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Algebras of Binary Isolating Formulas for Tensor Product Theories

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Abstract. Algebras of distributions of binary isolating and semi-isolating formulae are derived objects for a given theory and reflect binary formula relations between 1-type realizations. These algebras are related to the following natural classification questions: 1) for a given class of theories, determine which algebras correspond to theories from that class, and classify those algebras; 2) classify theories from the class according to the isolating and semi-isolating formulae algebras defined by those theories. The description of a finite algebra of binary isolating formulas unambiguously implies the description of an algebra of binary semi-isolating formulas, which makes it possible to trace the behavior of all binary formula relations of a given theory. The paper describes algebras of binary formulas for tensor products. The Cayley tables are given for the obtained algebras. Based on these tables, theorems are formulated describing all algebras of binary formulae distributions for tensor multiplication theory of regular polygons on an edge. It is shown that they are completely described by two algebras.

Keywords: algebra of binary isolating formulas, tensor product, model theory, Cayley tables

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Научная статья

Алгебры бинарных изолирующих формул для теорий тензорных произведений

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Аннотация. Алгебры распределений бинарных изолирующих и полуизолирующих формул являются производными объектами для данной теории и отражают бинарные формульные связи между реализациями 1-типов. Эти алгебры связаны со следующими естественными классификационными вопросами: 1) по данному классу теорий определить, какие алгебры соответствуют теориям из этого класса, и классифицировать эти алгебры; 2) классифицировать теории из класса в зависимости от определяемых этими теориями алгебр изолирующих и полуизолирующих формул. При этом описание конечной алгебры бинарных изолирующих формул однозначно влечет и описание алгебры бинарных полуизолирующих формул, что позволяет отслеживать поведение всех бинарных формульных связей данной теории.
В статье описаны алгебры бинарных формул для тензорных произведений. Для полученных алгебр приведены таблицы Кэли. На основании этих таблиц сформулированы теоремы, описывающие все алгебры распределений бинарных формул для теории тензорного умножения правильных многоугольников на ребро. Показано, что они полностью описываются двумя алгебрами.

Ключевые слова: алгебра бинарных изолирующих формул, тензорное произведение, теория моделей, таблицы Кэли

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1. Introduction

The present paper continues the study of algebras of distributions [1;2;8–10] binary isolation formulas: we describe such algebras for unary theories, Cartesian products of graphs, and some generalizations of them for semi-isolating formulas.

Algebras of distributions of binary isolating formulas are derived objects for these theories and, at the binary level, reflect relations between types as well as relations on the set of realizations of a given type. In [8] the general properties of these algebras are studied, allowing us to characterize a class of such algebras. Multiplication tables of algebras are shown on a number of basic examples, as well as the mutual influence of these algebras with the structures of initial theories.

The description of distribution algebras of binary isolating formulas allows us to classify theories from this class by derived algebraic objects. This paper proposes such a classification for structures of tensor products of graphs restricted to sets of realizations of fixed type. The theorem 1 shows that algebras are subdivided into two varieties, each of which, to a certain extent, makes it possible to reconstruct the binary structure of a given 1-type.

2. Algebras of binary isolating formulas for tensor product theories

Definition 1. *The tensor product $G \times H$ of graphs G and H is a graph whose set of vertices is the Cartesian product $V(G) \times V(H)$ with different vertices (u, u') and (v, v') adjacent in $G \times H$ when u is adjacent with v and u' is adjacent to v'. [7]*

Algebra for the tensor product of graphs of a triangle and an edge $T \times H$, with the set of labels $\rho_{\nu(p)} = \{0, 1, 2, 3\}$ is given by the following table:

$\cdot$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$
2	$\{2\}$	$\{1,3\}$	$\{0,2\}$	$\{1,3\}$
3	$\{3\}$	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$

By tensor product of the graph square and edge $Q \times H$ we obtain two identical algebras with a set of labels $\rho_{\nu(p)} = \{0, 1, 2\}$, which is given by the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0,2\}$	$\{1\}$
2	$\{2\}$	$\{1\}$	$\{0,2\}$

Algebra for the tensor product of graphs of a pentagon and an edge $P \times H$, with the set of labels $\rho_{\nu(p)} = \{0, 1, 2, 3, 4, 5\}$ is given by the following table:

$\cdot$	0	1	2	3	4	5
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
1	$\{1\}$	$\{0,2\}$	$\{1,3\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4\}$
2	$\{2\}$	$\{1,3\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4\}$	$\{1,3,5\}$
3	$\{3\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4\}$
4	$\{4\}$	$\{1,3,5\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4\}$	$\{1,3,5\}$
5	$\{5\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4\}$

When the tensor product of the hexagonal graph and the edge $Q \times H$ we obtain two identical algebras, with a set of labels $\rho_{\nu(p)} = \{0,1,2,3\}$, which are given by the following table:

$\cdot$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$
2	$\{2\}$	$\{1,3\}$	$\{0,2\}$	$\{1,3\}$
3	$\{3\}$	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$

The algebra for the tensor product of a heptagon and an edge graph $P \times H$, with the set of labels $\rho_{\nu(p)} = \{0,1,2,3,4,5,6,7\}$ is given by the following table:

$\cdot$	0	1	2	3	4	5	6	7
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$
1	$\{1\}$	$\{0,2\}$	$\{1,3\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$
2	$\{2\}$	$\{1,3\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$
3	$\{3\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$
4	$\{4\}$	$\{1,3,5\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$
5	$\{5\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$
6	$\{6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$
7	$\{7\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$	$\{1,3,5,7\}$	$\{0,2,4,6\}$

The tensor product algebra of graphs $G \times H$, where G is a polygon graph with graph diameter n and H an edge graph, denote by $\mathfrak{Tp}_{\mathfrak{o}}$, with labels

$\{0, 1, 2, 3, \ldots, n\}$, where n is–an even number equal to the diameter of the graph obtained by multiplication. The algebra is given by the following table:

$*$	0	1	2	3	4	...	n
0	{0}	{1}	{2}	{3}	{4}	...	$\{n\}$
1	{1}	{0,2}	{1,3}	{0,2}	{1,3,5}	...	$\{1,3,5,\ldots,n-1\}$
2	{2}	{1,3}	{0,2,4}	{1,3,5}	{0,2,4,6}	...	$\{0,2,4,\ldots,n\}$
3	{3}	{0,2}	{1,3,5}	{0,2,4,6}	$\{1,3,5,\ldots,n-1\}$	...	$\{1,3,5,\ldots,n-1\}$
4	{4}	{1,3,5}	{0,2,4,6}	{1,3,5, ..., n-1}	$\{0,2,4,\ldots,n\}$	...	$\{0,2,4,\ldots,n\}$
...	...	...	...	...	...	...	...
n	$\{n\}$	{1,3,5, ..., n-1}	{0,2,4,..., n}	{1,3,5, ..., n-1}	$\{0,2,4,\ldots,n\}$	...	$\{0,2,4,\ldots,n\}$

The tensor product algebra of graphs $G \times H$, where G is a polygon graph with graph diameter n and H an edge graph, denote by $\mathfrak{Tp}_{\mathfrak{o}}$, with labels $\{0, 1, 2, 3, \ldots, n\}$, where n is an odd number equal to the diameter of the graph obtained by multiplication. The algebra is given by the following table:

$*$	0	1	2	3	4	...	n
0	{0}	{1}	{2}	{3}	{4}	...	$\{n\}$
1	{1}	{0,2}	{1,3}	{0,2}	{1,3,5}	...	$\{0,2,4,\ldots,\ldots,n\}$
2	{2}	{1,3}	{0,2,4}	{1,3,5}	{0,2,4,6}	...	$\{1,3,5,\ldots,n-1\}$
3	{3}	{0,2}	{1,3,5}	{0,2,4,6}	$\{1,3,5,\ldots,n-1\}$	...	$\{0,2,4,\ldots,n\}$
4	{4}	{1,3,5}	{0,2,4,6}	{1,3,5, ..., n-1}	$\{0,2,4,\ldots,n\}$	...	$\{1,3,5,\ldots,n-1\}$
...	...	...	...	...	...	...	...
n	$\{n\}$	{0,2,4,..., n}	{1,3,5, ..., n-1}	{0,2,4,..., n}	$\{1,3,5,\ldots,\ldots,n-1\}$	...	$\{0,2,4,\ldots,n\}$

When $H_1 = H_2 = \ldots = H_k = H$ the tensor product $H \times H \times \ldots \times H$ is called *kth tensor degree* of the graph H and is denoted by H^k.

The distribution algebra of binary isolation formulas for the edge H with labels $\{0, 1\}$, will have the form:

$\cdot$	0	1
0	{0}	{1}
1	{1}	{0}

Given a tensor product of an edge on itself $H \times H$ or H^2, we obtain two graphs of the edge and the algebra for each graph with labels $\{0, 1\}$, each given by a table:

$\cdot$	0	1
0	{0}	{1}
1	{1}	{0}

Consider graph H^3, it consists of four edge graphs, and the same number of algebras each of which is isomorphic to an edge algebra H

For H^4 we get eight identical algebras and edge graphs, the algebra will be isomorphic to the algebra H.

When you elevate the graph of an edge H to the kth tensor degree, notice that the number of graphs and identical algebras will be a multiple of degree two, the relation will be such that for H^k the number of graphs and algebras will be 2^{k+1} and the algebras will be isomorphic to the algebra H.

The algebra for the tensor product of triangle graphs and edges in tensor degree two $T \times H^2$, with the set of labels $\rho_{\nu(p)} = \{0, 1, 2, 3\}$ is given by the following table:

$\cdot$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$
2	$\{2\}$	$\{1,3\}$	$\{0,2\}$	$\{1,3\}$
3	$\{3\}$	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$

In this case, we will obtain two identical algebras of the form $T \times H^2$, which will be isomorphic to the algebra $T \times H$. For other degrees, we will deal with increasing number of identical algebras, for $T \times H^3$ we will get four algebras, in case H^k we will get 2^{k+1} copies of algebra.

If tensor product of graphs of square and edge in tensor degree two $Q \times H^2$ we get four identical algebras, with set of labels $\rho_{\nu(p)} = \{0, 1, 2\}$, which is given by the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0,2\}$	$\{1\}$
2	$\{2\}$	$\{1\}$	$\{0,2\}$

For $Q \times H^3$ we obtain eight algebras isomorphic to $Q \times H$. The product of the form $Q \times H^k$ gives us 2^{k+2} identical algebras which are also isomorphic to the algebra Q.

The algebra for the tensor product of the graph pentagon and the edge in tensor degree two $P \times H^2$, with the set of labels $\rho_{\nu(p)} = \{0, 1, 2, 3, 4, 5\}$ is given by the following table:

$\cdot$	0	1	2	3	4	5
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
1	$\{1\}$	$\{0,2\}$	$\{1,3\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4\}$
2	$\{2\}$	$\{1,3\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4\}$	$\{1,3,5\}$
3	$\{3\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4\}$
4	$\{4\}$	$\{1,3,5\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4\}$	$\{1,3,5\}$
5	$\{5\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4\}$	$\{1,3,5\}$	$\{0,2,4\}$

For $P \times H^2$ we obtain eight algebras isomorphic to $P \times H$. The product of the form $P \times H^k$ gives us 2^{k+1} algebras which are isomorphic to the algebra $P \times H$

The number of algebras when multiplied by the kth tensor degree of an edge depends on the parity of the multiplied polygon graph; for an even number of vertices we get 2^{k+2} algebras, and for an odd 2^{k+1} identical algebras.

Note that the number of labels for the tensor product of polygons with even number of vertices per edge coincides with the graph diameter and is half of their number.

For the tensor product per edge of a polygon with odd number of vertices the number of labels and diameter of the graph is equal to the number of vertices.

Theorem 1. *If T is the theory of the tensor product of a graph on an edge, $\mathfrak{B}$ — algebra of binary isolating formulas of the theory T, then the algebra $\mathfrak{B}$ is given by exactly one of the following algebras: $\mathfrak{Tp}_{\mathfrak{e}}$ $\mathfrak{Tp}_{\mathfrak{o}}$.*

Remark 1. Algebra for a tensor product of the form $(((((G \times H) \times H) \times H) \ldots) \times H)$ is isomorphic to the algebra $\mathfrak{Tp}_{\mathfrak{e}}$ or $\mathfrak{Tp}_{\mathfrak{o}}$.

3. Conclusion

As can be seen from Theorem 1, the algebra for multiplication of the graph of a regular polygon by an edge is divided into two kinds for even and odd numbers of vertices, two kinds these algebras are exhausted.

References

1. Emel'yanov D.Yu. On the algebra distributions of binary formulas of unary theories. *The Bulletin of Irkutsk State University. Series Mathematics*, 2016, vol. 17, pp. 23–36. (in Russian)
2. Emel'yanov D.Yu. Algebras of binary isolating formulas for theories of Cartesian products of graphs. *Algebra and model theory 12. Collection of papers.* Novosibirsk, NSTU Publ., 2019, pp. 21–31.

3. Emelyanov D.Yu. Algebra distributions of binary formulas for theories of Archimedean bodies. *The Bulletin of Irkutsk State University. Series Mathematics*, 2019, vol. 28, pp. 36–52.
4. Emelyanov D.Y., Sudoplatov S.V. Structure of algebras of binary formulas of polygonometric theories with symmetry condition. *Siberian Electronic Mathematical Reports*, 2020, vol. 17, pp. 1–20.
5. Emelyanov D.Y. Algebras of binary isolating formulas for simplex theories. Algebra and Model Theory. Novosibirsk, Edition of NSTU, 2017, pp. 66–74.
6. Harari F. *Graph theory.* Moscow, Unitorial URSS Publ., 2003, 300 p.
7. Hahn G., Sabidussi G. (eds.). *Graph symmetry: algebraic methods and applications.* Springer, 1997, 418 p.
8. Shulepov I.V., Sudoplatov S.V. Algebras of distributions for isolating formulas of a complete theory. *Siberian Electronic Mathematical Reports*, 2014, vol. 11, pp. 380–407.
9. Sudoplatov S.V. *Classification of countable models of complete theories. Part 1.* Novosibirsk, NSTU Publ., 2018, 326 p.
10. Sudoplatov S.V. Hypergraphs of prime models and distributions of countable models of small theories. *J. Math. Sciences*, 2010, vol. 169, no. 5, pp. 680–695. https://doi.org/10.1007/s10958-010-0069-9

Список источников

1. Емельянов Д. Ю. Об алгебрах распределений бинарных формул теорий унаров // Известия Иркутского государственного университета. Серия Математика. 2016. Т. 17. С. 23–36.
2. Емельянов Д. Ю. Алгебры бинарных изолирующих формул для теорий декартовых произведений графов. *Algebra and model theory 12. Collection of papers.* Новосибирск : НГТУ, 2019. С. 21–31.
3. Емельянов Д. Ю. Алгебры распределений бинарных формул для теорий архимедовых тел // Известия Иркутского государственного университета. Серия Математика. 2019. Т. 28. С. 36—52.
4. Емельянов Д. Ю., Судоплатов С. В. Структура алгебр бинарных формул полигонометрических теорий с условием симметрии // Сибирские электронные математические известия. 2020. Т. 17. С. 1–20.
5. Емельянов Д. Ю. Алгебры бинарных изолирующих формул для теорий симплексов // Algebra and Model Theory. Новосибирск : Изд. НГТУ, 2017. С. 66–74.
6. *Харари Ф.* Теория графов. М. : Едиториал УРСС, 2003. 300 с.
7. Graph symmetry: algebraic methods and applications / eds. Hahn G., Sabidussi G. Springer, 1997. Vol. 497. 418 p.
8. Shulepov I. V., Sudoplatov S. V. Algebras of distributions for isolating formulas of a complete theory // Siberian Electronic Mathematical Reports. 2014. Vol. 11. P. 380–407.
9. Sudoplatov S. V. Classification of countable models of complete theories. Part 1. Novosibirsk : NSTU Publisher, 2018. 326 p.
10. Sudoplatov S.V. Hypergraphs of prime models and distributions of countable models of small theories // J. Math. Sciences. 2010. Vol. 169, N 5. P. 680–695. https://doi.org/10.1007/s10958-010-0069-9

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