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Sections of the Generating Series of a Solution to a Difference Equation in a Simplicial Cone

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Abstract. We consider a multidimensional difference equation in a simplicial lattice cone with coefficients from a field of characteristic zero and sections of a generating series of a solution to the Cauchy problem for such equations. We use properties of the shift and projection operators on the integer lattice $\mathbb{Z}^n$ to find a recurrence relation (difference equation with polynomial coefficients) for the section of the generating series. This formula allows us to find a generating series of a solution to the Cauchy problem in the lattice cone through a generating series of its initial data and a right-side function of the difference equation. We derived an integral representation for sections of the holomorphic function, whose coefficients satisfy the difference equation with complex coefficients. Finally, we propose a system of differential equations for sections that represent D-finite functions of two complex variables.

Keywords: generating series, difference equation, lattice cone, Stanley hierarchy, section

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Научная статья

Сечения производящего ряда решения разностного уравнения в симплициальном конусе

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Аннотация. Рассмотрено многомерное разностное уравнение в симплициальном решеточном конусе с коэффициентами из поля характеристики ноль и сечения производящего ряда решения задачи Коши для таких уравнений. Использованы свойства операторов сдвига и проекции на целочисленной решетке $\mathbb{Z}^n$, чтобы найти рекуррентное соотношение (разностное уравнение с полиномиальными коэффициентами) для сечения производящего ряда. Эта формула позволяет найти производящий ряд решения задачи Коши в решеточном конусе через производящий ряд его начальных данных и функцию в правой части разностного уравнения. Получено интегральное представление сечений голоморфной функции, коэффициенты которой удовлетворяют разностному уравнению с комплексными коэффициентами. Предложена система дифференциальных уравнений для сечений, представляющих D-конечные функции двух комплексных переменных.

Ключевые слова: производящий ряд, разностное уравнение, решеточный конус, иерархия Стенли, сечение

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1. Introduction

The notion of a section of a power series $F(z)$ appeared in [9] in connection with studying properties of D-finite power series in a subset $\mathbb{Z}^n_{\geqslant}$ of vectors with non-negative integer coordinates. In particular, it was proven that sections of D-finite power series are D-finite.

In the case of lattice paths problem (see, for example, [2] or [3]), sections represent a generating series of the number of paths ending on the hyperplane, parallel to the coordinate axis. In [1], $(n-1)$-dimensional sections of n-dimensional generating functions of numbers of lattice paths in $\mathbb{Z}^n_{\geqslant}$ were studied. In [11], k-dimensional sections, $k \leqslant n$, were considered in $\mathbb{Z}^n_{\geqslant}$ and a recurrence relation for such sections was derived, which lead to a simple algorithm for their computation. Additionally, it was shown that sections of the generating series in the lattice paths problem could represent famous recurrence polynomials, dependent on the choice of steps (see e.g. [10]).

Sections of a generating series $F(z)$ of a solution to a difference equation in a simplicial lattice cone were considered in [6] to prove a condition for $F(z)$ belonging to one of the classes of Stanley hierarchy: rational $\subset$ algebraic $\subset$ D-finite. Namely, it was proved that the generating series of the initial data function and solution to the Cauchy problem belong to the same class of the hierarchy.

If $F(z)$ is a holomorphic function, then there is an integral formula for coefficients of its power series (see, for example, [20]). Various integral representations for diagonals of power series were proved in [14–17], see also [4]. The integral representation for the generating function of a solution to the Cauchy problem and the generating function of its initial data was proved in [5; 7; 8].

This paper is dedicated to the sections of a generating series in a simplicial lattice cone. In §2, we give a definition of a section of a generating series of a solution to the multidimensional difference equation. In §3, we derive a difference equation (Theorem 1) and integral representation (Theorem 2) for the section of generating series of the Cauchy problem in a lattice cone. In §4, a simple example for Theorem 1 is given, and recurrence formulas for its section are derived. Consequently, a system of differential equations for such sections is derived in Corollary 1.

2. Definitions and notations

Let $\mathcal{F}$ be a field of characteristic zero and $\mathbb{Z}$, $\mathbb{Z}_{\geqslant}$ denote the integers and the non-negative integers, respectively, $\mathbb{Z}^n = \mathbb{Z}\times\ldots\times\mathbb{Z}$, $\mathbb{Z}^n_{\geqslant} = \mathbb{Z}_{\geqslant}\times\ldots\times\mathbb{Z}_{\geqslant}$ and $f, g : \mathbb{Z}^n \to \mathcal{F}$. Let $\Delta = \{\alpha^1, \alpha^2, \ldots, \alpha^N\} \subset \mathbb{Z}^n$ be a set of N vector-columns $\alpha^j = (\alpha^j_1, \ldots, \alpha^j_n)^\top, j = 1, \ldots, N$. Let K be a lattice cone spanned by vectors from Δ as follows:

$$K = \{\lambda \in \mathbb{Z}^n : \lambda = \lambda_1\alpha^1 + \cdots + \lambda_N\alpha^N, \lambda_1, \ldots, \lambda_N \in \mathbb{Z}_{\geqslant}\}.$$

We consider only simplicial lattice cones, see Fig. 1. A lattice cone K is called simplicial if each element of K is uniquely represented by vectors of Δ, which implies that $N \leqslant n$ and the vectors in Δ are linearly independent. The simplicial lattice cone K is always pointed which means that the rational cone

$$S = \{x \in \mathbb{R}^n : x = x_1\alpha^1 + \cdots + x_N\alpha^N, x_1, \ldots, x_N \in \mathbb{R}_+\}$$

does not contain any line.

We define a linear difference equation with coefficients $c_j \in \mathcal{F}, j = 0, \ldots, N$, by

$$c_0 f(\lambda) + c_1 f(\lambda - \alpha^1) + \ldots + c_N f(\lambda - \alpha^N) = g(\lambda), \quad \lambda \in K + m, \quad (2.1)$$

where $m = \alpha^1 + \cdots + \alpha^N$.

The Cauchy problem is to find function $f(\lambda)$, satisfying equation 2.1, which coincides with given initial data function $\varphi : \mathbb{Z}^n \to \mathcal{F}$ on the set $X_m = K \setminus (m + K)$, i.e.:

$$f(\lambda) = \varphi(\lambda), \lambda \in X_m. \tag{2.2}$$

Let $\mathbb{C}$ be the field of complex numbers, $\mathbb{C}^n = \mathbb{C}\times\ldots\times\mathbb{C}$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$. We define the generating series of $f(\lambda) = f(\lambda_1, \ldots, \lambda_n)$ in the cone K by:

$$F(z) = \sum_{\lambda \in K} f(\lambda) z^\lambda. \tag{2.3}$$

The collection of such series forms a ring $\mathcal{F}_K[[z]]$ with operations of sum and product.

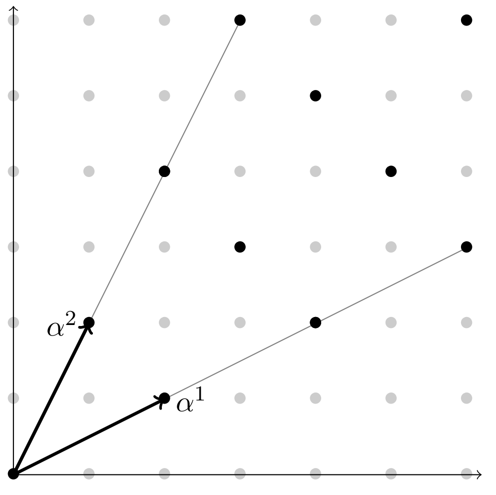

Figure 1. Cone $K \subset \mathbb{Z} \times \mathbb{Z}$ spanned by $\alpha^1 = (2, 1)^\top$ and $\alpha^2 = (1, 2)^\top$.

Let $\delta_j : \mathbb{Z}^n \to \mathbb{Z}^n, j = 1, \ldots, N$, be a shift operator defined by

$$\delta_j : \lambda \longmapsto \lambda + e^j,$$

where e^j is a unit vector with 1 only on jth position, then define $\delta^{\alpha^j} := \delta_1^{\alpha_1^j} \cdots \delta_n^{\alpha_n^j}$.

For a polynomial difference operator $Q(\delta) = Q(\delta_1, \ldots, \delta_N) = c_0 + c_1\delta^{\alpha^1} + \cdots + c_N\delta^{\alpha^N}$, we write difference equation 2.1 as

$$Q(\delta^{-1}) f(\lambda) = g(\lambda), \lambda \in m + K.$$

Let cones K_p and L_p, $p = 1, \ldots, N$, be spanned by vectors from $\Delta_1 = \{\alpha^1, \ldots, \alpha^p\}$ and $\Delta_2 = \{\alpha^{p+1}, \ldots, \alpha^N\}$, respectively. Since vectors in the

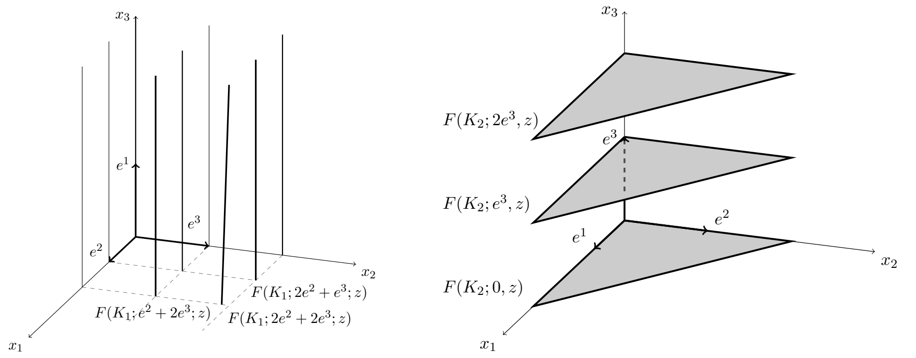

Figure 2. Sections in $K = \mathbb{Z}^3$ when (left) $K_1 = \langle e^1 \rangle$ and $L_1 = \langle e^2, e^3 \rangle$ and (right) $K_2 = \langle e^1, e^2 \rangle$ and $L_2 = \langle e^3 \rangle$.

set Δ are linearly independent, each element λ of the cone K can be represented uniquely as a sum of elements x and y from the cones L_p and K_p by

$$K \ni \lambda = x + y, \quad x \in L_p, \quad y \in K_p.$$

Consequently, the generating series 2.3 can be represented as the sum

$$F(z) = \sum_{x \in L_p} F(K_p; x; z)\, z^x, \quad F(K_p; x; z) := \sum_{y \in K_p} f(x+y) z^y.$$

We call $F(K_p; x; z), x \in L_p$, a section of generating series $F(z)$. If $\Delta = \{e^1, \ldots, e^N\}$ is the standard basis and $\mathcal{F} = \mathbb{C}$, we get the sections considered by L. Lipshitz in [9], see Fig. 2. Sometimes we use the notation

$$F(\langle \alpha_{j_1}, \ldots, \alpha_{j_s} \rangle; x; z) = \sum_{y \in \langle \alpha_{j_1}, \ldots, \alpha_{j_s} \rangle} f(x+y) z^y,$$

if the cone K is spanned by vectors $\{\alpha_{j_1}, \ldots, \alpha_{j_s}\} \subset \Delta$, $1 \leqslant j_1 < \ldots < j_s \leqslant N, s \leqslant N$. We note that $F(K; 0; z) = F(z)$ and $F(\varnothing; x; z) = f(x)$ for $x \in K$.

Let $\pi_j : K \to K, j = 1, \ldots, N$, be the projection operator

$$\pi_j \alpha^i = \begin{cases} 0, & i = j; \\ \alpha^i, & i \neq j \end{cases}$$

for $1 \leqslant i \leqslant N$. If $\lambda = x_1 \alpha^1 + \ldots + x_N \alpha^N$, then $\pi_j \lambda = \lambda - x_j \alpha^j$, $j = 1, \ldots, N$. For example, applying π_j to cone K spanned by vectors from Δ yields its $(N-1)$-facet $\pi_j K$ spanned by vectors from $\Delta \setminus \{\alpha^j\}$, see Fig. 3. We note that $K_p = \pi_{p+1} \ldots \pi_N K$ and $L_p = \pi_1 \ldots \pi_p K$.

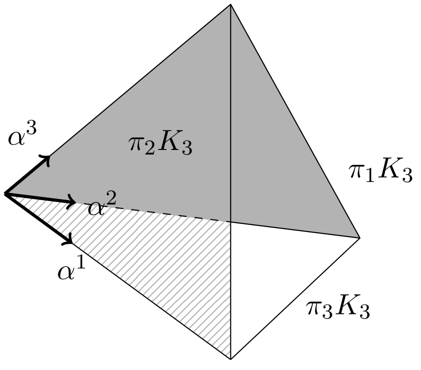

Figure 3. Facets on K_3 obtained by the projections π_i for $i = 1, 2, 3$.

We shall use the same symbol to define an induced operator $\pi_j, j \leqslant p \leqslant N$, on the ring of formal power series $\mathcal{F}_K[[z]]$ by

$$\pi_j : F(K_p; x; z) \longmapsto F(\pi_j K_p; x; z).$$

In particular, $\pi_j z^{\alpha^j} = 0$ for $j = 1, \ldots, N$. The function $F(\pi_j K_p; x; z)$ represents a generating series for the initial data function on the facet $\pi_j K_p$.

We now prove some useful properties of π_j.

Lemma 1. *If $1 \leqslant j \leqslant p$ and $x \in L_p$, then*

$$(1 - \pi_j) F(K_p; x; z) = F(K_p + \alpha^j; x; z) = z^{\alpha^j} F(K_p; x + \alpha^j; z).$$

Proof. By definition, we compute

$$(1 - \pi_j) F(K_p; x; z) = F(K_p; x; z) - \pi_j F(K_p; x; z) = \\ = \sum_{y \in K_p + \alpha^j} f(x + y) z^y = z^{\alpha^j} \sum_{y \in K_p} f(x + y + \alpha^j) z^y,$$

which proves the lemma. □

Lemma 2. *If $1 \leqslant j \leqslant p$ and $x \in L_p$, then*

$$(1 - \pi_j) z^{\alpha^j} F(K_p; x; z) = (1 - \pi_j) \delta^{-\alpha^j} F(K_p; x; z).$$

Proof. By definition we compute

$$(1-\pi_j)z^{\alpha^j}F(K_p;x;z) = z^{\alpha^j}\sum_{y\in K_p} f(x+y)z^y =$$
$$= \sum_{y\in K_p} f(x+y)z^{y+\alpha^j} = \sum_{y\in \alpha^j+K_p} f(x+y-\alpha^j)z^y =$$
$$= \sum_{y\in K_p} f(x+y-\alpha^j)z^y - \pi_j\sum_{y\in K_p} f(x+y-\alpha^j)z^y =$$
$$= (1-\pi_j)F(K_p;x-\alpha^j;z) = (1-\pi_j)\delta^{-\alpha^j}F(K_p;x;z),$$

which proves the lemma. □

Lemma 3. *If* $1 \leqslant p \leqslant N$*, then*

$$\pi_p F(K_p;x;z) = F(K_{p-1};x;z).$$

Proof. By definition we compute

$$\pi_p \sum_{y\in K_p} f(x+y)z^y = \sum_{y\in \pi_p K_p} f(x+y)z^y = \sum_{y\in K_{p-1}} f(x+y)z^y,$$

which proves the lemma. □

Lemma 4. *If* $G(z)$ *is the generating series for* $g(\lambda)$ *in 2.1, then*

$$(1-\pi_j)G(K_p;x;z) = G(K_p;x;z)$$

holds for $1 \leqslant j \leqslant p$ *and* $x \in L_p$.

Proof. Since $g(\lambda)$ is given for $\lambda \in K+m$, we acknowledge that $g(\lambda)=0$ for $\lambda \in X_m$, which yields $\pi_j G(K_p;x;z) = G(\pi_j K_p;x;z) = 0$ for $1 \leqslant j \leqslant p$ and $x \in L_p$. □

3. Difference equation and integral representation for the sections of generating series

Let $f(\lambda)$ be a solution to the Cauchy problem 2.1–2.2. We derive a difference equation (recurrence relation) for the sections of generating series of $f(\lambda)$.

Theorem 1. *Let* $\Pi_p = (1-\pi_1)\dots(1-\pi_p), p \leqslant N,$ *and let*

$$Q_p(z;\delta) = c_0 + c_1 z^{\alpha^1} + \ldots + c_p z^{\alpha^p} + c_{p+1}\delta^{\alpha^{p+1}} + \ldots + c_N\delta^{\alpha^N}$$

be a difference operator; if $p = 0$, then $Q_p(z;\delta) = Q(\delta)$. The sections $F(K_p;x;z)$ of generating series $F(z)$ satisfies the recurrence relation

$$\Pi_p Q_p(z;\delta^{-1})F(K_p;x;z) = \Pi_p G(K_p;x;z), \tag{3.1}$$

for $x \in L_p + \ell_p$, where $\ell_p = \alpha^{p+1} + \dots + \alpha^N$.

Proof. For $x \in L_p + \ell_p$ and $y \in K_p + m_p$ we multiply both sides of difference equation 2.1 by z^y and sum them over $y \in K_p + m_p$:

$$\sum_{y \in K_p + m_p} Q(\delta^{-1}) f(x+y) z^y = \sum_{y \in K_p + m_p} g(x+y) z^y, \quad x \in L_p + \ell_p.$$

Lemmas 1, 2 and 3 yield

$$\Pi_p Q_p(z;\delta^{-1})F(K_p;x;z) = \Pi_p Q(\delta^{-1}) \cdot \sum_{y \in K_p} f(x+y) z^y =$$

$$= (1-\pi_1)\cdots(1-\pi_p) \sum_{y \in K_p} Q(\delta^{-1}) f(x+y) z^y = \sum_{y \in K_p + m_p} Q(\delta^{-1}) f(x+y) z^y$$

for $x \in L_p + \ell_p$. Lemma 4 yields

$$\Pi_p G(K_p;x;z) = \sum_{y \in K_p + m_p} g(x+y) z^y$$

for $x \in L_p + \ell_p$. Equating these expressions proves the theorem. □

Remark 1. We note that for $p = 0$, $K_p = \varnothing$, $L_p = K$, then $F(K_0;x;z) = f(x)$ for $x \in K$, and 3.1 becomes the difference equation 2.1. For $p = N$ we get $K_p = K, L_p = \varnothing$, and formula 3.1 for a generating series $F(z)$ was proven in [12].

Let $\mathcal{F} = \mathbb{C}$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and generating (Laurent) series $F(z)$ of a solution to the Cauchy problem 2.1–2.2 converges at the origin. We derive an integral representation for sections of generating series $F(z)$ with support $K \subset \mathbb{Z}^n$.

Theorem 2. *Let K be a cone spanned on a set of linearly independent vectors $\Delta = \{\alpha^1, \dots, \alpha^N\}$ with integer coordinates. If $F(\zeta) = \sum\limits_{\lambda \in K} f(\lambda)\zeta^\lambda$, where $\zeta^\lambda = \zeta_1^{\lambda_1} \dots \zeta_n^{\lambda_n}$, is holomorphic at the origin, cone $K_p \subset \mathbb{Z}^n$ is spanned by vectors from $\Delta_1 = \{\alpha^1, \dots, \alpha^p\}$, and cone $L_p \subset \mathbb{Z}^n$ is spanned by vectors from $\Delta_2 = \{\alpha^{p+1}, \dots, \alpha^N\}$, then its section $F(K_p, x, z)$ at any $x \in L_p$ has the integral representation*

$$F(K_p;x;z) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma} \frac{F(\zeta)\zeta^{-x}}{\prod\limits_{j=1}^{p} \left(1 - \left(\dfrac{z}{\zeta}\right)^{\alpha^j}\right)} \frac{d\zeta}{\zeta}, \tag{3.2}$$

where $\zeta^{-x} = \zeta_1^{-x_1}\dots\zeta_n^{-x_n}$, $\left(\frac{z}{\zeta}\right)^{\alpha^j} = \left(\frac{z_1}{\zeta_1}\right)^{\alpha_1^j}\cdots\left(\frac{z_n}{\zeta_n}\right)^{\alpha_n^j}$, $\frac{\mathrm{d}\zeta}{\zeta} = \frac{\mathrm{d}\zeta_1}{\zeta_1}\wedge\dots\wedge\frac{\mathrm{d}\zeta_n}{\zeta_n}$, *the contour*

$$\Gamma = \{z \in \mathbb{C}^n : |z_1| = \varepsilon_1, \dots, |z_n| = \varepsilon_n\}$$

is chosen so that the closed polydisk contains no poles of $F(z)$, *and* $\left|\frac{z}{\zeta}\right| < 1$ *on* Γ.

Proof. Since $F(\zeta)$ converges on Γ and $\left(1-(z/\zeta)^{\alpha^j}\right)^{-1} = \sum_{k=0}^{\infty}(z/\zeta)^{k\alpha^j}$, we compute 3.2:

$$\frac{1}{(2\pi i)^n}\int\limits_\Gamma \frac{F(\zeta)\zeta^{-x}}{\prod_{j=1}^{p}\left(1-\left(\frac{z}{\zeta}\right)^{\alpha^j}\right)}\frac{\mathrm{d}\zeta}{\zeta} =$$

$$= \frac{1}{(2\pi i)^n}\int\limits_\Gamma \sum_{\lambda\in K} f(\lambda)\zeta^{\lambda-x}\sum_{k_1=0}^{\infty}\left(\frac{z}{\zeta}\right)^{k_1\alpha^1}\cdots\sum_{k_p=0}^{\infty}\left(\frac{z}{\zeta}\right)^{k_p\alpha^p}\frac{\mathrm{d}\zeta}{\zeta} =$$

$$= \frac{1}{(2\pi i)^n}\int\limits_\Gamma \sum_{\lambda\in K}\sum_{k_1,\dots,k_p=0}^{\infty} f(\lambda)\zeta^{\lambda-x-k_1\alpha^1-\dots-k_p\alpha^p-I}z^{k_1\alpha^1+\dots+k_p\alpha^p}\mathrm{d}\zeta =$$

$$= \frac{1}{(2\pi i)^n}\sum_{k_1,\dots,k_p=0}^{\infty}\left(z^{k_1\alpha^1+\dots+k_p\alpha^p}\cdot\underbrace{\sum_{\lambda\in K} f(\lambda)\int\limits_\Gamma \zeta^{\lambda-x-k_1\alpha^1-\dots-k_p\alpha^p-I}\mathrm{d}\zeta}_{=f(x+k_1\alpha^1+\dots+k_p\alpha^p)}\right) =$$

$$= \sum_{k_1,\dots,k_p=0}^{\infty} f(x+\underbrace{k_1\alpha^1+\dots+k_p\alpha^p}_{=y})z^{k_1\alpha^1+\dots+k_p\alpha^p} = \sum_{y\in K_p} f(x+y)z^y,$$

since

$$\frac{1}{(2\pi i)^n}\int\limits_\Gamma \zeta^{\lambda-x-k_1\alpha^1-\dots-k_p\alpha^p-I}\mathrm{d}\zeta = \begin{cases}1, & \text{if } \lambda = x+k_1\alpha^1+\dots+k_p\alpha^p;\\ 0, & \text{otherwise.}\end{cases}$$

The last sum coincides with the definition of $F(K_p; x; z)$, which proves the theorem. □

Remark 2. If $x = 0$, $\Delta_1 = \varnothing$, 3.2 yields the integral representation for coefficients $f(\lambda)$ (see [18]):

$$f(\lambda) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma} \frac{F(\zeta)\mathrm{d}\zeta}{\zeta^{\lambda+I}}.$$

Remark 3. Under the conditions of Theorem 2 there is an integral representation for $\pi_j F(z) = F(\Delta \setminus \{\alpha^j\}; 0; z)$, $j = 1, \dots, N$, as

$$\pi_j F(z) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma} \frac{F(\zeta)}{\prod\limits_{\alpha \in \Delta \setminus \{\alpha^j\}} \left(1 - \left(\frac{z}{\zeta}\right)^{\alpha}\right)} \frac{\mathrm{d}\zeta}{\zeta}.$$

Remark 4. Under the conditions of Theorem 2 we can prove Theorem 1 by substituting integral representation 3.2 into difference equation 3.1.

4. Example

Theorem 1 provides a recurrence relation between sections of generating series for $0 < p < N$. The left side of the relation is a polynomial and constant coefficient difference equation on the sections, and the right side is the difference equation for sections of generating series of the initial data function $\varphi(x)$ and $g(x)$. For $p = 0$ or $p = N$, see Remark 1.

For $n = N = 2$, $\Delta = \{\alpha_1, \alpha_2\}$ such that $K = \langle \alpha_1, \alpha_2 \rangle$ is pointed and simplicial, $c_0, c_1, c_2 \in \mathbb{C}$, and $m = \alpha^1 + \alpha^2$, we consider the Cauchy problem

$$\begin{cases} c_0 f(\lambda) + c_1 f(\lambda - \alpha^1) + c_2 f(\lambda - \alpha^2) = g(\lambda), \lambda \in K + m, \\ f(\lambda) = \varphi(\lambda), \lambda \in X_m. \end{cases} \tag{4.1}$$

For $p = 0$, cone K_0 is empty, $L_p = K$, then 3.1 yields the given difference equation 4.1 for $x \in K + m$:

$$c_0 F(K_0; x; z) + c_1 F(K_0; x - \alpha^1; z) + c_2 F(K_0; x - \alpha^2; z) = G(K_0; x; z).$$

For $p = 1$, we have $K_1 = \langle \alpha^1 \rangle$, $L_1 = \langle \alpha^2 \rangle$, and for $x \in L_1 + \alpha^2$ formula 3.1 yields

$$(1 - \pi_1)(c_0 + c_1 z^{\alpha^1} + c_2 \delta^{-\alpha^2}) F(K_1; x; z) = (1 - \pi_1) G(K_1; x; z),$$

then using Lemma 3 yields the difference equation

$$(c_0 + c_1 z^{\alpha^1}) F(K_1; x; z) + c_2 F(K_1; x - \alpha^2; z) = \\ = G(K_1; x; z) + c_0 F(K_0; x; z) + c_2 F(K_0; x - \alpha^2; z).$$

Since $F(K_0;x;z)=f(x)$, and $F(K_0;x-\alpha^2;z)=f(x-\alpha^2)$, we get

$$(c_0+c_1z^{\alpha^1})F(K_1;x;z)+c_2F(K_1;x-\alpha^2;z)= \\ =G(K_1;x;z)+c_0f(x)+c_2f(x-\alpha^2). \quad (4.2)$$

For $p=2$, cone $K_2=K$, $L_2=\varnothing$, and we get

$$(1-\pi_1)(1-\pi_2)(c_0+c_1z^{\alpha^1}+c_2z^{\alpha^2})F(K_2;x;z)= \\ =(1-\pi_1)(1-\pi_2)G(K_2;x;z), x\in L_2=\varnothing,$$

which implies

$$(c_0+c_1z^{\alpha^1}+c_2z^{\alpha^2})F(K;x;z)=G(K;x;z)+(c_0+c_2z^{\alpha^2})F(\pi_1K;x;z)+ \\ +(c_0+c_1z^{\alpha^1})F(\pi_2K;x;z)-c_0F(\pi_1\pi_2K;x;z)+c_0f(x),$$

and yields a formula for the generating series for the solution to 4.1 by the generating series of its initial data function on facets of the cone K. There are two one-dimensional facets π_1K,π_2K, and one facet of zero dimension $\pi_1\pi_2K$, which coincides with the origin.

Consider 4.2 and $G(z)=0$. Denoting $M=c_0f(x)+c_2f(x-\alpha^2)$ in 4.2 yields

$$(c_0+c_1z_1^{\alpha_1^1}z_2^{\alpha_2^1})F(K_1;x;z)+c_2F(K_1;x-\alpha^2;z)=M. \quad (4.3)$$

For $z\in\mathbb{C}^n$ and $j\in 1,\dots,n$ we consider the linear differential operator with polynomial coefficients

$$\Omega_j=P_s^j(z)\frac{\partial^s}{\partial z_j^s}+P_{s-1}^j(z)\frac{\partial^{s-1}}{\partial z_j^{s-1}}+\ldots+P_1^j(z)\frac{\partial}{\partial z_j}+P_0^j(z).$$

The function $F(z)$ is called D-finite (see [9; 19]) if it satisfies a system of differential equations

$$\Omega_1F(z)=\Omega_2F(z)=0. \quad (4.4)$$

Let $F(K_1;x-\alpha^2;z)$ be a D-finite function. Applying Ω_1 to 4.3 yields

$$F(K_1;x;z)\Omega_1(c_0+c_1z_1^{\alpha_1^1}z_2^{\alpha_2^1})+(c_0+c_1z_1^{\alpha_1^1}z_2^{\alpha_2^1})\Omega_1F(K_1;x;z)=\Omega_1M.$$

Since

$$\Omega_1(c_0+c_1z_1^{\alpha_1^1}z_2^{\alpha_2^1})=c_1z_2^{\alpha_2^1}\Omega_1z_1^{\alpha_1^1}+P_0^1(c_0+c_1z_1^{\alpha_1^1}z_2^{\alpha_2^1}) \\ \Omega_1M=MP_0^1(z_1,z_2),$$

we get

$$\left[c_1z_2^{\alpha_2^1}\Omega_1z_1^{\alpha_1^1}+(c_0+c_1z^{\alpha^1})(\Omega_1+P_0^1)\right]F(K_1;x;z)=MP_0^1.$$

Differentiating this equality $k_1 = \deg P_0^1 + 1$ times with respect to z_1, we get

$$\frac{\partial^{k_1}}{\partial z_1^{k_1}} \left[c_1 z_2^{\alpha_2^1} \Omega_1 z_1^{\alpha_1^1} + (c_0 + c_1 z^{\alpha^1})(\Omega_1 + P_0^1) \right] F(K_1; x; z) = 0.$$

Applying Ω_2 to 4.3 and differentiating $k_2 = \deg P_0^2 + 1$ times with respect to z_2 yields that $F(K_1; x; z)$ is D-finite function.

Thus, we have proved the following proposition.

Proposition 1. *If section $F(K_1; x - \alpha^2; z)$ is D-finite and satisfy the system of differential equations*

$$\Omega_j F(K_1; x - \alpha^2; z) = 0, j = 1, 2,$$

then the section $F(K_1; x; z)$ is D-finite and satisfies the system of differential equations

$$\frac{\partial^{k_j}}{\partial z_j^{k_j}} \left[c_1 z_{1-j}^{\alpha_2^1} \Omega_1 z_j^{\alpha_1^1} + (c_0 + c_1 z^{\alpha^1})(\Omega_1 + P_0^j) \right] F(K_1; x; z) = 0, j = 1, 2,$$

where $k_j = \deg P_0^j + 1$.

5. Conclusion

We offered a notion of the section of generating (Laurent) series of a solution to a difference equation with constant coefficients in a lattice cone. We found a difference equation for the sections of a generating series and an integral representation for sections when the generating series defines a holomorphic function. The results of this article will be useful in the enumerative combinatorial analysis, multidimensional recursive digital filters, and the theory of difference schemes.

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