Серия «Математика»
2022. Т. 39. С. 80—95

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского
государственного
университета

Research article

УДК 519.63
MSC 65M38
DOI https://doi.org/10.26516/1997-7670.2022.39.80

Numerical Experiments of the Dual Null Field Method for Dirichlet Problems of Laplace's Equation in Elliptic Domains with Elliptic Holes

Zi-Cai Li[1], Hung-Tsai Huang[2], Li-Ping Zhang[3], Anna A. Lempert[4], Ming-Gong Lee[5]✉

[1] National Sun Yat-sen University, Kaohsiung, Taiwan
[2] I-Shou University, Kaohsiung, Taiwan
[3] Zhejiang University of Technology, Hangzhou, China
[4] Matrosov Institute for System Dynamics and Control Theory SB RAS, Irkutsk, Russian Federation
[5] Chung Hua University, Hsin-Chu, Taiwan
✉ mglee@chu.edu.tw

Abstract. Dual techniques have been used in many engineering papers to deal with singularity and ill-conditioning of the boundary element method (BEM). In the first part of the two-part article, our efforts were focused on studying the theoretical aspects of this problem, including the analysis of errors and the study of stability. We provided the theoretical analysis for Laplace equation in elliptic domains with elliptic holes. To bypass the degenerate scales of Dirichlet problems, the second and the first kinds of the null field methods (NFM) are used for the exterior and the interior boundaries, simultaneously. This approach is called the dual null field method (DNFM).

This paper is the second part of the study. Numerical results for degenerate models of an elliptic domain with one elliptic hole at $a + b = 2$ are carried out to verify the theoretical analysis obtained. The collocation Trefftz method (CTM) is also designed for comparisons. Both the DNFM and the CTM can provide excellent numerical performances. The convergence rates are the same but the stability of CTM is excellent and can achieve the constant condition numbers, Cond $= O(1)$.

Keywords: boundary element method, degenerate scales, elliptic domains, dual null field method, collocation Trefftz methods, condition number

Acknowledgements: The reported study was funded by the Ministry of Science and Technology (MOST), Grant 109-2923-E-216-001-MY3 and RFBR, research project 20-51-S52003.

For citation: Li Z. C., Huang H. T., Zhang L. P., Lempert A. A., Lee M. G.Numerical Experiments of the Dual Null Field Method for Dirichlet Problems of Laplace's Equation in Elliptic Domains with Elliptic Holes. *The Bulletin of Irkutsk State University. Series Mathematics*, 2022, vol. 39, pp. 80–95.
https://doi.org/10.26516/1997-7670.2022.39.80

Научная статья

Численные эксперименты двойственным методом нулевого поля в задаче Дирихле для уравнения Лапласа в эллиптических областях с эллиптическими отверстиями

З. К. Ли[1], Х. Ц. Хуанг[2], Л. П. Жанг[3], А. А. Лемперт[4], М. Г. Ли[5]✉

[1] Государственный университет имени Сунь Ятсена, Гаосюн, Тайвань
[2] Университет Ишоу, Гаосюн, Тайвань
[3] Чжэцзянский технологический университет, Ханчжоу, Китай
[4] Институт динамики систем и теории управления имени В. М. Матросова СО РАН, Иркутск, Российская Федерация
[5] Университет Чунг Хуа, Синьчжу, Тайвань
✉ mglee@chu.edu.tw

Аннотация. Двойственные методы часто используются для решения проблемы сингулярности и плохой обусловленности метода граничных элементов (МГЭ). В первой части работы изучаются теоретические аспекты данной проблемы, включая анализ ошибок и исследование устойчивости. Так, авторами был выполнен анализ уравнения Лапласа в эллиптических областях с эллиптическими отверстиями. Для того чтобы преодолеть алгоритмическую сингулярность (degenerate scale problem) в задаче Дирихле, второй и первый виды МНП использовались для внешних и внутренних границ одновременно. Данный подход мы назвали методом двойственного нулевого поля (ДМНП).
Настоящая статья является второй частью исследования. Здесь представлены результаты вычислительных экспериментов для вырожденных моделей эллиптической области с одним эллиптическим отверстием при $a+b=2$ для проверки полученного теоретического анализа. Также для сравнения результатов разработан метод на основе коллокации Треффца (CTM). И DNFM, и CTM показывают отличные результаты. Их скорость сходимости одинакова, при этом устойчивость CTM превосходна и позволяет достичь постоянных чисел обусловленности, Cond $= O(1)$.

Ключевые слова: метод граничных элементов, вырожденные шкалы, эллиптическая область, двойственный метод нулевого поля, метод коллокации Треффца, число обусловленности

Благодарности: Работа выполнена при частичной финансовой поддержке Министерства науки и технологии Тайваня (грант 109-2923-E-216-001-MY3) и РФФИ (проект 20-51-S52003).

Ссылка для цитирования: Li Z. C., Huang H. T., Zhang L. P., Lempert A. A., Lee M. G. Numerical Experiments of the Dual Null Field Method for Dirichlet Problems of Laplace's Equation in Elliptic Domains with Elliptic Holes // Известия Иркутского государственного университета. Серия Математика. 2022. Т. 39. С. 80–95.
https://doi.org/10.26516/1997-7670.2022.39.80

1. Introduction

Dual techniques have been used in many engineering papers (see [1–3; 14]) to deal with singularity and ill-conditioning of the boundary element method (BEM). Our efforts are paid to explore theoretical analysis to fill up the gap between theory and computation. Our group provides the analysis for Laplace equation in circular domains with circular holes in [9] and elliptic domains with elliptic holes ([12]) in [10]. When the field nodes are located on the exterior elliptic boundary, the degenerate scales of algorithm singularity occurs at $a + b = 2$ [5], where a and b are two semi-axes of the exterior ellipse. It is too complicated to find all pitfall nodes of the null field method (NFM) causing algorithm singularity, as done in [5]. However, when the field nodes are confined on the same ellipses, the degenerate scales may be bypassed, see [10].

To guarantee the non-singularity of coefficient matrices obtained, other numerical algorithms should be solicited. In [1], a self-regularized method is proposed in the matrix level to deal with non-unique solutions of the Neumann and Dirichlet problems which contain rigid body mode and degenerate scale, respectively. In [3], they have examined the sufficient and necessary condition of boundary integral formulation for the uniqueness solution of 2D Laplace problem subject to the Dirichlet boundary condition by five regularization techniques, namely hypersingular formulation, method of adding a rigid body mode, rank promotion by adding the boundary flux equilibrium (direct BEM), CHEEF method and the Fichera's method (indirect BEM). In [4], they revisited the problem of an infinite plane with two identical circular holes by using the complex variables instead of using the degenerate kernel. The complex variables provide another way to solve these problems and it was easier than the degenerate kernel to understand. They have analytically derived the degenerate scale and compare the result with that of the degenerate kernel, and the equivalence is also proved.

The error analysis of the DNFM is made for elliptic domains with one elliptic hole to reach the optimal convergent rates and the bounds of condition numbers of the DNFM of a simple case are derived to display good stability in [10]. More important results are given in this paper. The dual null field method (DNFM) is studied to present numerical experiments for Laplace's equations with elliptic domain with elliptic holes to avoid the algorithm singularity. Numerical results for degenerate models at $a + b = 2$ are carried out to verify the theoretical analysis made in [10]. Besides, the collocation Trefftz method (CTM) in [8] is also designed for comparisons. Both the dual null field method and the CTM can provide excellent numerical performances. The CTM offers the constant condition numbers, but the dual null field method may link the BEM and the original

NFM for arbitrary solution domains, where the original NFM denotes the algorithms using piecewise polynomials as those in the BEM, instead of the semi-analytic solutions in this paper. This paper with [9] and [10] may shorten some gap between computation and theory of the dual null field method (DNFM).

This paper is organized as follows. In the next section, for elliptic domains with one elliptic hole, the dual null field method (DNFM) are described, and the algorithm of the DNFM is discussed. In Section 3, the theoretical outcomes of the errors and stability are stated. In Section 4, numerical experiments of the dual techniques and the CTM are reported. In the last section, a few concluding remarks are made.

2. The Dual Null Field Method in Elliptic Domains with Elliptic Holes

The elliptic coordinates are defined in [13] by

$$x = \sigma_0 \cosh \rho \cos \theta, \quad y = \sigma_0 \sinh \rho \sin \theta, \tag{2.1}$$

where $\sigma_0 > 0$ and two coordinates (ρ, θ) have the ranges: $0 \leq \rho < \infty$ and $0 \leq \theta \leq 2\pi$. More characteristics of elliptic coordinates are provided in [12]. Denote the large ellipse S_R with $\rho = R$, where the elliptic coordinates (ρ, θ) are given by (2.1) with the origin $(0, 0)$. Also denote a small ellipse $S_{R_1} \subset S_R$ with $\bar{\rho} = R_1$, where the other (i.e., local) elliptic coordinates $(\bar{\rho}, \bar{\theta})$ are given by

$$\bar{x} = \sigma_1 \cosh \bar{\rho} \cos \bar{\theta}, \quad \bar{y} = \sigma_1 \sinh \bar{\rho} \sin \bar{\theta}, \tag{2.2}$$

where $\sigma_1 > 0$. This Cartesian system $(\bar{x}, \bar{y})$ with the origin (x_1, y_1) is rotated from the axis X, by a counter-clockwise angle Θ as in Figure 1.

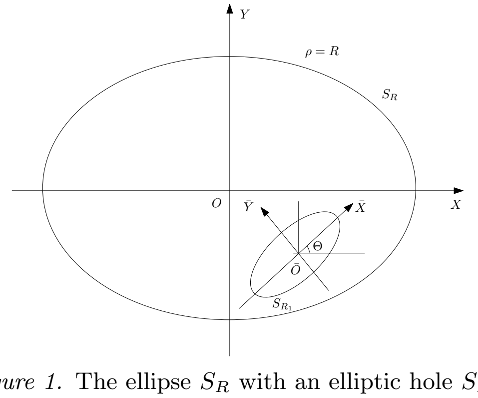

Figure 1. The ellipse S_R with an elliptic hole S_{R_1}.

The coordinate transformations between (ρ, θ) and $(\bar{\rho}, \bar{\theta})$ are given in (2.10) and (2.14) below.

Denote the annular domain by $S = S_R \setminus S_{R_1}$, and its boundary by $\partial S = \partial S_R \cup \partial S_{R_1}$. In this paper, consider the Dirichlet problem only,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in} \quad S, \tag{2.3}$$

$$u = f \quad \text{on} \quad \partial S_R, \quad u = g \quad \text{on} \quad \partial S_{R_1}, \tag{2.4}$$

where f and g are the known functions. On the exterior elliptic boundary ∂S_R, suppose that there exist approximations of series expansions (see [12]),

$$u = f \approx a_0 + \sum_{k=1}^{M} \{a_k \cos k\theta + b_k \sin k\theta\} \quad \text{on} \quad \partial S_R, \tag{2.5}$$

$$\frac{\partial u}{\partial \nu} = f_\nu^* \approx \frac{1}{\sigma_0 \tau_0(\theta)} \Big\{ p_0 + \sum_{k=1}^{M} \{p_k \cos k\theta + q_k \sin k\theta\} \Big\} \quad \text{on} \quad \partial S_R, \tag{2.6}$$

where a_k, b_k, p_k and q_k are coefficients, and $\tau_0(\theta) = \sqrt{\sinh^2 R + \sin^2 \theta}$. On the interior elliptic boundary ∂S_{R_1}, similarly

$$\bar{u} = g \approx \bar{a}_0 + \sum_{k=1}^{N} \{\bar{a}_k \cos k\bar{\theta} + \bar{b}_k \sin k\bar{\theta}\} \quad on \quad \partial S_{R_1} \tag{2.7}$$

$$\frac{\partial \bar{u}}{\partial \bar{\nu}} = -\frac{\partial \bar{u}}{\partial \bar{\rho}} \approx \frac{1}{\sigma_1 \tau_1(\bar{\theta})} \left\{ \bar{p}_0 + \sum_{k=1}^{N} \{\bar{p}_k \cos k\bar{\theta} + \bar{q}_k \sin k\bar{\theta}\} \right\} \quad on \quad \partial S_{R_1}, \tag{2.8}$$

where $\bar{a}_k, \bar{b}_k, \bar{p}_k$ and $\bar{q}_k$ are coefficients, and $\tau_1(\bar{\theta}) = \sqrt{\sinh^2 R_1 + \sin^2 \bar{\theta}}$. For the Dirichlet problem, the coefficients a_k and b_k in (2.5) and $\bar{a}_k$ and $\bar{b}_k$ in (2.7) are known, but the coefficients p_k and q_k in (2.6) and $\bar{p}_k$ and $\bar{q}_k$ in (2.8) are unknown to be sought.

For the numerical computation of explicit algebraic equations, the coordinate transformations between different elliptic coordinates are needed. In general, the axes of the small ellipse are not along the X and Y axes. The local Cartesian coordinates $X'O'Y'$ are located from the standard Cartesian coordinates XOY by rotating a counter-clockwise angle $\Theta \in [0, \pi)$, see Figure 1. There exist the relations of coordinates,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Denote the local elliptic coordinates $(\bar{\rho}, \bar{\theta})$ in $X'O'Y'$ as

$$\bar{x} = \sigma_1 \cosh \bar{\rho} \cos \bar{\theta}, \quad \bar{y} = \sigma_1 \sinh \bar{\rho} \sin \bar{\theta}, \tag{2.9}$$

where $\sigma_1 > 0$. We cite from [12] the explicit formulas of the transformations between two different elliptic coordinates. The transformation from (ρ, θ)

of (2.1) to $(\bar{\rho}, \bar{\theta})$ of (2.2) is given by

$$T : \{(\rho, \theta) \to (\bar{\rho}, \bar{\theta})\}, \tag{2.10}$$

where

$$\bar{\rho} = \sinh^{-1}(F(\bar{x}, \bar{y}; \sigma_1)), \quad \bar{\theta} = \cos^{-1}(\frac{\bar{x}}{\sigma_1 \cosh \bar{\rho}}), \tag{2.11}$$

where the function $F(x, y; \sigma)$ is defined as

$$F(x, y; \sigma) = \frac{1}{\sqrt{2}\sigma} \sqrt{(x^2 + y^2 - \sigma^2) + \sqrt{(x^2 + y^2 - \sigma^2)^2 + 4\sigma^2 y^2}}, \tag{2.12}$$

and the coordinates are

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix} \begin{pmatrix} \sigma \cosh\rho \cos\theta - x_1 \\ \sigma \sinh\rho \sin\theta - y_1 \end{pmatrix}. \tag{2.13}$$

The inverse transformation from $(\bar{\rho}, \bar{\theta})$ to (ρ, θ) is given by

$$T^{-1} : \{(\bar{\rho}, \bar{\theta}) \to (\rho, \theta)\}, \tag{2.14}$$

where

$$\rho = \sinh^{-1}(F(x, y; \sigma)), \quad \theta = \cos^{-1}(\frac{x}{\sigma \cosh \rho}), \tag{2.15}$$

and

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix} \begin{pmatrix} \sigma_1 \cosh\bar{\rho} \cos\bar{\theta} \\ \sigma_1 \sinh\bar{\rho} \sin\bar{\theta} \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{2.16}$$

2.1. The dull null field methods (DNFM)

From [10], we have derived the dual null field method with explicit algebraic equations and are given below,

$$\frac{\partial}{\partial \nu} \mathcal{L}_{ext}(\rho, \theta; \bar{\rho}, \bar{\theta}) \tag{2.17}$$

$$= \frac{1}{\sigma_0 \tau_0(\rho, \theta)} \Big\{ \sum_{k=1}^{M} k e^{-k\rho} (a_k \sinh kR \cos k\theta + b_k \cosh kR \sin k\theta)$$

$$-p_0 - \sum_{k=1}^{M} e^{-k\rho} (p_k \cosh kR \cos k\theta + q_k \sinh kR \sin k\theta) \Big\}$$

$$-\frac{1}{\sigma_1 \tau_1(\bar{\rho}, \bar{\theta})} \Big\{ \cos(\eta - \bar{\eta}) \bar{p}_0 + \sum_{k=1}^{N} k e^{-k\bar{\rho}} \{ \bar{a}_k \sinh kR_1 \cos[k\bar{\theta} - \eta + \bar{\eta}]$$

$$+\bar{b}_k \cosh kR_1 \sin[k\bar{\theta} - \eta + \bar{\eta}]\} + \sum_{k=1}^{N} e^{-k\bar{\rho}} \{ \bar{p}_k \cosh kR_1 \cos[k\bar{\theta} - \eta + \bar{\eta}]$$

$$+\bar{q}_k \sinh kR_1 \sin[k\bar{\theta} - \eta + \bar{\eta}]\} \Big\} = 0, \quad \rho \geq R, \bar{\rho} > R_1,$$

and

$$\mathcal{L}_{int}(\rho,\theta;\bar{\rho},\bar{\theta}) = -[R+\ln(\frac{\sigma_0}{2})]p_0 \tag{2.18}$$
$$+\sum_{k=1}^{M}\frac{1}{k}e^{-kR}\{p_k\cosh k\rho\cos k\theta + q_k\sinh k\rho\sin k\theta\}$$
$$-[R_1+\ln(\frac{\sigma_1}{2})]\bar{p}_0 + \sum_{k=1}^{N}\frac{1}{k}e^{-kR_1}\{\bar{p}_k\cosh k\bar{\rho}\cos k\bar{\theta} + \bar{q}_k\sinh k\bar{\rho}\sin k\bar{\theta}\}$$
$$+a_0+\sum_{k=1}^{M}e^{-kR}\{a_k\cosh k\rho\cos k\theta + b_k\sinh k\rho\sin k\theta\}$$
$$-\bar{a}_0-\sum_{k=1}^{N}e^{-kR_1}\{\bar{a}_k\cosh k\bar{\rho}\cos k\bar{\theta} + \bar{b}_k\sinh k\bar{\rho}\sin k\bar{\theta}\} = 0, \quad \bar{\rho}\le R_1.$$

2.2. Algorithms of Dual Null Field Method

Denote the explicit equations (2.17) and (2.18) by

$$\mathcal{D}_{ext}(\rho,\theta;\bar{\rho},\bar{\theta}) = 0, \quad \mathcal{L}_{int}(\rho,\theta;\bar{\rho},\bar{\theta}) = 0. \tag{2.19}$$

Choose the uniform nodes on the same ellipses,

$$(\rho,\theta) = (R+\epsilon, j\Delta\theta), \quad j=0,1,...,2M, \tag{2.20}$$
$$(\bar{\rho},\bar{\theta}) = (R_1-\bar{\epsilon}, j\Delta\bar{\theta}), \quad j=0,1,...,2N, \tag{2.21}$$

where $\epsilon \ge 0$, $0<\bar{\epsilon}<R_1$, $\Delta\theta = \frac{2\pi}{2M+1}$ and $\Delta\bar{\theta} = \frac{2\pi}{2N+1}$. We obtain $2(M+N)+2$ collocation equations of the NFM,

$$\frac{\sqrt{w_j}}{M}\mathcal{D}_{ext}(R+\epsilon, j\Delta\theta;\bar{\rho}_j,\bar{\theta}_j) = \frac{\sqrt{w_j}}{M}f(j\Delta\theta), \quad j=0,1,...,2M, \tag{2.22}$$
$$\sqrt{w_j}\mathcal{L}_{int}(\rho_j,\theta_j;R_1-\bar{\epsilon}, j\Delta\bar{\theta}) = \sqrt{w_j}g(j\Delta\bar{\theta}), \quad j=0,1,...,2N, \tag{2.23}$$

where the corresponding coordinates (ρ_j,θ_j) and $(\bar{\rho}_j,\bar{\theta}_j)$ can be evaluated from $(R+\epsilon, j\Delta\theta)$ and $(R_1-\bar{\epsilon}, j\Delta\bar{\theta})$, based on the coordinate transformations (2.10) and (2.14). The wights $w_0=1$ and $w_j=2$ for $j\ge 1$. By following [8], Eqs. (2.22) and (2.23) are called the collocation Trefftz method (CTM). Eqs. (2.22) and (2.23) lead to the following linear equations,

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{2.24}$$

where the matrix $\mathbf{A}\in\mathbb{R}^{n\times n}$, the vector $\mathbf{x}(\in\mathbb{R}^n) = [p_0,p_k,q_k,\bar{p}_0,\bar{p}_k,\bar{q}_k]^T$ and $n=2(M+N)+2$. The unknown coefficients can be obtained by solving (2.24) if matrix $\mathbf{A}$ is nonsingular. When $\rho=R$ and $\bar{\rho}=R_1$, $\epsilon=\bar{\epsilon}=0$, and the collocation equations of the NFM lead to those of the interior field method (IFM) [6].

2.3. Removal of Algorithm Singularity

Let us discuss the degenerate scales of the DNFM. We have a proposition without proof, see [10].

Proposition 1. *For Laplace's equation in elliptic domains with one elliptic hole, when $a+b=2$, there do not exist degenerate scales of the DNFM. When $a+b \neq 2$, the statement is true if constant ρ $(\geq R)$ and not small M are chosen.*

3. Analysis of Errors and Stability

3.1. Error Bounds

We have explored in [10] the analysis for elliptic domains with one elliptic hole, and here we just present their convergence and stability outcomes for reference.

Define the norm $\|v\|_0^* = \sqrt{\omega^2 \int_{\partial S_R} v_\nu^2 ds + \int_{\partial S_{R_1}} v^2 ds}$, we have the following theorem.

Theorem 1. *Suppose $u \in H^p(\partial S_R)$, $u_\nu \in H^{p-1}(\partial S_R)$ $(p \geq 2)$, $u \in H^\sigma(\partial S_{R_1})$ and $u_\nu \in H^{\sigma-1}(\partial S_{R_1})$ $(\sigma \geq 2)$, and the exact coefficients of the Dirichlet conditions in (2.5) and (2.7) be given. Then the solutions from the DNFM (2.17) and (2.18) have the following bound,*

$$\|u - u_{M-N}\|_{0,\Gamma}^* \leq C\Big\{\frac{1}{M^p}(\|u\|_{p,\partial S_R} + \|u_\nu\|_{p-1,\partial S_R}) \qquad (3.1)$$
$$+\frac{1}{N^\sigma}(\|u\|_{\sigma,\partial S_{R_1}} + \|u_{\bar\nu}\|_{\sigma-1,\partial S_{R_1}})\Big\}.$$

We choose $\rho = R$ and $\bar\rho = R_1$. For simplicity, consider the simple case: (1) the symmetric cases $q_k = \bar q_k = 0$ and $M = N$, and (2) the same elliptic coordinates with $(\rho,\theta) = (\bar\rho,\bar\theta)$ are used, i.e., $\sigma_0 = \sigma_1, x_1 = y_1 = 0, \Theta = 0$ and $\bar\eta = \eta$. We have a theorem about the condition number of the simple case.

Theorem 2. *Under the simple case of elliptic domains with one elliptic hole, for the DNFM (2.22) and (2.23) at $\epsilon = \bar\epsilon = 0$, there exist the bounds,*

$$Cond(\mathbf{A}) = O(M). \qquad (3.2)$$

4. Numerical Experiments

4.1. Two Computed Models

Consider a large elliptic domain with a small elliptic hole, as shown in Figure 2. The large ellipse is defined by $a = 2.5,\ b = 2$, $\sigma_0 = \sqrt{a^2 - b^2} = 1.5$ and

$$x = \sigma_0 \cosh R \cos\theta, \qquad y = \sigma_0 \sinh R \sin\theta. \tag{4.1}$$

From $\tanh R = \frac{b}{a} = \frac{2}{2.5} = 0.8$, we have $R = 1.0986$. The small ellipse is

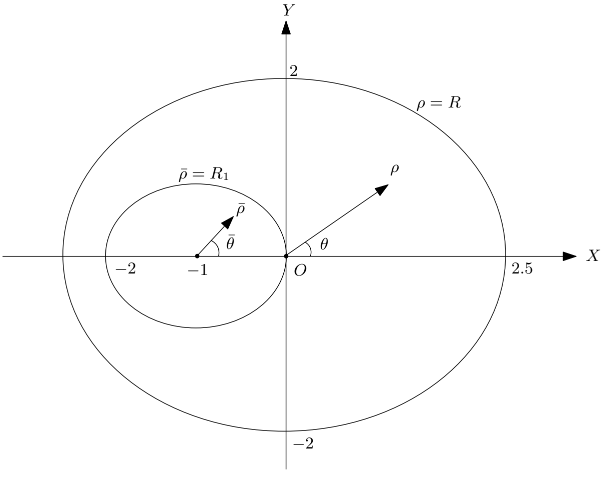

Figure 2. Model Problem on the ellipse S_R with an elliptic hole S_{R_1}.

defined by $\bar a = 1, \bar b = \frac{1}{2}$, and $\sigma_1 = \sqrt{\bar a^2 - \bar b^2} = \frac{\sqrt{3}}{2}$, where the local elliptic coordinates with origin $(-1, 0)$ are given by

$$x = \sigma_1 \cosh R_1 \cos\bar\theta - 1, \qquad y = \sigma_1 \sinh R_1 \sin\bar\theta. \tag{4.2}$$

For simplicity in computation, the long axis of the small ellipse is chosen along semiaxis X with $\Theta = 0$. From $\tanh R_1 = \frac{1}{2}$, we have $R_1 = 0.5493$. The Dirichlet condition is given by

$$u = a_0 = 1 \ \text{ on } \partial S_R, \qquad u = \bar a_0 = 0 \ \text{ on } \partial S_{R_1}. \tag{4.3}$$

Such a problem is called Model Problem in this paper.

We design Degenerate Model as follows. The exterior large ellipse ∂S_R with $a + b = 2$ as in Figure 3. Choose $\sigma_0 = 1.5$, and obtain from (2.1)

$$R = \ln\frac{a+b}{\sigma_0} = \ln\frac{2}{\sigma_0} = 3.79367, \tag{4.4}$$
$$a = \sigma_0 \cosh R = 1.5625, \quad b = 2 - a = 0.4375.$$

Choose the interior ellipse ∂S_{R_1} with origin $(-0.5, 0)$ and $(\bar{a}, \bar{b}) = (0.5, 0.1)$. We have from (2.9)

$$\tanh R_1 = \frac{1}{5}, \quad R_1 = 0.2040, \ \sigma_1 = \sqrt{\bar{a}^2 - \bar{b}^2} = 0.4899. \tag{4.5}$$

When the Dirichlet condition is also given by (4.3), Degenerate Model is called in this paper.

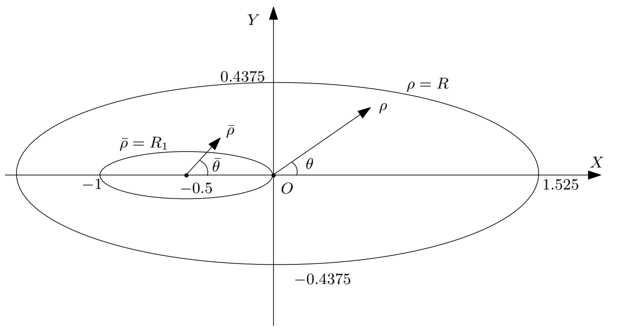

Figure 3. Degenerate Model on the ellipse S_R at $a + b = 2$ with an elliptic hole S_{R_1}.

4.2. By the Dual Null Field Method

The interior solutions are given, see [10].

$$u_{M-N}(\rho, \theta; \bar{\rho}, \bar{\theta}) = a_0 - [R + \ln(\frac{\sigma_0}{2})]p_0 - [\bar{\rho} + \ln(\frac{\sigma_1}{2})]\bar{p}_0 \tag{4.6}$$
$$+ \sum_{k=1}^{M} \frac{p_k}{k} e^{-kR} \cosh k\rho \cos k\theta + \sum_{k=1}^{N} \frac{\bar{p}_k}{k} e^{-k\bar{\rho}} \cosh kR_1 \cos k\bar{\theta}, \quad \text{in} \quad S.$$

Two explicit equations of the DNFM are obtained from (2.17) and (2.18)

$$\frac{\partial}{\partial \nu} \mathcal{L}_{ext}(\rho, \theta; \bar{\rho}, \bar{\theta}) := \frac{1}{\sigma_0 \tau_0(\rho, \theta)} \Big\{ - p_0 - \sum_{k=1}^{M} p_k e^{-k\rho} \cosh kR \cos k\theta \Big\}$$
$$- \frac{1}{\sigma_1 \tau_1(\bar{\rho}, \bar{\theta})} \Big\{ \cos(\eta - \bar{\eta})\bar{p}_0 + \sum_{k=1}^{N} \bar{p}_k e^{-k\bar{\rho}} \cosh kR_1 \cos(k\bar{\theta} - \eta + \bar{\eta}) \Big\} = 0,$$
$$\rho \geq R, \tag{4.7}$$

and

$$\mathcal{L}_{int}(\rho, \theta; \bar{\rho}, \bar{\theta}) := a_0 - \bar{a}_0 - [R + \ln(\frac{\sigma_0}{2})]p_0 - [R_1 + \ln(\frac{\sigma_1}{2})]\bar{p}_0 \tag{4.8}$$
$$+ \sum_{k=1}^{M} \frac{p_k}{k} e^{-kR} \cosh k\rho \cos k\theta + \sum_{k=1}^{N} \frac{\bar{p}_k}{k} e^{-kR_1} \cosh k\bar{\rho} \cos k\bar{\theta} = 0, \quad \bar{\rho} \leq R_1.$$

The $(N+M+2)$ collocation equations with $\rho = R$, $\bar{\rho} = R_1$ are chosen as

$$\sqrt{w_j}\frac{1}{M}\frac{\partial}{\partial \nu}\mathcal{L}_{ext}(R, j\Delta\theta; \bar{\rho}_j, \bar{\theta}_j) = 0, \quad j = 0, 1, ..., M, \tag{4.9}$$

$$\sqrt{w_j}\mathcal{L}_{int}(\rho_j, \theta_j; R_1, j\Delta\bar{\theta}) = 0, \quad j = 0, 1, ..., N, \tag{4.10}$$

where weights $w_0 = 1$ and $w_j = 2(j \geq 1)$. Eqs. (4.9) and (4.10) are expressed by $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{x} \in \mathbb{R}^n$, $n = M+N+2$. The condition number and the effective condition number are defined in [7] by Cond $= \frac{\sigma_{\max}}{\sigma_{\min}}$ and Cond_eff $= \frac{\|\mathbf{b}\|}{\sigma_{\min}\|\mathbf{x}\|}$, where $\sigma_{\max}$ and $\sigma_{\min}$ are the maximal and the minimal singular values of matrix $\mathbf{A}$, respectively, and $\|\mathbf{x}\|$ is the 2-norm of vector $\mathbf{x}$. We compute the errors, $\|\varepsilon\|_{\infty,\partial S_R}$, $\|\varepsilon\|_{\infty,\partial S_{R_1}}$, $\|\varepsilon\|_{0,\partial S_R}$, $\|\varepsilon\|_{0,\partial S_{R_1}}$ and the norm $\|\varepsilon\|_h = \sqrt{\|\varepsilon\|^2_{0,\partial S_R} + \|\varepsilon\|^2_{0,\partial S_{R_1}}}$.

For Model Problem, Table 1 shows the errors and the condition numbers.

Table 1

Errors and condition numbers of Model Problem by the DNFM, where $\varepsilon = u - u_{M-N}$.

M	$\|\varepsilon\|_{0,\partial S_R}$	$\|\varepsilon\|_{0,\partial S_{R_1}}$	$\|\varepsilon\|_h$	$\|\varepsilon\|_{\infty,\partial S_R}$	$\|\varepsilon\|_{\infty,\partial S_{R_1}}$	Cond	Cond_eff
6	1.270E-02	4.977E-03	1.364E-02	1.394E-02	5.461E-03	15.97	8.72
12	3.665E-04	2.847E-04	4.640E-04	4.643E-04	3.384E-04	29.97	16.66
18	1.733E-05	1.629E-05	2.378E-05	2.145E-05	2.006E-05	43.94	24.55
24	9.649E-07	1.036E-06	1.416E-06	1.184E-06	1.294E-06	57.91	32.42
30	5.870E-08	7.040E-08	9.166E-08	7.184E-08	8.873E-08	71.87	40.28

From Table 1, there exist the asymptotes,

$$\|\varepsilon\|_h = O(0.62^M), \; \|\varepsilon\|_{\infty,\partial S_R} = O(0.62^M), \; \|\varepsilon\|_{\infty,\partial S_{R_1}} = O(0.63^M), \tag{4.11}$$

$$\text{Cond} = O(M), \quad \text{Cond\_eff} = O(M). \tag{4.12}$$

For Degenerate Model, when $M = 20$, the errors and the condition numbers are listed in Table 2 for different N. It can be seen the compatible maximal errors on two boundaries are found as $N = 5$. Hence in computation, we choose the ratios $(M : N) = (4 : 1)$. The errors and the condition numbers are listed in Table 3.

From Table 3, there exist the asymptotes,

$$\|\varepsilon\|_h = O(0.69^M), \; \|\varepsilon\|_{\infty,\partial S_R} = O(0.69^M), \; \|\varepsilon\|_{\infty,\partial S_{R_1}} = O(0.69^M), \tag{4.13}$$

$$\text{Cond} = O(M), \quad \text{Cond\_eff} = O(M). \tag{4.14}$$

Eqs. (4.11)-(4.14) coincide with the error and stability analysis in Section 3. In particular, by the DNFM not only can the algorithm singularity be bypassed, but also the good stability as Cond $= O(M)$ may be achieved.

Table 2
Errors and condition numbers of Degenerate Model by the DNFM at $M = 20$.

N	$\|\varepsilon\|_{0,\partial S_R}$	$\|\varepsilon\|_{0,\partial S_{R_1}}$	$\|\varepsilon\|_h$	$\|\varepsilon\|_{\infty,\partial S_R}$	$\|\varepsilon\|_{\infty,\partial S_{R_1}}$	Cond	Cond_eff
3	1.83E-04	1.39E-03	1.40E-03	1.74E-04	2.50E-03	42.26	10.94
4	1.83E-04	1.95E-04	2.67E-04	1.75E-04	4.59E-04	48.10	12.50
5	1.83E-04	1.05E-04	2.11E-04	1.75E-04	1.78E-04	53.32	13.88
8	1.83E-04	1.74E-06	1.83E-04	1.75E-04	2.68E-06	66.58	17.39
10	1.83E-04	2.25E-08	1.83E-04	1.75E-04	3.11E-08	74.11	19.38
12	1.83E-04	4.45E-09	1.83E-04	1.75E-04	8.16E-09	80.95	21.18
15	1.83E-04	3.92E-11	1.83E-04	1.75E-04	6.90E-11	90.24	23.63
20	1.83E-04	3.47E-15	1.83E-04	1.75E-04	1.49E-14	103.90	27.22

Table 3
Errors and condition numbers for Degenerate Model by the DNFM with $(M : N) = (4 : 1)$.

M	$\|\varepsilon\|_{0,\partial S_R}$	$\|\varepsilon\|_{0,\partial S_{R_1}}$	$\|\varepsilon\|_h$	$\|\varepsilon\|_{\infty,\partial S_R}$	$\|\varepsilon\|_{\infty,\partial S_{R_1}}$	Cond	Cond_eff
12	3.55E-03	1.51E-03	3.85E-03	3.66E-03	2.55E-03	33.04	8.51
16	6.09E-04	1.85E-04	6.37E-04	6.23E-04	4.42E-04	43.15	11.20
20	1.83E-04	1.05E-04	2.11E-04	1.75E-04	1.78E-04	53.32	13.88
24	2.90E-05	2.83E-05	4.05E-05	3.02E-05	3.38E-05	63.50	16.57
28	1.19E-05	3.09E-06	1.23E-05	1.09E-05	7.24E-06	73.69	19.25
32	1.54E-06	1.58E-06	2.21E-06	1.68E-06	2.78E-06	83.87	21.93
36	8.17E-07	3.01E-07	8.71E-07	6.95E-07	4.92E-07	94.04	24.61
40	9.65E-08	8.10E-08	1.26E-07	1.05E-07	1.04E-07	104.22	27.29

4.3. Degenerate Model by the CTM

We may use the following particular solutions (PS) from [12],

$$u^{CTM}_{M-N} = u^{CTM}_{M-N}(\rho, \theta; \bar{\rho}, \bar{\theta}) = \alpha_0 + \bar{\alpha}_0\, \bar{\rho} \tag{4.15}$$
$$+ \sum_{k=1}^{M} e^{-kR}\{\alpha_k \cosh k\rho \cos k\theta + \beta_k \sinh k\rho \sin k\theta\}$$
$$+ \sum_{k=1}^{N} e^{-k\bar{\rho}}\{\bar{\alpha}_k \cosh kR_1 \cos k\bar{\theta} + \bar{\beta}_k \sinh kR_1 \sin k\bar{\theta}\},$$

where $\alpha_k, \beta_k, \bar{\alpha}_k$ and $\bar{\beta}_k$ are the coefficients. For the simple case, the solutions (4.15) are simplified as

$$u_{M-N}^{CTM} = u_{M-N}^{CTM}(\rho,\theta;\bar{\rho},\bar{\theta}) = \alpha_0 + \bar{\alpha}_0\,\bar{\rho} \tag{4.16}$$
$$+\sum_{k=1}^{M}\alpha_k e^{-kR}\cosh k\rho\cos k\theta + \sum_{k=1}^{N}\bar{\alpha}_k e^{-k\bar{\rho}}\cosh kR_1\cos k\bar{\theta}.$$

Compared (4.16) with (4.6), we find the relations of coefficients,

$$\bar{\alpha}_0 = -\bar{p}_0, \quad \alpha_0 = a_0 - [R + \ln(\frac{\sigma_0}{2})]p_0 - \ln(\frac{\sigma_1}{2})\bar{p}_0,$$
$$\alpha_k = \frac{1}{k}p_k, \quad \bar{\alpha}_k = \frac{1}{k}\bar{p}_k. \tag{4.17}$$

From the Dirichlet condition (4.3), we have

$$u_{M-N}^{CTM}(R,\theta;\bar{\rho},\bar{\theta}) = \alpha_0 + \bar{\alpha}_0\,\bar{\rho} \tag{4.18}$$
$$+\sum_{k=1}^{M}\alpha_k e^{-kR}\cosh kR\cos k\theta + \sum_{k=1}^{N}\bar{\alpha}_k e^{-k\bar{\rho}}\cosh kR_1\cos k\bar{\theta} = 1,$$
$$u_{M-N}^{CTM}(\rho,\theta;R_1,\bar{\theta}) = \alpha_0 + \bar{\alpha}_0\,R_1 \tag{4.19}$$
$$+\sum_{k=1}^{M}\alpha_k e^{-kR}\cosh k\rho\cos k\theta + \sum_{k=1}^{N}\bar{\alpha}_k e^{-kR_1}\cosh kR_1\cos k\bar{\theta} = 0.$$

The corresponding collocation equations at the nodes in (4.9) and (4.10) can be obtained. For Degenerate Model, the errors and the condition numbers are listed in Table 4.

Table 4

Errors and condition numbers of Degenerate Model by the CTM for $(M : N) = (4 : 1)$.

M	$\Vert\varepsilon\Vert_{0,\partial S_R}$	$\Vert\varepsilon\Vert_{0,\partial S_{R_1}}$	$\Vert\varepsilon\Vert_h$	$\Vert\varepsilon\Vert_{\infty,\partial S_R}$	$\Vert\varepsilon\Vert_{\infty,\partial S_{R_1}}$	Cond	Cond_eff
12	3.13E-03	1.52E-03	3.48E-03	2.81E-03	2.58E-03	13.99	3.58
16	4.99E-04	1.88E-04	5.33E-04	4.59E-04	4.48E-04	13.95	3.59
20	1.74E-04	1.06E-04	2.03E-04	1.53E-04	1.79E-04	13.94	3.60
24	2.51E-05	2.83E-05	3.78E-05	2.46E-05	3.38E-05	13.93	3.60
28	1.14E-05	3.09E-06	1.19E-05	1.00E-05	7.24E-06	13.92	3.61
32	1.36E-06	1.58E-06	2.08E-06	1.43E-06	2.78E-06	13.91	3.61
36	7.88E-07	3.01E-07	8.43E-07	6.52E-07	4.92E-07	13.91	3.61
40	8.70E-08	8.10E-08	1.19E-07	9.42E-08	1.04E-07	13.91	3.61

From Table 4, there exist the asymptotes,

$$\Vert\varepsilon\Vert_h = O(0.69^M),\ \Vert\varepsilon\Vert_{\infty,\partial S_R} = O(0.69^M),\ \Vert\varepsilon\Vert_{\infty,\partial S_{R_1}} = O(0.66^M), \tag{4.20}$$
$$\text{Cond} = O(1), \quad \text{Cond\_eff} = O(1). \tag{4.21}$$

Compared (4.20) and (4.21) with (4.13) and (4.14), the convergence rates are the same, but the stability of the CTM is excellent, because both Cond and Cond_eff remain unchangeable, when $M \to \infty$. Note that the coefficients from the DNFM and the CTM indeed satisfy the coefficient relations (4.17), due to limited space, we did not show the coefficients from these two methods in this paper. Both the CTM and the DNFM offer the excellent numerical performance, but the DNFM may link the BEM and the original NFM, see [11].

5. Concluding Remarks

Let us give a few remarks, to address the novelties of this paper.

1. The DNFM for Laplace's equation in circular domains with circular holes was first proposed in [9]; and the first part of the current study in [10] is devoted to the analysis of the DNFM for Laplace's equation in elliptic domains with elliptic holes. This paper with [9] and [10] may establish a theoretical foundation and numerical validation to fill up some gap between theory and computation.

2. For the degenerate case of an elliptic domain with an elliptic hole at $a + b = 2$ of the IFM, however, the renovated methods should be adapted. The algorithms of the DNFM are proposed in [10], and the removal of algorithm singularity is verified by the numerical experiments in Section 4.

3. Numerical experiments are carried out in Section 4, to support the theoretical analysis made in [10]. Moreover, the collocation Trefftz methods (CTM) is also used for comparisons. Both the CTM and the DNFM offer the excellent numerical performance. Although the CTM yields the excellent stability with Cond $= O(1)$, the DNFM may link the BEM and the original NFM for arbitrary domains, see [11].

References

1. Chen J.T., Han H., Kuo S.R., Kao S.K. Regularization for ill-conditioned systems of integral equation of first kind with logarithmic kernel. *Inverse Problems in Science and Engineering*, 2014, vol. 22, no. 7, pp. 1176-1195. https://doi.org/10.1080/17415977.2013.856900
2. Chen J.T., Hong H.K., Review of dual finite element methods with emphasis on hypersingular integrals and divergent series. *Appl. Mech. Rev.*, 1999, vol. 52, no. 1, pp. 17-33. https://doi.org/10.1115/1.3098922
3. Chen J.T., Lee Y.T., Chang Y.L., Jian J. A self-regularized approach for rank-deficient systems in the BEM of 2D Laplace problems. *Inverse Problems in Science and Engineering*, 2017, vol. 25, no. 1, pp. 89-113. https://doi.org/10.1080/17415977.2016.1138948
4. Kuo S.R., Kao S.K., Huang Y.L., Chen J.T. Revisit of the degenerate scale for an infinite plane problem containing two circular holes using con-

formal mapping. *Applied Mathematics Letters*, 2019, vol. 92, pp. 99-107. https://doi.org/10.1016/j.aml.2018.11.023

5. Lee M.G., Li Z.C., Zhang L.P., Huang H.T., Chiang J.Y. Algorithm singularity of the null-field method for Dirichlet problems of Laplace's equation in annular and circular domains. *Eng. Anal. Bound. Elem.*, 2014, vol. 41, pp. 160-172. https://doi.org/10.1016/j.enganabound.2014.01.013
6. Li Z.C., Chiang J.Y., Huang H.T., Lee M.G. The new interior field method for Laplace's equation on circular domains with circular holes. *Eng. Anal. Bound. Elem.*, 2016, vol. 67, pp. 173-185. https://doi.org/10.1016/j.enganabound.2016.03.006
7. Li Z.C., Huang H.T., Wei Y., Cheng A.H.-D. *Effective Condition Number for Numerical Partial Differential Equations*. Beijing, Science Press, 2014, 286 p.
8. Li Z.C., Lu T.T., Hu H.Y., Cheng A.H.-D. *Trefftz and Collocation Methods*. Southampton, Boston, WIT press, 2008, 432 p.
9. Lee M.G., Zhang L.P., Li Z.C., Kazakov A.L. Dual Null Field Method for Dirichlet Problems of Laplace's Equation in Circular Domains with Circular Holes. *Siberian Electronic Mathematical Reports*, 2021, vol. 19(1), pp. 393-422. https://doi.org/10.33048/semi.2021.18.028
10. Li Z.C., Huang H.T., Zhang L.P., Lempert A.A., Lee M.G. Analysis of Dual Null Field Methods for Dirichlet Problems of Laplace's Equation in Circular Domains with Circular Holes: Bypassing the degenerate scales. *The Bulletin of Irkutsk State University. Series Mathematics*, submitted, June 2021.
11. Li Z.C., Zhang L.P., Lee M.G., Interior field methods for Neumann problems of Laplace's equation in elliptic domains, Comparisons with degenerate scales. *Eng. Anal. Bound. Elem*, 2016, vol. 71, pp. 190-202. https://doi.org/10.1016/j.enganabound.2016.07.003
12. Li Z.C., Zhang L.P., Wei Y., Lee M.G., Chiang J.Y. Boundary methods for Dirichlet problems of Laplace's equation in elliptic domains with elliptic holes. *Eng. Anal. Bound. Elem.*, 2015, vol. 61, pp. 92-103. https://doi.org/10.1016/j.enganabound.2015.07.001
13. Morse P.M., Feshbach H. *Methods of Theoretical Physics*. New York, McGraw-Hill, Inc., 1953, 997 p.
14. Portela A., Aliabadi M.H., Rooke D.P. The dual boundary element method: effective implementation for crack problems. *Int. J. Numer. Methods Engrg.*, 1992, vol. 33, pp. 1269–1287. https://doi.org/10.1002/nme.1620330611
15. Krnić L. Types of Bases in the Algebra of Logic. *Glasnik Matematicko-Fizicki i Astronomski*, series 2, 1965, vol. 20, pp. 23–32.
16. Lau D., Miyakawa M. Classification and enumerations of bases in $P_k(2)$. *Asian-European Journal of Mathematics*, 2008, vol. 1, no. 2, pp. 255-282.
17. Miyakawa M., Rosenberg I., Stojmenović I. Classification of Three-valued logical functions preserving 0. *Discrete Applied Mathematics*, 1990, vol. 28, pp. 231-249. https://doi.org/10.1016/0166-218X(90)90005-W

Об авторах

Ли Зи-Кай, проф., Государственный университет имени Сунь Ятсена, Тайвань, 80424, г. Гаосюн, zicili1@gmail.com

About the authors

Zi-Cai Li, Prof., National Sun Yat-sen University, Kaohsiung, 80424, Taiwan, zicili1@gmail.com

Хуанг Хунг-Цай, профессор, Университет Ишоу, Тайвань, 84001, Гаосюн, huanght@isu.edu.tw

Жанг Ли-Пинг, проф., Чжэцзянский технологический университет, Китай, 310023, г. Ханчжоу, zhanglp@zjut.edu.cn

Лемперт Анна Ананьевна, канд. физ.-мат. наук, Институт динамики систем и теории управления имени В. М. Матросова СО РАН, Российская Федерация, 664033, г. Иркутск, lempert@icc.ru, https://orcid.org/0000-0001-9562-7903

Ли Минг-Гонг, проф., Университет Чунг Хуа, Тайвань, 30012, г. Синьчжу, mglee@chu.edu.tw, https://orcid.org/0000-0001-9405-2247

Hung-Tsai Huang, Prof., I-Shou University, Kaohsiung, 84001, Taiwan, huanght@isu.edu.tw

Li-Ping Zhang, Prof., Zhejiang University of Technology, Hangzhou, 310023, China, zhanglp@zjut.edu.cn

Anna A. Lempert, Cand. Sci. (Phys.Math.), Matrosov Institute for System Dynamics and Control Theory SB RAS, Irkutsk, 664033, Russian Federation, lempert@icc.ru, https://orcid.org/0000-0001-9562-7903

Ming-Gong Lee, Prof., Chung Hua University, Hsin-Chu, 30012, Taiwan, mglee@chu.edu.tw, https://orcid.org/0000-0001-9405-2247

Поступила в редакцию / Received 25.10.2021
Поступила после рецензирования / Revised 23.12.2021
Принята к публикации / Accepted 17.01.2022