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Multi-Period Loan Interest Rate Nash Model with Basel II Solvency Constraint

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Abstract. This paper introduces multi–period loan interest rate Nash game models in the banking sector under regulatory solvency constraints. By taking solvency constraint as Basel II and modelling economic condition as AR(1) process, we obtain results regarding the existence of loan interest rate equilibrium. A sensitivity analysis for the solvency constraint model and some numerical results are presented.

Keywords: Nash equilibrium model, one factor KMV/Riskmetrics model, Basel II solvency constraint, credit rating, loan interest rate

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Научная статья

Периодическая процентная ставка по кредиту модели Нэша с ограничением платежеспособности Базель II

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Аннотация. В статье представлены модели игры Нэша с процентной ставкой по периодическим кредитам в банковском секторе в условиях нормативных ограничений платежеспособности. Принимая ограничение платежеспособности Базель II и моделируя экономическое состояние как процесс AR (1), мы получаем результаты относительно существования равновесия процентной ставки по кредиту. Представлены анализ чувствительности модели ограничения платежеспособности и некоторые численные результаты.

Ключевые слова: модель равновесия Нэша, однофакторная модель KMV/Riskmetrics, ограничение платежеспособности Базель II, кредитный рейтинг, процентная ставка по кредиту

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1. Introduction

In the insurance industry, there are plenty of game theory models. However, Dutang et al. [8] first introduced Nash and Stackelberg equilibrium models with solvency constraints for insurance premium. They also included a sensitivity analysis. Battulga et al. [2] extended the one–period (cf. [8]) model to a multi–period one by using a transition probability matrix dependent on several economic factors. Under the assumption that the solvency constraints will be updated each period, they provided similar results regarding the existence of premium equilibrium and sensitivity analysis to those of [8].

In the banking industry, the capital impact in the composition of the asset side of the bank balance sheet was studied in [11; 15]. For instance, Kim et al. [15] considered the mean–variance portfolio problem with equity to asset ratio and investigated the impact of this variable. Furfine [11] developed the bank's dynamic asset allocation model by considering factors such as assets, deposits, and some costs, including risk based capital cost. By using the US real data, he showed that the regulatory standard is a significant impact on optimal asset allocation. Moreover, it would be interesting to analyze the capital impact on loan interest rates of the banking industry. Drumond and Jorge [7] obtained the equilibrium loan interest rates under Basel I and risk–based capital requirements. However, they did not take into account solvency constraints which guarantee bank still having solvency after its lending business.

Basel Committee on Bank Supervision (BCBS) issued Basel I in 1988. According to Basel I, banks are required to keep capital of at least 8% of risk weighted assets of their credit portfolio. Basel II first released in 2004 by the BCBS and it expanded the rules for minimum capital requirements established under Basel I. Basel II uses a "three pillars" concept, namely, minimum capital requirements (addressing credit risk, market risk, and operational risk), supervisory review, and market discipline. Process of Basel II attracted a lot of interest in quantitative credit risk models in industry, academia, and among regulators.

Recently, following the idea of [2; 8], Battulga et al. [3] presented loan interest rate Nash game models with solvency constraints in the banking industry. By taking solvency constraints as Basel I, Basel II, and ES, they obtained results regarding the existence of loan interest rate equilibrium and sensitivity analysis. In this paper, we introduced multi–period loan interest rate Nash game model with Basel II solvency constraint, whose default probability modeled by one factor KMV/Riskmetrics model and we obtained results regarding the existence of loan interest rate equilibrium. A sensitivity analysis for the solvency model and some numerical results are also presented.

The rest of the paper is structured as follows. Section 2 is devoted to one factor KMV/Riskmetrics model for default probability of bank customers. In Section 3, we investigate the Nash game models with a Basel II solvency constraint and an assertion related to the existence of loan interest rate equilibrium. A sensitivity analysis is also presented. Numerical results based on the data from two domestic systematically important Mongolian banks are given in Section 4.

2. One Factor KMV/Riskmetrics Model

In the paper, we modeled default probabilities of companies by the well known one factor KMV/Riskmetrics model. Idea of the KMV model can be found in Crosbie and Bohn [5] and Crouhy et al. [6] and the Riskmetrics model was introduced by Gupton et al. [14]. Both of the models are included in class of threshold credit risk models and are equivalent under some conditions, see McNeil et al. [16]. The KMV/Riskmetrics model assumes that asset return at time t of a generic company with credit rating i is modeled by

$$R_{i,t} = \sqrt{\rho_i}\Psi_t + \sqrt{1-\rho_i}\varepsilon_{i,t}, \tag{2.1}$$

where mixing random process Ψ_t represents economic condition at time t, which will affect all the loaners at time t of banks, $\varepsilon_{i,t}$ is a specific random variable, and ρ_i is a uniform correlation between the asset returns at time t of companies with credit rating i. Therefore, the asset return of the company is partially explained by a systematic factor representing economic condition, and the rest of the asset return is explained by company specific random shock. Under the assumption of the KMV/Riskmetrics model, random variables Ψ_t and $\varepsilon_{i,t}$ are independent and both are standard normally distributed. This assumption implies that $R_{i,t}$ also to be standard normal.

Let c_i be a default threshold corresponding to loans with credit rating i and $\bar{p}_i = \mathbb{P}[R_{i,t} < c_i]$ be an average default probability, which does not depend on time t. Then, it can be shown that conditional on the mixing random variable Ψ_t default probability at time t of loans with credit rating i is given by the following equation (see, e.g., [4], [10], [12], and [16])

$$p_i(\Psi_t) = \Phi\bigg(\frac{\Phi^{-1}(\bar{p}_i) + \sqrt{\rho_i}\Psi_t}{\sqrt{1-\rho_i}}\bigg), \quad i = 1, \dots, R, \ t = 1, \dots, T,$$

where $\Phi(x) := \int_0^x \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$ is the standard normal cumulative distribution function, $\Phi^{-1}(x)$ is an inverse function of $\Phi(x)$, $\bar{p}_i$ is the average default probability and ρ_i is the asset correlation between loans at time t with credit rating i.

After modeling of default probabilities of companies, we need to model the mixing random process Ψ_t. For that reason, here we assume that the mixing random process Ψ_t follows standard normally distributed AR(1) process, that is,

$$\Psi_t = \sqrt{\varphi}\Psi_{t-1} + \sqrt{1-\varphi} Z_t, \quad t = 0, \pm 1, \pm 2, \dots, \tag{2.2}$$

where we assume that $|\varphi| < 1$ and Z_t are independent identically standard normally distributed random variables.

Since $|\varphi| < 1$, it follows that the random variable Ψ_t can be written by

$$\Psi_t = \sqrt{1-\varphi} \sum_{j=0}^{\infty} \varphi^{j/2} Z_{t-j} \sim \mathcal{N}(0, 1). \tag{2.3}$$

Therefore, covariance between random variables Ψ_t and Ψ_s is given by $\operatorname{Cov}[\Psi_t, \Psi_s] = \varphi^{|t-s|}$. As a result, it follows from equations (2.1)–(2.3) that correlations between asset returns at time t of companies with credit rating i and asset returns at time s of companies with credit rating j are obtained as

$$\operatorname{Corr}[R_{i,t}, R_{j,s}] = \sqrt{\rho_i \rho_j} \varphi^{|t-s|}.$$

From above equation, one can deduce that asset correlation between loans at time t with credit rating i equals ρ_i.

Let us denote a two dimensional cumulative normal distribution function at point (x, y) with correlation ϱ by $\Phi_2\big(x, y; \varrho\big)$. Then, the following

Lemma, which will be used to estimate parameters of our model is a direct extension of result of Bluhm et al. [4].

Proposition 1. *Let for each rating* $i = 1, \dots, R$, $\mu_i(\Psi) = \frac{1}{T}\sum_{t=1}^T p_i(\Psi_t)$, $\gamma_{i,0}(\Psi) = \frac{1}{T}\sum_{t=1}^T p_i^2(\Psi_t)$, *and* $\gamma_{i,1}(\Psi) = \frac{1}{T-1}\sum_{t=2}^T p_i(\Psi_t)p_i(\Psi_{t-1})$ *be sample means and* $s_i^2(\Psi) := \frac{1}{T-1}\sum_{t=1}^T \big(p_i(\Psi_t) - \mu_i(\Psi)\big)^2$ *be a sample variance of random default probabilities, respectively. Then, it holds*

(a.) $\mathbb{E}\big[\bar{\mu}_i(\Psi)\big] = \bar{p}_i$

(b.) $\mathbb{E}\big[\gamma_{i,0}(\Psi)\big] = \Phi_{i,2}^*(0)$

(c.) $\mathbb{E}\big[\gamma_{i,1}(\Psi)\big] = \Phi_{i,2}^*(1)$

(d.) $\mathbb{E}\big[s_i^2(\Psi)\big] = \Phi_{i,2}^*(0) - \dfrac{2}{(T-1)T}\displaystyle\sum_{j=1}^{T-1}(T-j)\Phi_{i,2}^*(j)$

where $\Phi_{i,2}^*(j) := \Phi_2\big(\Phi^{-1}(\bar{p}_i), \Phi^{-1}(\bar{p}_i); \rho_i\varphi^j\big)$.

Proof. Let us fix the credit rating i for $i = 1, \dots, R$. Let X_t and X_s be independent standard normally distributed random variables and also independent of the following random variables

$$\xi_t := \frac{\Phi^{-1}(\bar{p}_i) + \sqrt{\rho_i}\Psi_t}{\sqrt{1-\rho_i}} \quad \text{and} \quad \xi_s := \frac{\Phi^{-1}(\bar{p}_i) + \sqrt{\rho_i}\Psi_s}{\sqrt{1-\rho_i}} \quad \text{for } t, s = 1, \dots, T.$$

As the random variables X_t, X_s, Ψ_t and Ψ_s are standard normally distributed, and former two independent and also independent of last two, according to equation (2.3) we have

$$\mathbb{E}[X_t - \xi_t] = \mathbb{E}[X_s - \xi_s] = \frac{\Phi^{-1}(\bar{p}_i)}{\sqrt{1-\rho_i}}, \quad \mathrm{Var}[X_t - \xi_t] = \mathrm{Var}[X_s - \xi_s] = \frac{1}{1-\rho_i} \tag{2.4}$$

and

$$\mathrm{Cov}[X_t - \xi_t, X_s - \xi_s] = \frac{\rho_i}{1-\rho_i}\varphi^{|t-s|}. \tag{2.5}$$

Following the idea in Gordy [12] (see also Bluhm et al. [4]), it can be shown that

$$\mathbb{E}\big[p_i(\Psi_t)p_i(\Psi_s)\big] = \mathbb{P}\big[\eta_t \le \Phi^{-1}(\bar{p}_i), \eta_s \le \Phi^{-1}(\bar{p}_i)\big],$$

where random variables

$$\eta_t := \frac{X_t - \xi_t - \mathbb{E}(X_t - \xi_t)}{\sqrt{X_t - \xi_t}} \quad \text{and} \quad \eta_s := \frac{X_s - \xi_s - \mathbb{E}(X_s - \xi_s)}{\sqrt{X_s - \xi_s}}$$

are both standard normally distributed. Now we move to calculation of correlation between the random variables η_t and η_s. It follows from equations (2.4) and (2.5), we obtain that

$$\mathrm{Corr}[\eta_t, \eta_s] = \mathrm{Cov}[\eta_t, \eta_s] = \rho_i\varphi^{|t-s|}, \quad \varphi \neq 0, \ |\varphi| < 1.$$

Note that if $\varphi = 0$, then $\mathrm{Corr}[\eta_t, \eta_t] = \rho_i$. As a result, we get that

$$\begin{aligned} \Phi^*_{i,2}(|t-s|) &= \mathbb{E}\big[p_i(\Psi_t)p_i(\Psi_s)\big] \\ &= \begin{cases} \Phi_2\big(\Phi^{-1}(\bar{p}_i), \Phi^{-1}(\bar{p}_i); \rho_i\varphi^{|t-s|}\big) & \text{if} \quad \varphi \neq 0,\ |\varphi| < 1 \\ \Phi_2\big(\Phi^{-1}(\bar{p}_i), \Phi^{-1}(\bar{p}_i); \rho_i\big) & \text{if} \quad \varphi = 0,\ t = s. \end{cases} \end{aligned} \tag{2.6}$$

Thus, if we take $t = s$ and $t = s + 1$ in equation (2.6), then $(b.)$ and $(c.)$ hold. Using the above idea for $p_i(\Psi_t)$ one can show that for all $t = 1, \dots, T$, $\mathbb{E}[p_i(\Psi_t)] = \bar{p}_i$. Therefore, the relation $(a.)$ is true. As the sample variance $s_i^2(\Psi)$ is represented by

$$s_i^2(\Psi) = \frac{1}{T-1}\Bigg[\sum_{t=1}^{T} p_i^2(\Psi_t) - \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T} p_i(\Psi_t)p_i(\Psi_s)\Bigg],$$

if we take expectation, then it follows from equation (2.6) one can obtain the relation $(d.)$. It should be note that if $\varphi = 0$, then $\mathbb{E}[s_i^2(\Psi)] = \Phi^*_{i,2}(0) - \bar{p}_i^2$, cf. [4; 12]. Which completes the proof of the Proposition. □

Let us denote a realization of the mixing random variable Ψ_t by ψ_t for $t = 1, \dots, T$. Then, from the above Proposition 1 one can obtain moment based parameter estimations of KMV/Riskmetrics model: the average default probability $\bar{p}_i$ is estimated by $\bar{p}_i = \frac{1}{T}\sum_{t=1}^{T} p_i(\psi_t)$, the asset correlation ρ_i is estimated from nonlinear equation $\frac{1}{T}\sum_{t=1}^{T} p_{i,t}^2(\psi_t) = \Phi^*_{i,2}(0)$, and after substituting the estimation of the asset correlation into right hand side of the following equation

$$\frac{1}{T-1}\sum_{t=2}^{T} p_{i,t}(\psi_t)p_{i,t}(\psi_{t-1}) = \Phi^*_{i,2}(1),$$

the parameter φ is estimated from above nonlinear equation. Let us denote the estimation corresponding to credit rating i of the parameter φ by $\hat{\varphi}_i$. Then, since estimation of the parameter φ obtained for each rating i, one can naturally estimate the parameter φ by average of the estimations $\hat{\varphi}_i$, $i = 1, \dots, R$, namely, $\hat{\varphi} = \frac{1}{R}\sum_{i=1}^{R} \hat{\varphi}_i$.

3. Nash Model with Basel II Solvency Constraint

In this section, we introduce the loan interest rate Nash game model with Basel II solvency constraint. The similar models were investigated in the non–life insurance market [2; 8] and banking industry [3]. We assume that in the market, N banks are competing on loan interest rates for T period and that there are R credit ratings for loans. For the loan with credit rating i, conditional on mixing random process Ψ_t, which represents economic conditions let $p_i(\Psi_t)$ be its default probability and Δ_i^t be the loss given default at time t. For $1 \leq i \leq R$, we assume that the loss given defaults Δ_i^t have values in $(0,1]$. For the n–th bank, let $L_{i,n}^t$ be an amount at time t of loan portfolio with i–th credit rating. Let $x_{i,n}^t$ be a loan interest rate at time t of bank n corresponding to i–th credit rating. Then, taking into account the approach in [8] (see also [2; 3]), the n–th bank maximizes a sum of discounted future net incomes before taxes defined as

$$O_n(x) = \sum_{t=1}^{T}\sum_{i=1}^{R} v^t L_{i,n}^t \left(1 - \beta_{i,n}^{t,L}\left(\frac{x_{i,n}^t}{m(x_{i,n}^t)} - 1\right)\right)\left(x_{i,n}^t - \pi_n^t - e_n^t - \Delta_i^t p_i(\Psi_t)\right), \tag{3.1}$$

where $v = 1/(1+r)$ is the discount rate corresponding to the risk–free interest rate r, for the n–th bank with i–th credit rating, $\beta_{i,n}^{t,L} > 0$ is the loan interest rate elasticity parameter at time t, π_n^t is the break–even interest rate at time t, e_n^t is the expense rate at time t, $x = (x_1^T, \dots, x_N^T)^T$, $x_n = (x_{1,n}^1, \dots, x_{R,n}^1, \dots, x_{1,n}^T, \dots, x_{R,n}^T)^T$, $n = 1, \dots, N$ and $m(z_{i,n}^t) = \frac{1}{N-1}\sum_{k\neq n} z_{i,k}^t$ is the market interest proxy at time t, i.e., for the n–th bank with i–th credit rating, $m(z_{i,n}^t)$ is the average of the loan interest rates at time t of other banks with i–th credit rating.

It should be noted that the break–even interest rate π_n^t and expense rate e_n^t are defined by a ratio between deposit interest expense and net loan, and the ratio difference of non–interest expense (excluding tax expense and loan loss expense) and non–interest income to net loan, respectively, in numerical experiments section, see below.

The solvency constraint aims to require banks to hold a certain amount of capital in order to protect depositors against adverse economic conditions. Therefore, in addition to maximizing an objective function (3.1), banks must satisfy a solvency constraint imposed by the regulator. In this paper, we consider Basel II constraint.

Now we move to the solvency constraint. For the n–th bank, we assume that there are $m_{i,n}^s$ loans at time s with credit rating i and for the k–th lender, its corresponding default indicator is $Y_{i,k,n}^s$, and its exposure is $\theta_{i,k,n}^s$. Observe that for all banks and all the lenders with same credit rating, conditional default probabilities of the lenders are same, that is, $\mathbb{P}(Y_{i,k,n}^s = 1|\Psi_s) = p_i(\Psi_s)$, $s = 1, \dots, T$, $k = 1, \dots, m_{i,n}^s$, $i = 1, \dots, R$ and $\sum_{k=1}^{m_{i,n}^s} \theta_{i,k,n}^s = L_{i,n}^s$. For the n–th bank, the random loss at time t from the loan portfolio is given by

$$Q_n^t := \sum_{s=1}^{t}\sum_{i=1}^{R}\sum_{k=1}^{m_{i,n}^s} v^s \theta_{i,k,n}^s \Delta_i^s Y_{i,k,n}^s,$$

and the expected loss conditional on mixing random vector

$$\Psi := (\Psi_1, \dots, \Psi_T)$$

becomes

$$\mathbb{E}\big[Q_n^t|\Psi\big] := \sum_{s=1}^t \sum_{i=1}^R \sum_{k=1}^{m_{i,n}^s} v^s \theta_{i,k,n}^s \Delta_i^s \mathbb{E}\big[Y_{i,k,n}^s|\Psi\big] = \sum_{s=1}^t \sum_{i=1}^R v^s L_{i,n}^s \Delta_i^s p_i\big(\Psi_s\big).$$

According to [10] (see also [16]), if the loan portfolio size is sufficiently large, then for $t = 1, \dots, T$, the α–quantile of the loan loss random variable Q_n^t can be approximated by

$$q_\alpha(Q_n^t) \approx q_\alpha\big(\mathbb{E}\big[Q_n^t|\Psi\big]\big) = q_\alpha\Bigg(\sum_{s=1}^t \sum_{i=1}^R v^s L_{i,n}^s \Delta_i^s p_i(\Psi_s)\Bigg),$$

where $q_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq \alpha\}$ is the α-quantile of a random variable X. For the portfolio loss ratio, a similar result was given by Gordy in [13]. This is one of the most influential works in the area of capital charges in the Basel II proposal on credit risk. For the n–th bank, the Basel II solvency constraint at time t conditional on Ψ can be defined as

$$g_{t,n}(x_n) = K_n + \sum_{s=1}^t \sum_{i=1}^R v^s L_{i,n}^s \big(x_{i,n}^s - \pi_n^s - e_n^s - \Delta_i^s p_i(\Psi_s)\big) - q_\alpha\big(\mathbb{E}\big[Q_n^t|\Psi\big]\big) \tag{3.2}$$

for $n = 1, \dots, N$ and $t = 1, \dots, T$. Therefore, the strategy set of the n–th player, $n \in \{1, \cdots, N\}$ is given by

$$\begin{aligned} X_n \;:=\; \Bigg\{ & x_n \in \prod_{t=1}^T \prod_{i=1}^R [\underline{x}_i^t, \overline{x}_i^t] \;\Big|\; K_n + \sum_{s=1}^t \sum_{i=1}^R v^s L_{i,n}^s \big(x_{i,n}^s - \pi_n^s - e_n^s \\ & - \;\Delta_i^s p_i(\Psi_s)\big) \geq q_\alpha\Bigg(\sum_{s=1}^t \sum_{i=1}^R v^s L_{i,n}^s \Delta_i^s p_i(\Psi_s)\Bigg),\; t = 1, \dots, T\Bigg\}, \end{aligned} \tag{3.3}$$

where for $i = 1, \dots, R$ and $t = 1, \dots, T$, $\underline{x}_i^t$, and $\overline{x}_i^t$ represent minimum, and maximum loan interest rate at time t of loan with credit rating i. The minimum and maximum loan interest rates could be set by a regulators (some countries have loan interest rate ceiling). For equation (3.3), which guarantees that n–th bank still has solvency after its lending business, $\sum_{s=1}^t \sum_{i=1}^R v^s L_{i,n}^s \big(x_{i,n}^s - \pi_n^s - e_n^s - \Delta_i^s p_i(\Psi_s)\big)$ expresses capital increment of n–th bank.

The loan interest rate Nash game model with Basel II solvency constraint consists in finding $x = (x_1^T, \dots, x_N^T)^T$ such that for the player n it holds:

$$\begin{aligned} (\mathrm{NEP}_n) \quad & \text{maximize } O_n(x) \\ & \text{subject to } x_n \in X_n,\; x_n = (x_{1,n}^1, \dots, x_{R,n}^1, \dots, x_{1,n}^T, \dots, x_{R,n}^T)^T. \end{aligned} \tag{3.4}$$

The following Proposition is devoted to dealing with the existence of premium equilibrium. Because a proof of the Proposition is similar as in [2; 3; 8], we omit the proof.

Proposition 2. *The bank game with N players whose objective functions are defined by* (3.1) *and solvency constraints are defined by* (3.3) *admits a unique Nash interest rate equilibrium.*

To keep notation simple, let us define $\alpha^t_{k,n} := \beta^{t,L}_{k,n}\big[\pi^t_n + e^t_n + \Delta^t_n p_k(\Psi_t)\big]$ for $k = 1, \ldots, R$, $n = 1, \ldots, N$, and $t = 1, \ldots, T$ and

$$A^t_k = \frac{1}{N-1}\begin{pmatrix} 2(N-1)\beta^{t,L}_{k,1} & -\big(1+\beta^{t,L}_{k,1}\big) & \cdots & -\big(1+\beta^{t,L}_{k,1}\big) \\ -\big(1+\beta^{t,L}_{k,2}\big) & 2(N-1)\beta^{t,L}_{k,2} & \cdots & -\big(1+\beta^{t,L}_{k,2}\big) \\ \vdots & \vdots & \ddots & \vdots \\ -\big(1+\beta^{t,L}_{k,N}\big) & -\big(1+\beta^{t,L}_{i,N}\big) & \cdots & 2(N-1)\beta^{t,L}_{k,N} \end{pmatrix}$$

for $k = 1, \ldots, R$ and $t = 1, \ldots, T$. For the Nash problem (3.4), if we assume that there are ℓ $(1 \le \ell \le T)$ solvency constraints active, namely, $t_0 := 0 < t_1 < \cdots < t_\ell \le T$, then we have $g_{t_1,n}(x_n) = 0, \ldots, g_{t_\ell,n}(x_n) = 0$. In this case, it follows from the Basel II solvency constraint equation (3.2) that

$$\sum_{s=1}^{t_1}\sum_{i=1}^{R} v^s L^s_{i,n} x^s_{i,n} = \sum_{s=1}^{t_1}\sum_{i=1}^{R} v^s L^s_{i,n}\big(\pi^s_{i,n}+e^s_n+\Delta^s_i p_i(\Psi_s)\big)+q_\alpha\big(\mathbb{E}\big[Q^{t_1}_n|\Psi\big]\big)-K_n \tag{3.5}$$

and for $q = 2, \ldots, \ell$,

$$\begin{aligned}\sum_{s=t_{q-1}+1}^{t_q}\sum_{i=1}^{R} v^s L^s_{i,n} x^s_{i,n} \;=\; & \sum_{s=t_{q-1}+1}^{t_q}\sum_{i=1}^{R} v^s L^s_{i,n}\big(\pi^s_{i,n} + e^s_n + \Delta^s_i p_i(\Psi_s)\big) \\ & +\; q_\alpha\big(\mathbb{E}\big[Q^{t_q}_n|\Psi\big]\big) - q_\alpha\big(\mathbb{E}\big[Q^{t_{q-1}}_n|\Psi\big]\big).\end{aligned} \tag{3.6}$$

Then, the following Proposition is used to the sensitivity analysis for our model with Basel II solvency constraint and following the ideas in [2; 3; 8] and Rau–Bredow [17], and equations (3.5)–(3.6) one can prove it.

Proposition 3. *Let x^* be the loan rates equilibrium of the Basel II bank game with N players and T periods. Then, the following results hold:*

1) If ℓ $(1 \le \ell \le T)$ solvency constraints are active, namely, $t_0 := 0 < t_1 < \cdots < t_\ell \le T$, then we have:

(i) for $q = 1, \ldots, \ell$ and $s = t_{q-1} + 1, \ldots, t_q$, the corresponding loan rate equilibrium $x^{s*}_{k,n} \in]\underline{x}^s_k, \overline{x}^s_k[$ increases with the break–even interest rate π^u_n, expense rate e^u_n, loss given default Δ^u_k, and mixing random variable Ψ_u for all $u = t_{q-1} + 1, \ldots, t_q$

(ii) for $s = 1, \ldots, t_1$, the corresponding loan rate equilibrium $x_{k,n}^{s*} \in]\underline{x}_k^s, \overline{x}_k^s[$ increases with the average default probability $\bar{p}_k$ and confidence level α, and decreases with the capital K_n.

2) If the solvency constraint functions are all inactive, then we have:

(i) the loan rates equilibrium is a solution of the linear system of equations

$$M_\beta x^* = b,$$

where

$$M_\beta := diag\{A_1^1, \ldots, A_R^1, \ldots, A_1^T, \ldots, A_R^T\},$$

and

$$\begin{aligned} b \; := \; & \big(\alpha_{1,1}^1, \ldots, \alpha_{1,N}^1, \ldots, \alpha_{R,1}^1, \ldots, \alpha_{R,N}^1, \ldots, \\ & \alpha_{1,1}^T, \ldots, \alpha_{1,N}^T, \ldots, \alpha_{R,1}^T, \ldots, \alpha_{R,N}^T\big)^T \end{aligned}$$

(ii) and for each player n, rating k, and period s, the corresponding loan rate equilibrium $x_{k,n}^{s*} \in]\underline{x}_k^s, \overline{x}_k^s[$ depends on the parameters in the following way:

(a) it increases with break–even interest rate π_n^s, expense rate e_n^s, loss given default Δ_k^s, average default probability $\bar{p}_k$, and mixing random variable Ψ_s and

(b) it decreases with the sensitivity parameter $\beta_{k,n}^{s,L}$.

3) If $x_{k,n}^{s*} = \underline{x}_k^s$ or $\overline{x}_k^s$, then the loan interest rate equilibrium is independent of those parameters.

Remark 1. The bank loan interest rate Nash problem can be extended to the bank loan deposit interest rate Nash game in the following way. Let us assume that there are S deposit products, for the n–th bank, $D_{j,n}^t$ is the amount of the j–th deposit product at time t, and $y_{j,n}$ is the corresponding interest rate. Then

$$\begin{aligned} O_n(x, y) \; = \; & \sum_{t=1}^{T} \sum_{i=1}^{R} v^t L_{i,n}^t \left[1 - \beta_{i,n}^{t,L} \left(\frac{x_{i,n}^t}{m(x_{i,n}^t)} - 1\right)\right] \big(x_{i,n}^t - e_n^t - \Delta_i^t p_i(\Psi_t)\big) \\ - \; & \sum_{t=1}^{T} \sum_{j=1}^{S} v^t D_{j,n}^t \left[1 + \beta_{j,n}^{t,D} \left(\frac{y_{j,n}^t}{m(y_{j,n}^t)} - 1\right)\right] y_{j,n}^t, \end{aligned}$$

where $\beta_{j,n}^{t,D} > 0$ is the deposit rate elasticity parameter at time t, $y = (y_1^T, \ldots, y_N^T)^T$ and $y_n = (y_{1,n}^1, \ldots, y_{S,n}^1, \ldots, y_{1,n}^T, \ldots, y_{S,n}^T)^T$ for $n = 1, \ldots, N$. In this case, the Basel II solvency constraint conditional on Ψ is given by

$$\begin{aligned} g_{t,n}(x_n, y_n) \;=\; & K_n + \sum_{s=1}^{t}\sum_{i=1}^{R} v^s L^s_{i,n}(x^s_{i,n} - e^s_n - \Delta^s_i p_i(\Psi_s)) - \sum_{s=1}^{t}\sum_{j=1}^{S} D^s_{j,n} y^s_{j,n} \\ & - \; q_\alpha\Big(\sum_{s=1}^{t}\sum_{i=1}^{R} v^s L^s_{i,n}\Delta^s_i p_i(\Psi_s)\Big), \quad t = 1, \dots, T. \end{aligned}$$

This loan deposit interest rate Nash game has a unique Nash equilibrium and the sensitivity analysis can be investigated in a similar way.

Remark 2. For a game with one leader and $N-1$ followers with payoff function O_n and the strategy set X_n, the Stackelberg equilibrium is a problem that consists of finding a vector $\bar{x} = (\bar{x}_1^T, \cdots, \bar{x}_N^T)^T$, $\bar{x}_n = (\bar{x}^1_{1,n}, \dots, \bar{x}^1_{R,n}, \cdots, \bar{x}^T_{1,n}, \dots, \bar{x}^T_{R,n})^T$ such that $\bar{x}_1$ solves the problem

$$\sup_{y \in X_1} O_1(y; x_2, \cdots, x_N),$$

where $(x_2^T, \cdots, x_N^T)^T$ is a Nash equilibrium for the game with $N-1$ followers and given strategy x_1 for bank 1, which is assumed to be a leader. In this case, one can show the existence of the Stackelberg equilibrium (cf. [8]).

4. Numerical experiments

In this section, we give some numerical results based on the Mongolian two commercial banks which are domestic systematically important. It is the well known fact that the Nash equilibrium model can be reduced into the variational inequality problem, which consists of finding $x \in \Omega := X_1 \times X_2 \times \cdots \times X_N$ such that

$$(VI) \qquad \langle F(x), y - x\rangle \geq 0, \;\forall y \in \Omega,$$

where $F(x) = \big(\nabla_{x_n} O_n(x)\big)_{n=1}^N$, which can be solved by the hyperplane projection algorithm (see [9] and [19]). This algorithm has also been used in [1–3].

In Mongolia, in 2020, there are 11 commercial banks, which are actively competing for financial products. Mongolbank (Mongolian central bank) classified Mongolian commercial banks by domestic systemically important banks and rests. The systemically important banks composed of 93.4% of total banking system assets. For 12/31/2020, 1 USD equals 2,849.89 MNT (Mongolian tugrik), and by aggregate bank balance report of Mongolbank, banking system asset was 36,685.49 billion MNT (equals to 12,872.60 million USD). The fourth–quarter financial results of these three banks for 2018 are summarized in Table 1. For the capital adequacy ratio, the Mongolian banking system still uses the Basel I ratio at 12%.

Table 1

Main Financial Results of the Two Banks, Q4, 2020, in million USD

	MS_n	L_n	K_n	K_n/L_n	CAR%	RWA	π_n	e_n
Bank1	19.8%	1,063.67	221.61	20.83%	14.16%	1,565.03	12.07%	2.90%
Bank2	9.5%	615.49	106.55	17.31%	15.84%	672.68	13.34%	4.16%

(Note: MS_n is the market share of the selected banks in the banking system total assets, L_n is the bank's net loan and advances, K_n is the bank's capital, K_n/L_n is the ratio between capital and net loan, CAR% is the capital adequacy ratio, RWA is a risk–weighted asset calculated from banks net loan and advance and CAR%)

According to Mongolbank website, on 07/29/21 S&P Global Ratings affirmed that Mongolian credit rating is B, which is the latest rating of Mongolia. However we do not know the default probabilities of the two banks' customers, because of the latest country rating, we assume that their customers' default probabilities are the same as S&P's ratings of B+, B, and B–. For the ratings, using Proposition 1 we estimated the parameters of the model that correspond to the annual default rate data from S&P for the period 1981–2020, see [18]. The parameter estimations are listed in Table 2. It should be noted that in order to increase the rating data and to estimate the parameter φ, we additionally estimated parameters that correspond to ratings BB+, BB, BB–, and CCC/C. From the last row of the Table 2, one can obtain estimation of the parameter φ by $\hat{\varphi} = \frac{1}{7}\sum_{i=1}^{7}\hat{\varphi}_i = 0.416$

Table 2

Parameter Estimation of the Model

	BB+	BB	BB–	B+	B	B–	CCC/C
$\hat{\bar{p}}_i$	0.005	0.007	0.012	0.021	0.058	0.087	0.249
$\hat{\rho}_i$	0.195	0.123	0.170	0.124	0.147	0.179	0.131
$\hat{\varphi}_i$	0.330	0.170	0.372	0.424	0.691	0.557	0.365

(Note: For loans with credit rating i, $\hat{\bar{p}}_i$ is the parameter estimation of the average default probability $\bar{p}_i$, $\hat{\rho}_i$ is the parameter estimation of the asset correlation ρ_i, and $\hat{\varphi}_i$ is the parameter estimation of the AR(1) economic condition process Ψ_t)

Now, we define the base case parameters of the model. Here we assume that all the parameters of model are same for the 1st and 2nd period. As we do not know the loan portfolio structure of the two banks, we assume for all banks that the loan portfolio consists of $s_1 = 30\%$ B+ rated loans, $s_2 = 40\%$ B rated loans, and $s_3 = 30\%$ B– rated loans. For the capital multiplier, we assume that its value is equal to $K = 1$ for the base case. For the interest rate elasticity, we assume that for $i = 1, 2, 3$ and $t = 1, 2$, $\beta_{i,1}^t = 4.5$ and $\beta_{i,2}^t = 3.5$. For example, if the 2nd bank's interest rate increases by 1% compared to other bank's interest rates, then the 2nd bank will lose 3.5% of the loan portfolio. For the expense rate, we assume that its value is given by the seventh column of Table 1, i.e. $e_1^t = 2.90\%$ and $e_2^t = 4.16\%$ for $t = 1, 2$. For the break–even interest rate, we assume that the value is given by the sixth column in Table 1, i.e. $\pi_1^t = 12.07\%$ and $\pi_2^t = 13.34\%$ for $t = 1, 2$. For economic condition variables at times $t = 1, 2$, Ψ_t corresponding to the base case, we take their values equal to zero. We assume that the confidence level is equal to $\alpha = 99.9\%$ and that for LGD it is equal to $\Delta_i^t = 50\%$ for $i = 1, 2, 3$ and $t = 1, 2$. For base case risk–free interest rate, we suppose that its value equals to $r = 6\%$. Moreover, for all the ratings and banks, let the interest rates be unconstrained, i.e. $\prod_{i=1}^{3} [\underline{x}_i^t, \overline{x}_i^t] = \mathbb{R}^3$ for $t = 1, 2$. All parameters for the base case are given in the first column of Table 3.

Table 3

BIS–II Solvency Constraint Nash Equilibrium Results

P/NE	BIS–II–1	BIS–II–2	BIS–II–3	BIS–II–4	BIS–II–5	BIS–II–6	BIS–II–7	BIS–II–8
$s\%$	30–40–30	30–40–30	30–40–30	30–40–30	30–40–30	30–40–30	30–40–30	30–40–30
K	1	**0.8**	1	1	1	1	1	1
β	4.5–3.5	4.5–3.5	**5.0–4.0**	4.5–3.5	4.5–3.5	4.5–3.5	4.5–3.5	4.5–3.5
$e\%$	2.9–4.2	2.9–4.2	2.9–4.2	**3.9–5.2**	2.9–4.2	2.9–4.2	2.9–4.2	2.9–4.2
$\pi\%$	12.1–13.3	12.1–13.3	12.1–13.3	12.1–13.3	**11.1–12.3**	12.1–13.3	12.1–13.3	12.1–13.3
Ψ	0	0	0	0	0	**−1**	0	0
α	99.9%	99.9%	99.9%	99.9%	99.9%	99.9%	**95.0%**	99.9%
Δ	50%	50%	50%	50%	50%	50%	50%	**30%**
r	6%	6%	6%	6%	6%	6%	6%	6%
$x_{1,1}^{1*}$	0.2148	0.2219	0.2072	0.2281	0.2015	0.2086	0.2148	0.2109
$x_{2,1}^{1*}$	0.2343	0.2420	0.2259	0.2475	0.2210	0.2163	0.2343	0.2226
$x_{3,1}^{1*}$	0.2495	0.2578	0.2407	0.2628	0.2363	0.2214	0.2495	0.2317
$x_{1,1}^{2*}$	0.2315	0.2391	0.2232	0.2447	0.2182	0.2252	0.2315	0.2276
$x_{2,1}^{2*}$	0.2509	0.2592	0.2420	0.2642	0.2376	0.2330	0.2509	0.2392
$x_{3,1}^{2*}$	0.2662	0.2750	0.2567	0.2794	0.2529	0.2381	0.2662	0.2484
$x_{1,2}^{1*}$	0.2230	0.2346	0.2144	0.2365	0.2095	0.2167	0.2230	0.2190
$x_{2,2}^{1*}$	0.2428	0.2555	0.2334	0.2564	0.2293	0.2245	0.2428	0.2309
$x_{3,2}^{1*}$	0.2584	0.2719	0.2484	0.2719	0.2449	0.2298	0.2584	0.2403
$x_{1,2}^{2*}$	0.2400	0.2525	0.2307	0.2535	0.2265	0.2336	0.2400	0.2360
$x_{2,2}^{2*}$	0.2598	0.2734	0.2498	0.2733	0.2463	0.2415	0.2598	0.2479
$x_{3,2}^{2*}$	0.2754	0.2898	0.2647	0.2889	0.2618	0.2467	0.2754	0.2572

(Note: P/NE means parameters and Nash equilibrium, $s\%$ is the loan portfolio composition, K is the capital multiplier, β is the loan interest rate elasticity, $e\%$ is the expense, π is the break even interest rate, Ψ is the economic condition, α is the confidence level, Δ is the loss given default, r is the risk–free rate, and $x_{i,n}^{t*}$ is the loan interest Nash equilibrium at time t for n—th bank's loans with credit rating i)

In the BIS–II–1 column in Table 3, we have Nash equilibriums corresponding to the base case, where $x_{i,n}^{t*}$ represents a Nash equilibrium at period t for a loan interest rate of the n–th bank with i–th credit rating. In column BIS–II–2, we assume that for all banks capital decreased by 20% at periods 1 and 2. In the BIS–II–3 column, we give results that correspond to the interest rate elasticity of the loans are increased by 0.5 unit for the two banks. In the BIS–II–4 column, the expense of the two banks increase by 1%. If we decrease the base case break–even interest rates by 1%, we get the BIS–II–5 column. BIS–II–6 column in Table 3 represents the economic conditions that correspond to $\Psi_1 = -1$ and $\Psi_2 = -1$ and shows a better scenario compared to the base case. Column BIS–II–7 represents the scenario that corresponds to the confidence level being equal to $\alpha = 95\%$. Lastly, if we assume LGD decreased to 30%, then the loan rate equilibrium is given in the BIS–II–8 column. After defining the parameters, in order to calculate approximate α–quantile of the random loss from loan portfolio, $q_\alpha(\mathbb{E}[Q_n^t|\Psi])$ we made 10,000 Monte–Carlo simulation.

5. Conclusion

This paper devoted to investigate multi–period loan interest rate Nash game models in the banking sector under Basel II regulatory solvency constraints. The suggested model can be used for other banking system. It would be interesting to consider the models in full stochastic environment, namely, stochastic Nash equilibrium problems and to connect Begre equilibrium problems.

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