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Approximations of Acyclic Graphs

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Abstract. In this paper, approximations of acyclic graphs are studied. It is proved that any theory of an acyclic graph (tree) of finite diameter is pseudofinite with respect to acyclic graphs (trees), that is, any such theory is approximated by theories of finite structures (acyclic graphs, trees). It is also proved that an acyclic graph of infinite diameter with infinite number of rays is pseudofinite.

Keywords: approximation of theory, tree, acyclic graph, pseudofinite theory

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Научная статья

Аппроксимации ациклических графов

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Аннотация. В статье исследуются аппроксимации теории ациклических графов. Доказано, что теория любого ациклического графа (дерева) конечного диаметра псевдоконечна, то есть любая такая теория аппроксимируется теориями конечных структур (ациклических графов, деревьев). Также доказано, что ациклический граф бесконечного диаметра с бесконечным числом лучей псевдоконечен.

Ключевые слова: аппроксимация теории, дерево, ациклический граф, псевдоконечная теория

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1. Preliminaries

Throughout the article, we use the terminology from [1; 3; 8]. We also continue to study the approximations of various structures [4–7].

1.1. Preliminaries from graph theory

Recall the following basic concepts. Let $L = \{R\}$, where R is a binary relation symbol. A *graph* is a structure $\Gamma = \langle G, R\rangle$. The elements of the universe G are called the *vertices* of the graph Γ, and the elements of the binary relation $R \subseteq G^2$ are *edges*. The edge connecting the vertices a and b is denoted by $[a, b]$. If a vertex $a \in G$ is an endpoint of edge $u \in R$, then a and u are *incident*. The *degree* of vertex a in a graph Γ, written $deg_\Gamma(a)$ or simply $deg(a)$, is the number of edges incident to a, except that each loop at a counts twice. A vertex of degree 0 is called *isolated*, a vertex of degree 1 is called *leaf* or *hanging vertex*. A graph that contains no cycles is called an *acyclic graph*. A connected acyclic graph is called a *tree*. Any graph without cycles is also called a *forest* so that the components of a forest are trees. Subsystems $\Gamma' = \langle G', R'\rangle$ of the graph Γ are called *subgraphs*. If $\Gamma' \subseteq \Gamma$ and Γ' contains all the edges $[x, y] \in R'$ with $x, y \in G'$, then Γ' is an *induced subgraph* of Γ.

Theorem 1. [3] *Any nontrivial finite tree has at least two hanging vertices.*

Recall that a *connected component* of the graph $\Gamma = \langle G, R\rangle$ is a connected subgraph of the graph Γ that is maximal by inclusion. Any connected component C of the graph Γ is uniquely determined by any of its elements $a \in G$ and is denoted by $C(a, G)$ or by $C(a, R)$, when it is clear what universe G go speech.

A *route* in the graph Γ is any non-empty sequence

$$\bar{r}(a_0, a_n) = (a_0, \dots, a_n)$$

of vertices for which $\Gamma \models R(a_i, a_{i+1}), i = 0, \dots, n$. In this case, the route $\bar{r}(a_0, a_n)$ will also be called (a_0, a_{n-1})-route (or n-route), and the natural number n is the length of the route $\bar{r}(a_0, a_n)$.

A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. If for two vertices $a, b \in G$ there is a path connecting them, then there is sure to be a minimal path connecting these vertices. We denote the length of this path by $\rho(a, b)$. If Γ has no such path, then $\rho(a, b) = \infty$. A tree is a path if and only if $deg(a) \leq 2$ for all vertice a of the tree.

Definition 1. [8] For a fixed vertex a, the value $e(a) \rightleftharpoons max\{\rho(a, b) | b \in G\}$ is called the *eccentricity* of the vertex a. The eccentricity of the vertex is equal to the distance from this vertex to the most distant from it. The maximum among all the eccentricities of the vertices is called the *diameter* of the graph Γ and is denoted by $d(\Gamma) : d(\Gamma) \rightleftharpoons max\{e(a) | a \in G\}$. The vertex a is called *peripheral* if $e(a) = d(\Gamma)$. The minimal eccentricity of the graph Γ is called its *radius* and is denoted by $r(\Gamma) : r(\Gamma) \rightleftharpoons min\{e(a) | a \in G\}$. The vertex a is called *central* if $e(a) = r(\Gamma)$. The set of all central vertices of a graph is called its *center*.

Definition 2. [2] An infinite graph $\Gamma = \langle G, R \rangle$ of the form $G = \{a_0, a_1, a_2, \dots\}, R = \{(a_0, a_1), (a_1, a_2), (a_2, a_3), \dots\}$ is called a *ray*, and a *double ray* is an infinite graph $\Gamma = \langle G, R \rangle$ of the form $G = \{\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots\}$, $R = \{\dots, (a_{-2}, a_{-1}), (a_{-1}, a_0), (a_0, a_1), (a_1, a_2), \dots\}$; in both cases the a_n are assumed to be distinct.

1.2. Preliminaries from model theory and approximations of theories

In 1968, James Ax, in his work [1], first introduced the concept of *pseudofiniteness* to show the decidability of the theory of all finite fields, i.e. there is an algorithm to decide whether a given statement is true for all finite fields. It was proved that pseudofinite fields are exactly those infinite fields that have every elementary property common to all finite fields, that is, pseudofinite fields are infinite models of the theory of finite fields. Pseudofinite structures after Ax have not been studied in general for a long time. Until the 1990s, only a few results were obtained on this topic. After a long break, pseudofinite structures began to be studied in the joint works of E. Hrushovski and G. Cherlin.

Definition 3. [1] An L-structure $\mathcal{M}$ is *pseudofinite* if for all L-sentences φ, $\mathcal{M} \models \varphi$ implies that there is a finite $\mathcal{M}_0$ such that $\mathcal{M}_0 \models \varphi$. The theory $T = Th(\mathcal{M})$ of the pseudofinite structure $\mathcal{M}$ is called pseudofinite.

In classical logic, the following property is a straightforward consequence of pseudofiniteness.

Proposition 1. [1] *Let $\mathcal{M}$ be a pseudofinite structure and $f : M^k \to M^k$ be a definable function. Then f is injective if and only if f is surjective.*

The structure of relational language L with universe $\bigsqcup_{i\in I} \mathcal{M}_i$ and interpretations of relation symbols from L represented as a combination of their interpretations in the structures $\mathcal{M}_i$, $i \in I$, is called a *disjoint union* $\bigsqcup_{i\in I} \mathcal{M}_i$ of disjoint structures $\mathcal{M}_i$, $i \in I$.

Definition 4. [7] Let $\mathcal{T}$ be a family of theories and T be a theory such that $T \notin \mathcal{T}$. The theory T is said to be *$\mathcal{T}$-approximated,* or *approximated by the family $\mathcal{T}$*, or a *pseudo-$\mathcal{T}$-theory,* if for any formula $\varphi \in T$ there exists $T' \in \mathcal{T}$ for which $\varphi \in T'$.

If the theory T is $\mathcal{T}$-approximated, then $\mathcal{T}$ is said to be an *approximating family* for T, and theories $T' \in \mathcal{T}$ are said to be *approximations* for T.

2. Classes of acyclic graphs (trees)

We denote the following classes of acyclic graphs and consider various approximations for these classes.

Let $\mathcal{G}_{fin}(\lambda)$, for arbitrary cardinality λ, be the family of all infinite acyclic graphs consisting of λ connected components of bounded in aggregate diameters.

Acyclic graph from the class $\mathcal{G}_{fin}(1)$ is an *tree*, that is, it consists of a single connected component with infinitely many hanging vertices.

Acyclic graphs from the class $\mathcal{G}_{fin}(\lambda)$, for arbitrary cardinality λ, can contain both finite and infinite components of bounded diameters.

Let $\mathcal{G}_{inf}(\lambda)$, for arbitrary cardinality λ, be the family of all infinite acyclic graphs consisting of λ connected components of infinite diameters.

It is clear that any component of the graph from $\mathcal{G}_{inf}(\lambda)$ is infinite. An infinite component has an infinite diameter if it contains at least one ray. Acyclic graphs from the class $\mathcal{G}_{inf}(1)$ may contain a finite or infinite number of hanging vertices or not have hanging vertices. An example of an acyclic graph from this class with only one hanging vertex is *ray.* And an example of an acyclic graph from this class without hanging vertices is *double ray.*

Acyclic graphs from the class $\mathcal{G}_{inf}(2)$ consists of two connected components of infinite diameters. Each component can have an unbounded number of hanging vertices.

For theories of graphs of the above-defined classes, we will consider various approximations (approximation by theories of acyclic graphs or trees and cycles).

3. Approximations of acyclic graphs from the class $\mathcal{G}_{inf}(\lambda)$

Let us consider how the theories of an acyclic graphs from the class $\mathcal{G}_{inf}(\lambda)$ with a finite number of rays are approximated by theories of finite structures. By the number of rays, we mean the number of rays emanating from a fixed number of vertices.

Theorem 2. *Let T be the theory of an acyclic graph Γ from the class $\mathcal{G}_{inf}(\lambda)$, for finite cardinality λ, with finitely many rays. If the number of rays in Γ is even then T is pseudofinite theory.*

Proof. Let $\Gamma = \langle G, R\rangle$, $G = \coprod_{i\in\lambda} G_i$, where G_i are trees. We fix a partition of the set of rays of the graph Γ into pairs. For an arbitrary $n \in \omega$ and a tuple $\bar{a} = \langle a_0, \ldots, a_{n-1}\rangle \in G$, where a_i lies to different connected components of the graph Γ, we define a finite graph $\Gamma_{\bar{a},n}$ as follows: to the union of n-neighborhoods of vertices $a_i, 0 \leq i \leq n-1$, we add the edges that connect the ends of pairs of rays belonging to the union of n-neighborhoods of vertices $a_i, 0 \leq i \leq n-1$. Then the theory T is approximated by the set $\{Th(\Gamma_{\bar{a},n}) | n \in \omega, \bar{a} = \langle a_0, \ldots, a_{n-1}\rangle \in G, \forall i(a_i \in G_{k_i}), G_{k_i} \cap G_{k_j} = \emptyset, (i \neq j)\}$. □

Theorem 3 immediately implies the following:

Corollary 1. *For any $\mu > 1$, there is a pseudofinite acyclic graph with μ hanging vertices.*

Theorem 3. *The theory T of an infinite acyclic graph $\Gamma \in \mathcal{G}_{inf}(\lambda)$ for arbitrary cardinality λ with an infinitely many rays is pseudofinite.*

Proof. Let T be a theory of an acyclic graph Γ with infinite number of rays. The set of theories of all subgraphs Γ' of graph Γ such that $\Gamma' \in \mathcal{G}_{inf}(\lambda)$ for some finite $\mu \leq \lambda$, Γ' with an even number of rays, approximates the theory $T = Th(\Gamma)$, so, by Theorem 2, T is pseudofinite. □

Example 1. We consider the theory T of an acyclic graph, which is a double ray. We denote the double ray by $\Gamma = \langle G, R\rangle$, where $G = \{\ldots, a_{-2}, a_{-1}, a_0, a_1,$
$a_2, \ldots\}$, $R = \{\ldots, (a_{-2}, a_{-1}), (a_{-1}, a_0), (a_0, a_1), (a_1, a_2), \ldots\}$. The thoery T

can be approximated by a set of theories of cycles of increasing length. So the theory T is pseudofinite.

Example 2. We consider the theory T of an infinite acyclic graph Γ from the class $\mathcal{G}_{inf}(\lambda)$, consisting of λ infinite components of infinite diameters without hanging vertices. It is clear that components of Γ are double rays. As in Example 1, the theory T can be approximated by a set of theories of cycles of increasing length. So the theory T is pseudofinite.

4. Approximations of acyclic graphs from the class $\mathcal{G}_{fin}(\lambda)$

The following theorem shows that any theory of an acyclic graph from the class $\mathcal{G}_{fin}(\lambda)$ is approximated by theories of finite structures.

Theorem 4. *Theory T of any infinite acyclic graph Γ from the class $\mathcal{G}_{fin}(\lambda)$, for arbitrary cardinality λ, is pseudofinite.*

Proof. Let the theory T have an infinite model $\Gamma = \langle G; R\rangle$ from the class $\mathcal{G}_{fin}(\lambda)$. Let $\Gamma = \Gamma_0 \sqcup \Gamma_1$ with universes G_0, G_1 respectively, where Γ_0 is a subgraph consisting of κ finite components of bounded diameters, for some cardinality κ, and Γ_1 is a subgraph consisting of μ infinite components of bounded diameters, for some cardinality μ.

Let Γ_0 be the disjoint union of trees $\Gamma'_i, i \in \kappa$. Then the theory $Th(\Gamma_0)$ is approximated by the set $\{Th(\coprod_{i\in I}\Gamma'_i) \mid I \subseteq \kappa, |I| < \omega\}$.

Let Γ_1 be the disjoint union of trees $\Gamma''_i, i \in \kappa$, and a_i be the vertices from the centers of the trees Γ''_i. Then the theory $Th(\Gamma_1)$ is approximated by the set $\{Th(\coprod_{i\in I}\Gamma'''_i) \mid I \subseteq \kappa, |I| < \omega,\ \Gamma'''_i$ is the finite subgraph of Γ''_i and a_i is the vertex from the center of $\Gamma'''_i\}$. □

As the following theorem shows, a theory of acyclic graphs from the class $\mathcal{G}_{fin}(\lambda), \lambda \geq 2$, does not approximated by theories of finite trees.

Theorem 5. *Let Γ be an acyclic graph from class $\mathcal{G}_{fin}(\lambda)$, for a cardinality λ. The theory $T = Th(\Gamma)$ is approximated by theories of finite trees if and only if $\lambda = 1$.*

Proof. ($\Rightarrow$) Let T be the theory of an acyclic graph $\Gamma = \langle G; R\rangle$ consisting of λ connected components of bounded diameters and suppose that T is approximated by theories of finite trees. Suppose d is the maximum diameter of the connected components of the graph Γ and let $\lambda > 1$. Consider the sentence Φ: the distance between any two vertices is either greater than $d+1$ or less than or equal to d; there are two vertices, the distance between which is greater than $d+1$. It is clear that the sentence Φ is true in Γ but false in any tree, which contradicts the approximability of $Th(\Gamma)$ by the set of theories of trees.

($\Leftarrow$) Proved as in the proof of Theorem 4. □

5. Conclusion

The work investigates the approximations of theories of acyclic graphs of the classes $\mathcal{G}_{fin}(\lambda)$, $\mathcal{G}_{inf}(\lambda)$. It is proved that any acyclic graph from the class $\mathcal{G}_{fin}(\lambda)$ is pseudofinite. It is also proved that an acyclic graph of infinite diameter with infinite number of rays is pseudofinite.

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