}} s^2\frac{d}{ds}\frac{1}{y(s)}\,ds=\frac{T_\varphi^2}{4\,y(T_\varphi/2)}-2\int_0^{\frac{T_\varphi}{2}}\frac{s}{y(s)}\,ds,$$

$$\int_{\frac{T_\varphi}{2}}^{T_\varphi}(T_\varphi-s)^2\frac{d}{ds}\frac{1}{y(s,T_\varphi)}\,ds=-\frac{T_\varphi^2}{4\,y(T_\varphi/2)}+2\int_{\frac{T_\varphi}{2}}^{T_\varphi}\frac{T_\varphi-s}{y(s,T_\varphi)}\,ds.$$

From the definition of the function Φ

$$\Phi(T_\varphi) = \int_0^{\frac{T_\varphi}{2}} \frac{s\,ds}{y(s)} + \int_{\frac{T_\varphi}{2}}^{T_\varphi} \frac{T_\varphi - s}{y(s, T_\varphi)}\,ds.$$

As a result, we find

$$\Phi(T_\varphi) = \frac{T_\varphi^2}{4\big(J_1 + J_2 + m_2\,x^2(T_\varphi/2)\big)} + \frac{F_0}{M_0^2}\Big(x(T_\varphi, T_\varphi) + x_0 - 2x(T_\varphi/2)\Big) + \\ + \frac{m_2}{M_0^2}\Big(\dot{x}^2(T_\varphi/2) - \frac{1}{2}\dot{x}^2(T_\varphi, T_\varphi)\Big).$$

for the parameters $x(T_\varphi/2)$, $y(T_\varphi/2)$, the following equalities are fulfilled

$$\frac{x(T_\varphi/2)}{d\,T_\varphi} = \frac{1}{2}\dot{x}(T_\varphi/2), \qquad \frac{d\,\dot{x}(T_\varphi/2)}{d\,T_\varphi} = \frac{1}{2}\ddot{x}(T_\varphi/2),$$

$$\frac{d\,y(T_\varphi/2)}{d\,T_\varphi} = m_2\,x(T_\varphi/2)\ddot{x}(T_\varphi/2), \quad \ddot{x}(T_\varphi/2) = \frac{F_0}{m_2} + \frac{M_0^2\,T_\varphi^2\,x(T_\varphi/2)}{4\,y^2(T_\varphi/2)}.$$

Using the equalities above, as a result we get

$$\frac{d\,\Phi(T_\varphi)}{d\,T_\varphi} = \frac{T_\varphi}{2\big(J_1 + J_2 + m_2\,x^2(T_\varphi/2)\big)} + \\ + \frac{F_0}{M_0^2}\frac{d}{d\,T_\varphi}\Big(x(T_\varphi, T_\varphi) - \frac{m_2}{2\,F_0}\dot{x}^2(T_\varphi, T_\varphi)\Big).$$

Restoring the derivative of the function

$$\Phi(T_\varphi) = 2\int_0^{\frac{T_\varphi}{2}} \frac{s\,ds}{J_1 + J_2 + m_2\,x^2(s)} + \frac{F_0}{M_0^2}\Big(x(T_\varphi, T_\varphi) - \frac{m_2}{2\,F_0}\dot{x}^2(T_\varphi, T_\varphi)\Big).$$

Theorem 3. *Let $x(t, t_x, T_\varphi)$, $0 \le t \le T_\varphi$, — solution of the differential equation*

$$\ddot{x} - \frac{x\,M_0^2\,\alpha^2(t, T_\varphi)}{\big(J_1 + J_2 + m_2 x^2\big)^2} = \frac{1}{m_2}\,F(t) \tag{4.14}$$

with initial conditions $x(0) = x_0$, $\dot{x}(0) = 0$, satisfying the condition $\dot{x}(T_\varphi, t_x, T_\varphi) > 0$. Then the parameters of optimal controls (2.1), (2.2) are determined by the equations $t_\varphi = T_\varphi/2$,

$$T_x = T_\varphi + \frac{m_2}{F_0}\dot{x}(T_\varphi, t_x, T_\varphi), \tag{4.15}$$

$$x(T_\varphi, t_x, T_\varphi) + \frac{m_2\,\dot{x}^2(T_\varphi, t_x, T_\varphi)}{2\,F_0} = x_T, \tag{4.16}$$

$$\int_0^{T_\varphi} \frac{\alpha(s, T_\varphi)\,ds}{J_1 + J_2 + m_2\,x^2(s, t_x, T_\varphi)} = \frac{\varphi_T}{M_0}. \tag{4.17}$$

Proof. For $T_\varphi \leq t \leq T_x$, the solution of the equation (1.2) is determined by the formula

$$x(t) = x_T - \frac{F_0\,(t - T_x)^2}{2\,m_2}.$$

Its continuous conjugation with the solution $x(t, t_x, T_\varphi)$, $0 \leq t \leq T_\varphi$, defines the equations (4.15) and (4.16). The change over time of the generalized coordinate φ is determined by the solution of the equation

$$\dot{\varphi} = \frac{M_0\,\alpha(t, T_\varphi)}{J_1 + J_2 + m_2\,x^2(t, t_x, T_\varphi)}.$$

It satisfies the required boundary conditions if $t_\varphi = T_\varphi/2$ and the equation (4.17) is satisfied. □

5. Numerical simulation

When numerically modeling the movement of the manipulator, the following values of the parameters of the mechanical system were used: $J_1 = 3$, $J_2 = 2$, $m_2 = 1$. The problem of transferring the center of mass by the manipulator from the initial position ($x_0 = 0.25$, $\varphi_0 = 0$) to the final position ($x_T = 1.5$, $\varphi_T = \pi/6$) was considered.

In fig. 2 a graph of the change in the position of the grip $x(t)$ of the manipulator is given. The purple thick line shows the change in the function $x(t)$ at the values of the control parameters $M_0 = 4$, $F_0 = 0.25$. The red thick line shows the change in the function $x(t)$ at the values of the control parameters $M_0 = 8$, $F_0 = 0.25$. Both cases correspond to the conditions of the theorem 2. The purple dashed line shows the change in the function $x(t)$ at the values of the control parameters $M_0 = 4$, $F_0 = 0.25$, with the assumption of slow rotation of the hand and fulfillment of the conditions of the theorem 1. The red dashed line shows the change in the function $x(t)$ at the values of the control parameters $M_0 = 8$, $F_0 = 0.25$, assuming slow rotation of the hand and fulfillment of the conditions of the theorem 1. The figure shows that using the results of the theorem 1 leads to a noticeable error in reaching the final position. The table 1 shows the obtained values of the control switching moments. It can be seen from the results that the movement occurs faster when building controls in accordance with the theorem 2.

If the conditions of the theorem 3 are met, there is also a noticeable difference in the results of the controls constructed in accordance with the theorem 3 and the theorem 1.

Fig. 3 shows a graph of the function $\Phi(T_\varphi)$. The purple thick line shows the change in the function $\Phi(T_\varphi)$ at the values of the control parameters $M_0 = 4$, $F_0 = 0.25$. The red thick line shows the change in the function $\Phi(T_\varphi)$ at the values of the control parameters $M_0 = 8$, $F_0 = 0.25$. The figure shows that the function $\Phi(T_\varphi)$ is monotonic, regardless of the control values. The function $\Phi(T_\varphi)$ has no significant differences when the conditions of theorem 1 or theorem 2 are met.

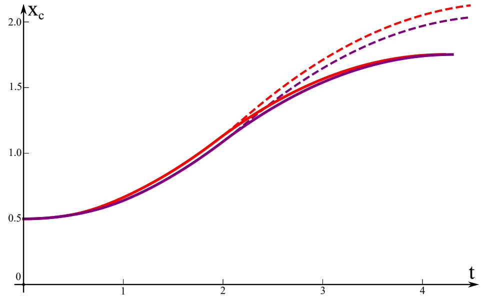

Figure 2. Coordinate of the grip $x(t)$ for different values of controls

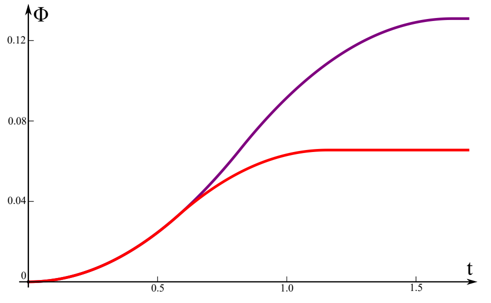

Figure 3. Function $\Phi(t)$ for different values of controls

Table 1

Numerical simulation results

Line	F_0	M_0	t_φ	T_φ	t_x	T_x
Purple Thick	0.25	4	0.82	1.64	1.99	4.3
Red Thick	0.25	8	0.58	1.16	1.92	4.22
Purple Dashed	0.25	4	0.82	1.64	2.24	4.47
Red Dashed	0.25	8	0.58	1.15	2.24	4.47

6. Conclusion

The design of the manipulator allows translational and rotational movements of its components. Relay controls are used to solve the nonlinear time-optimal problem. Equations for finding unknown parameters of optimal controls are obtained. The conditions of their solvability are studied.

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