ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ И ФУНКЦИОНАЛЬНЫЙ АНАЛИЗ

INTEGRO-DIFFERENTIAL EQUATIONS AND FUNCTIONAL ANALYSIS

Серия «Математика»
2021. Т. 38. С. 3—18

Онлайн-доступ к журналу:
http://mathizv.isu.ru

И З В Е С Т И Я
Иркутского
государственного
университета

УДК 517.95
MSC 35Q51
DOI https://doi.org/10.26516/1997-7670.2021.38.3

On the Construction and Integration of a Hierarchy for the Periodic Toda Lattice with a Self-Consistent Source

B. A. Babajanov[1], M. M. Ruzmetov[1]

[1] *Urgench State University, Urgench, Republic of Uzbekistan*

Abstract. In this paper, it is derived a rich hierarchy for the Toda lattice with a self-consistent source in the class of periodic functions. We discuss the complete integrability of the constructed systems that is based on the transformation to the spectral data of an associated discrete Hill's equation with periodic coefficients. In particular, Dubrovin-type equations are derived for the time-evolution of the spectral data corresponding to the solutions of any system in the hierarchy. At the end of the paper, we illustrate our theory on concrete example with analytical and numerical results.

Keywords: periodic Toda lattice hierarchy, Hill's equation, self-consistent source, inverse spectral problem, trace formulas.

1. Introduction

Our goal is to construct a hierarchy for the periodic Toda lattice with a self-consistent source in the class of periodic functions that can be integrated via the inverse spectral method. Toda lattice [29] is the model of a nonlinear one-dimensional crystal. It describes how the chain of particles with exponential interactions of the nearest neighbors move. It is widely known that, by means of the Flaschka variables [12], the Toda lattice has the form

$$\begin{cases} \dot{a}_n = a_n(b_{n+1} - b_n)\,, \\ \dot{b}_n = 2\,(a_n^2 - a_{n-1}^2), \quad n \in Z. \end{cases}$$

The Toda lattice has several applications. For instance, the Toda lattice model of Deoxyribonucleic acid (DNA) in biology [26]. In addition, another significant property of the Toda lattice type equations is the existence of so-called soliton solutions. The presence of soliton solutions and the integrability of equations have tight connections. The research results present that all the integrable systems have soliton solutions [22]. For further development of the periodic Toda lattice we refer to [9; 11; 17; 18].

Over the last few years, the interest has been growing in the soliton equations with self-consistent source [1–4; 8; 20; 23; 27; 28; 32]. These equations have essential applications in plasma physics, hydrodynamics, solid state physics, etc. [7; 13; 14; 19; 24; 25].

The fisrt investigation of the discrete soliton equations with a self-consistent sources has been considered in [21]. In this work the Toda lattice with a self-consistent sources are formulated and calculated by using the Darboux transformation. In [6; 30; 31], a scattering method was developed for integrating the Toda lattice with a self-consistent source.

In [5], it is obtained algebro-geometric quasi-periodic finite-gap solutions of the sourceless Toda lattice hierarchy. In [1], the authors demonstrated integrability of the periodic Toda lattice hierarchy with an integral-type source which comes from the eigenfunctions of the continuous spectrum of the discrete Hill‘s equation.

In this work, we consider the N- periodic Toda lattice hierarchy with a self-consistent source where source is developed by the eigenfunctions of the discrete spectrum of the discrete Hill‘s equation.

The goal of this work is to obtain formulations for the solutions of the constructed new system in the framework of the inverse spectral problem for discrete Hill‘s equation. In the one-gap case, we write the explicit formulas for the solutions of under consideration problem, expressed in terms of Jacobi elliptic functions.

The new system, similarly to [13], [10] can be applied to some models of special types of electric transmission line.

2. Formulation of the problem

In the present paper, we consider the N-periodic Toda lattice hierarchy with self-consistent source

$$\begin{cases} \dot{a}_n = P_m(a_n, b_n) + a_n \sum_{i=1}^{2N} \tilde{\theta}_{N+1}(\lambda_i, t) \left[(f_{n+1}^i)^2 - (f_n^i)^2 \right] , \\ \dot{b}_n = Q_m(a_n, b_n) + 2 \sum_{i=1}^{2N} \tilde{\theta}_{N+1}(\lambda_i, t) f_n^i (a_n f_{n+1}^i - a_{n-1} f_{n-1}^i), \\ a_{n-1} f_{n-1}^i + b_n f_n^i + a_n f_{n+1}^i = \lambda_i f_n^i, \quad i = 1,\, 2,\, ...,\, 2N, \\ a_{n+N} = a_n,\, b_{n+N} = b_n,\, a_n > 0,\, (f_{n+N}^i)^2 = (f_n^i)^2,\, n \in Z,\, t \in R, \end{cases} \tag{2.1}$$

and the initial conditions

$$a_n(0) = a_n^0, \quad b_n(0) = b_n^0, \quad n \in Z, \tag{2.2}$$

with the given N-periodic sequences a_n^0 and b_n^0, $\quad n \in Z$. Where

$$P_m(a_n, b_n) = a_n[-\beta_{n,m} - \beta_{n+1,m} + b_{n+1}\alpha_{n+1,m}],$$

$$Q_m(a_n, b_n) = a_n^2 \alpha_{n+1,m} - a_{n-1}^2 \alpha_{n-1,m} - 2b_n\beta_{n,m} + b_n^2 \alpha_{n,m},\ m \in N,\ t \in R,$$

and $\{\alpha_{n,s}(t)\}_{0\le s\le m}$, $\{\beta_{n,s}(t)\}_{0\le s\le m}$ satisfy the recursion relations

$$\alpha_{n,0} = 0, \quad \beta_{n,0} = c_0,\ \alpha_{n,1} = 2c_0,\ c_0 = const,$$

$$\beta_{n,s-1} - \beta_{n-1,s-1} = b_n(\beta_{n,s-2} - \beta_{n-1,s-2}) - a_n^2 \alpha_{n+1,s-2} + a_{n-1}^2 \alpha_{n-1,s-2},$$

$$\alpha_{n,s} = b_n \alpha_{n,s-1} - \beta_{n-1,s-1} - \beta_{n,s-1}, \quad 2 \le s \le m\,,$$

$$\beta_{n,m} = \frac{a_{n-1}^2}{2}\alpha_{n-1,m-1} - \frac{a_n^2}{2}\alpha_{n+1,m-1} + \frac{b_n^2}{2}\alpha_{n,m-1} - b_n\beta_{n-1,m-1}.$$

Varying $m \in N$ yields Toda lattice hierarchy with self-consistent source (2.1). The function sequences

$$\{a_n(t)\}_{-\infty}^{\infty}, \{b_n(t)\}_{-\infty}^{\infty}, \{f_n^1(t)\}_{-\infty}^{\infty}, \{f_n^2(t)\}_{-\infty}^{\infty}, \ldots, \{f_n^{2N}(t)\}_{-\infty}^{\infty}$$

– are unknown vector-functions, besides $\{f_n^i(t)\}_{-\infty}^{\infty}$ are the Floquet-Bloch solutions for the discrete Hill's equation and

$$(L(t)y)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n \tag{2.3}$$

normalized by conditions

$$f_1^i(t) = 1, \quad i = 1,\ 2,\ ...,\ 2N. \tag{2.4}$$

The eigenvalues λ_i of (2.3) are solutions of equation $\Delta^2(\lambda) - 4 = 0$, where $\Delta(\lambda) = \theta_N(\lambda, t) + \varphi_{N+1}(\lambda, t)$, and $\theta_n(\lambda, t)$, $n \in Z$ and $\varphi_n(\lambda, t)$, $n \in Z$ are solutions of equation (2.3) under the initial conditions

$$\theta_0(\lambda, t) = 1, \quad \theta_1(\lambda, t) = 0, \quad \varphi_0(\lambda, t) = 0, \quad \varphi_1(\lambda, t) = 1.$$

The factor $\tilde{\theta}_{N+1}(\lambda, t)$ in system (2.1) is defined from the equality

$$\tilde{\theta}_{N+1}(\lambda, t) = \prod_{j=1}^{N-1} (\lambda - \mu_j(t)),$$

where $\mu_1(t),\ \mu_2(t),\ ...,\ \mu_{N-1}(t)$ are roots of the equation $\theta_{N+1}(\lambda, t) = 0$.

The main aim of this work is to construct the N - periodic Toda lattice hierarchy with integral-type self-consistent sources and derive representations for the solutions $\{a_n(t)\}_{-\infty}^{\infty}$, $\{b_n(t)\}_{-\infty}^{\infty}$, $\{f_n^1(t)\}_{-\infty}^{\infty}$, $\{f_n^2(t)\}_{-\infty}^{\infty}, \ldots,$ $\{f_n^{2N}(t)\}_{-\infty}^{\infty}$ of the N-periodic Toda lattice hierarchy with an integral source under the initial conditions (2.2) by means of the inverse spectral problem for discrete Hill equation.

3. The basic information about the theory of Direct and Inverse Spectral Problem for the discrete Hill's equation

In this section we give basic information about the theory of direct and inverse spectral problem for the discrete Hill's equation [2; 29].

We start with the following discrete Hill's equation

$$(Ly)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n \tag{3.1}$$

$$a_{n+N} = a_n, \quad b_{n+N} = b_n, \quad n \in Z\,,$$

with spectral parameter λ, and with period $N > 0$. Let's $\theta_n(\lambda)$, $n \in Z$ and $\varphi_n(\lambda)$, $n \in Z$ be the solutions of equation (3.1) under the initial conditions $\theta_0(\lambda) = 1,\ \ \theta_1(\lambda) = 0,\ \ \ \varphi_0(\lambda) = 0,\ \ \varphi_1(\lambda) = 1.$

Let $\lambda_1,\ \lambda_2,\ ...,\ \lambda_{2N}$ be the roots of equation $\Delta^2(\lambda) - 4 = 0$. We define the auxiliary spectrum $\mu_1,\ \mu_2,\ ...,\ \mu_{N-1}$ as the roots of equation $\theta_{N+1}(\lambda) = 0$. As it is known [29], all $\lambda_i,\ \ i = 1,\ 2,\ ...,\ 2N$ and $\mu_j,\ j = 1,\ 2,\ ...,\ N-1$ are real, the roots μ_j are simple, but among the roots λ_i may occur the roots of multiplicity two. It is easy to show, that

$$\Delta^2(\lambda) - 4 = \left(\prod_{j=1}^{N} a_j\right)^{-2} \prod_{j=1}^{2N}(\lambda - \lambda_j) \tag{3.2}$$

$$\theta_{N+1}(\lambda) = -a_0 \left(\prod_{j=1}^{N} a_j\right)^{-1} \prod_{j=1}^{N-1} (\lambda - \mu_j) \tag{3.3}$$

We shall introduce

$$\sigma_j = sign\left[\theta_N(\mu_j) - \frac{1}{\theta_N(\mu_j)}\right],\ j = 1,\ 2,\ ...,\ N-1.$$

Definition 1. *The set of the numbers* μ_j, $j = 1, 2, ..., N-1$ *and sequences of signs* σ_j, $j = 1, 2, ..., N-1$ *is called spectral parameters of the discrete Hill's equation* (3.1).

Definition 2. *System of spectral parameters* $\{\mu_j, \sigma_j\}_{j=1}^{N-1}$ *and numbers* λ_i, $i = 1, 2, ..., 2N$ *is called spectral data of the discrete Hill's equation* (3.1).

To find the coefficients $a_n,\ b_n,\ n \in Z$ by the spectral data, we shift all the suffixes n by a constant k in (3.1) to get

$$a_{n+k-1}y_{n-1} + b_{n+k}y_n + a_{n+k}y_{n+1} = \lambda y_n, n \in Z \tag{3.4}$$

then we get the following trace formulas [2]

$$b_{k+1} = \frac{\lambda_1 + \lambda_{2N}}{2} + \frac{1}{2}\sum_{j=1}^{N-1}\left(\lambda_{2j} + \lambda_{2j+1} - 2\mu_{j,k}\right), \tag{3.5}$$

$$a_k^2 = \frac{\lambda_1^2 + \lambda_{2N}^2}{8} + \frac{1}{8}\sum_{j=1}^{N-1}\left(\lambda_{2j}^2 + \lambda_{2j+1}^2 - 2\mu_{j,k}^2\right) -$$

$$-\frac{1}{4}\left[\frac{\lambda_1 + \lambda_{2N}}{2} + \frac{1}{2}\sum_{j=1}^{N-1}\left(\lambda_{2j} + \lambda_{2j+1} - 2\mu_{j,k}\right)\right]^2 - \frac{1}{2}\sum_{j=1}^{N-1}\frac{\sigma_{j,k}\sqrt{\prod_{i=1}^{2N}(\mu_{j,k} - \lambda_i)}}{\prod_{\substack{i=1\\ i\neq j}}^{N-1}(\mu_{j,k} - \mu_{i,k})} \tag{3.6}$$

where $\mu_{j,k}$, $j = 1, 2, ..., N-1$ are the roots of equation $\theta_{N+1,k}(\lambda) = 0$. Here $\theta_{n,k}(\lambda)$, $n \in Z$ is the solution of (3.4), under the initial conditions $\theta_{0,k}(\lambda) = 1$, $\theta_{1,k}(\lambda) = 0$.

4. Constructing a hierarchy for the periodic Toda lattice with a self-consistent source

In this section, we present our method for constructing a hierarchy of the periodic Toda lattice with a self-consistent source by using spectral theory of the discrete Hill's equation. We consider the system

$$\begin{cases} \dot{a}_n = P_m(a_n, b_n) + a_n \sum_{i=1}^{2N} \tilde{\theta}_{N+1}(\lambda_i, t)\left[(f_{n+1}^i)^2 - (f_n^i)^2\right], \\ \dot{b}_n = Q_m(a_n, b_n) + 2\sum_{i=1}^{2N} \tilde{\theta}_{N+1}(\lambda_i, t) f_n^i (a_n f_{n+1}^i - a_{n-1} f_{n-1}^i), \\ a_{n-1}f_{n-1}^i + b_n f_n^i + a_n f_{n+1}^i = \lambda_i f_n^i,\ i = 1, 2, ..., 2N, \\ a_{n+N} = a_n,\ b_{n+N} = b_n,\ a_n > 0,\ (f_{n+N}^i)^2 = (f_n^i)^2,\ n \in Z,\ t \in R, \end{cases} \tag{4.1}$$

and initial conditions

$$a_n(0) = a_n^0, \;\; b_n(0) = b_n^0, \quad n \in Z, \tag{4.2}$$

with the given N - periodic sequences a_n^0 and b_n^0, $\; n \in Z$. Where P_m and Q_m are unknown functions of a_n and b_n. The aim is to find all functions P_m and Q_m, $m \in N$ so that the Cauchy problem (4.1)-(4.2) should be completely integrable in the framework of the inverse spectral problem of discrete Hill's equation

$$(L(t)y)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \;\; n \in Z. \tag{4.3}$$

Let $y^j(t) = (y_0^j(t), \; y_1^j(t), \; ..., \; y_N^j(t))^T$, $j = 1, \; 2, \; ..., \; N-1$ be the normalized eigenvectors for the corresponding eigenvalues $\lambda = \mu_j(t)$, $j = 1, \; 2, \; ..., \; N-1$, associated with the following boundary problem

$$\begin{cases} (L(t)y)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \;\; 1 \le n \le N \\ y_1 = 0, \;\; y_{N+1} = 0. \end{cases}$$

Using (4.1), we obtain the following equality

$$\dot{\mu}_j(t) = \sum_{n=1}^{N} [2P_m(a_n, \, b_n) y_n^j y_{n+1}^j + Q_m(a_n, \, b_n)(y_n^j)^2] + \sum_{i=1}^{2N} F_i^j(t), \tag{4.4}$$

where

$$F_i^j(t) = \tilde{\theta}_{N+1}(\lambda_i, t) \sum_{n=1}^{N} \Big\{ 2a_n[(f_{n+1}^i)^2 - (f_n^i)^2] y_n^j y_{n+1}^j + \\ +2a_n(f_n^i f_{n+1}^i)(y_n^j)^2 - 2a_{n-1}(f_{n-1}^i f_n^i)(y_n^j)^2 \Big\} .$$

For convenience, let us put

$$H_n = 2P_m(a_n, \, b_n) y_n^j y_{n+1}^j + Q_m(a_n, \, b_n)(y_n^j)^2. \tag{4.5}$$

We will find sequences u_n, that

$$u_{n+1} - u_n = H_n. \tag{4.6}$$

We seek for u_n as following

$$u_n = A_n (y_n^j)^2 + 2a_n(t) \, B_n y_n^j y_{n+1}^j + a_n^2(t) \, C_n (y_{n+1}^j)^2, \tag{4.7}$$

where $A_n = A_n(\mu_j(t))$, $\; B_n = B_n(\mu_j(t))$ and $C_n = C_n(\, \mu_j(t))$ are unknown coefficients yet. Substituting (4.7) in (4.6) we get

$$-A_n + a_n^2 C_{n+1} = Q_m,$$

$$A_{n+1} - a_n^2 C_n + 2B_{n+1}(\mu_j - b_{n+1}) + C_{n+1}(\mu_j - b_{n+1})^2 = 0,$$
$$-a_n B_n - a_n B_{n+1} - a_n C_{n+1}(\mu_j - b_{n+1}) = P_m.$$

Consiquently,

$$P_m = -a_n B_n - a_n B_{n+1} - a_n C_{n+1}(\mu_j - b_{n+1}), \tag{4.8}$$

$$Q_m = a_n^2 C_{n+1} - a_{n-1}^2 C_{n-1} + 2B_n(\mu_j - b_n) + C_n(\mu_j - b_n)^2. \tag{4.9}$$

The left side of (4.8) and (4.9) independent of μ_j, according to this we seek for B_n and C_n in the form

$$B_n = \sum_{k=0}^{m} \beta_{n,k}\mu_j^{m-k}, \quad C_n = \sum_{k=0}^{m} \alpha_{n,k}\mu_j^{m-k}. \tag{4.10}$$

Putting (4.10) into (4.8)-(4.9) and comparing left and right sides of the last equality we find

$$\alpha_{n,0} = 0, \quad \beta_{n,0} = c_0 = const, \quad \alpha_{n,1} = 2c_0, \tag{4.11}$$

$$\begin{aligned} \beta_{n,k-1} - \beta_{n-1,k-1} =& b_n(\beta_{n,k-2} - \beta_{n-1,k-2}) - a_n^2 \alpha_{n+1,k-2} + \\ &+ a_{n-1}^2 \alpha_{n-1,k-2},\ 2 \le k \le m, \end{aligned} \tag{4.12}$$

$$\alpha_{n,k} = b_n \alpha_{n,k-1} - \beta_{n-1,k-1} - \beta_{n,k-1}, \quad 2 \le k \le m, \tag{4.13}$$

$$\beta_{n,m} = \frac{a_{n-1}^2}{2}\alpha_{n-1,m-1} - \frac{a_n^2}{2}\alpha_{n+1,m-1} + \frac{b_n^2}{2}\alpha_{n,m-1} - b_n\beta_{n-1,m-1}, \tag{4.14}$$

and so

$$\begin{cases} P_m(a_n, b_n) =& a_n(-\beta_{n,m} - \beta_{n+1,m} + b_{n+1}\alpha_{n+1,m}), \\ Q_m(a_n, b_n) =& a_n^2\alpha_{n+1,m} - a_{n-1}^2\alpha_{n-1,m} - 2b_n\beta_{n,m} + b_n^2\alpha_{n,m}, m \in N. \end{cases} \tag{4.15}$$

Then, varying $m \in N$ yields the Toda lattice hierarchy with a self-consistent source

$$\begin{cases} \dot{a}_n =& a_n(-\beta_{n,m} - \beta_{n+1,m} + b_{n+1}\alpha_{n+1,m}) + a_n \sum_{i=1}^{2N} \tilde{\theta}_{N+1}(\lambda_i, t)\left[(f_{n+1}^i)^2 - (f_n^i)^2\right], \\ \dot{b}_n =& a_n^2\alpha_{n+1,m} - a_{n-1}^2\alpha_{n-1,m} - 2b_n\beta_{n,m} + b_n^2\alpha_{n,m} + \\ & + 2\sum_{i=1}^{2N} \tilde{\theta}_{N+1}(\lambda_i, t) f_n^i(a_n f_{n+1}^i - a_{n-1}f_{n-1}^i),\ m \in N. \end{cases} \tag{4.16}$$

Explicitly, one obtains from (4.11)-(4.16) few equations of the periodic Toda lattice hierarchy with a self-consistent source, $m = 1$, $c_0 = -1$,

$$\begin{cases} \dot{a}_n =& a_n(b_{n+1} - b_n) + a_n \sum_{i=1}^{2N} \tilde{\theta}_{N+1}(\lambda_i, t)\left[(f_{n+1}^i)^2 - (f_n^i)^2\right], \\ \dot{b}_n =& 2(a_n^2 - a_{n-1}^2) + 2\sum_{i=1}^{2N} \tilde{\theta}_{N+1}(\lambda_i, t) f_n^i(a_n f_{n+1}^i - a_{n-1}f_{n-1}^i), \end{cases}$$

$m = 2,$

$$\begin{cases} \dot a_n = & c_1 a_n(b_n - b_{n+1}) + a_n(a_{n+1}^2 - a_{n-1}^2) + a_n(b_{n+1}^2 - b_n^2) + \\ & + a_n \sum_{i=1}^{2N} \tilde\theta_{N+1}(\lambda_i, t)\left[(f_{n+1}^i)^2 - (f_n^i)^2\right], \\ \dot b_n = & 2c_1(a_{n-1}^2 - a_n^2) - 2a_{n-1}^2(b_n + b_{n-1}) + 2a_n^2(b_n + b_{n+1}) + \\ & + 2\sum_{i=1}^{2N} \tilde\theta_{N+1}(\lambda_i, t) f_n^i (a_n f_{n+1}^i - a_{n-1} f_{n-1}^i), \end{cases} \tag{4.17}$$

$m = 3,$

$$\begin{cases} \dot a_n = & c_2 a_n(b_n - b_{n+1}) - c_1 a_n(a_{n+1}^2 - a_{n-1}^2 + b_{n+1}^2 - b_n^2) + a_n(b_{n+1}^3 - b_n^3) + \\ & + a_n[a_{n+1}^2(b_{n+2} + 2b_{n+1}) + a_n^2(b_{n+1} - b_n) - a_{n-1}^2(b_{n-1} + 2b_n) + \\ & + a_n \sum_{i=1}^{2N} \tilde\theta_{N+1}(\lambda_i, t)\left[(f_{n+1}^i)^2 - (f_n^i)^2\right], \\ \dot b_n = & 2c_2(a_{n-1}^2 - a_n^2) + 2c_1[a_{n-1}^2(b_n + b_{n-1}) - a_n^2(b_n + b_{n+1})] + 2a_n^2(a_n^2 + \\ & + a_{n+1}^2) + 2a_n^2(b_n^2 + b_n b_{n+1} + b_{n+1}^2) - 2a_{n-1}^2(b_n^2 + b_n b_{n-1} + b_{n-1}^2) - \\ & - 2a_{n-1}^2 a_{n-2}^2 - 2a_{n-1}^4 + 2\sum_{i=1}^{2N} \tilde\theta_{N+1}(\lambda_i, t) f_n^i (a_n f_{n+1}^i - a_{n-1} f_{n-1}^i), \end{cases}$$

etc.

5. Evolution of spectral parameters

In this section, we prove the basic result of this paper.

Theorem 1. *If the functions* $a_n(t)$, $b_n(t)$, $f_n^k(t)$, $n \in Z$, $k = 1, 2, ..., 2N$ *are solutions of the problem* (2.1)-(2.4)*, then the spectrum of discrete Hill operator* (2.3) *is independent of* t*, and the spectral parametrs* $\mu_j(t)$, $j = 1, 2, ..., N-1$*, satisfy the system of equations*

$$\dot\mu_j(t) = 2\frac{\sigma_j(t) \cdot \sqrt{\prod_{k=1}^{2N}(\mu_j(t) - \lambda_k)}}{\prod_{k=1, k\neq j}^{N-1}(\mu_j(t) - \mu_k(t))}\left\{C_1(\mu_j(t)) - \sum_{i=1}^{2N}\frac{\tilde\theta_{N+1}(\lambda_i, t)}{\lambda_i - \mu_j(t)}\right\} \tag{5.1}$$

where

$$C_1(\mu_j(t)) = \sum_{k=0}^{m} \alpha_{1,k}\mu_j^{m-k}(t).$$

Proof. Putting (4.7) into (4.6) and summing over n, we get

$$\sum_{n=1}^{N} H_n = u_{N+1} - u_1 = A_{N+1}(y^j_{N+1})^2 + 2a_{N+1}(t)\, B_{N+1} y^j_{N+1} y^j_{N+2} + \\ + a^2_N(t)\, C_N (y^j_{N+2})^2 - A_1(y^j_1)^2 - 2a_1(t)\, B_1 y^j_1 y^j_2 - a^2_1(t)\, C_1 (y^j_2)^2. \tag{5.2}$$

Due to

$$a_{N+1}(t) y^j_{N+2} = (\mu_j(t) - b_{N+1}(t)) y^j_{N+1} - a_N(t) y^j_N,$$

and $y^j_1 = 0,\ \ y^j_{N+1} = 0$, from (5.2) we find

$$\sum_{n=1}^{N} H_n = a^2_0(t) C_1 (y^j_0)^2 - a^2_N(t) C_{N+1} (y^j_N)^2 = a^2_0(t) C_1 [(y^j_0)^2 - (y^j_N)^2] \tag{5.3}$$

Substituting (5.3) in (4.4) we obtain

$$\dot\mu_j(t) = a^2_0(t) C_1(\mu_j(t))[(y^j_0)^2 - (y^j_N)^2] + \sum_{i=1}^{2N} F^j_i(t). \tag{5.4}$$

Now, we will calculate second term of right-hand side of (5.4).

$$F^j_i(t) = \tilde\theta_{N+1}(\lambda_i, t) \sum_{n=1}^{N} [2a_n f^i_{n+1} y^j_{n+1} (y^j_n f^i_{n+1} - y^j_{n+1} f^i_n) + \\ + 2a_n f^i_n y^j_n (y^j_n f^i_{n+1} - f^i_n y^j_{n+1})] = 2\tilde\theta_{N+1}(\lambda_i, t) \sum_{n=1}^{N} f^i_{n+1} y^j_{n+1} T_n + \\ 2\tilde\theta_{N+1}(\lambda_i, t) \sum_{n=1}^{N} f^i_{n+1} y^j_{n+1} T_{n+1} = 2\tilde\theta_{N+1}(\lambda_i, t) \sum_{n=1}^{N} f^i_{n+1} y^j_{n+1} (T_n + T_{n+1}) = \\ \sum_{n=1}^{N} \frac{2\tilde\theta_{N+1}(\lambda_i, t)}{\lambda_i - \mu_j(t)} (T_{n+1} - T_n)(T_{n+1} + T_n) = \frac{2\tilde\theta_{N+1}(\lambda_i, t)}{\lambda_i - \mu_j(t)} (T^2_{N+1} - T^2_1), \tag{5.5}$$

where $T_n = a_n (y^j_n f^i_{n+1} - y^j_{n+1} f^i_n)$. It is easy to see that

$$F^j_i = \frac{2\tilde\theta_{N+1}(\lambda_i, t) a^2_0 (f^i_1)^2}{\lambda_i - \mu_j(t)} [(y^j_N)^2 - (y^j_0)^2] \tag{5.6}$$

Therefore, according to (5.5), we have

$$\sum_{i=1}^{2N} F^j_i(t) = \sum_{i=1}^{2N} \frac{2\tilde\theta_{N+1}(\lambda_i, t) a^2_0 (f^i_1)^2}{\lambda_i - \mu_j(t)} [(y^j_N)^2 - (y^j_0)^2] \tag{5.7}$$

Substituting (5.7) in (5.4) we obtain

$$\dot{\mu}_j(t) = -2a_0^2[(y_N^j)^2 - (y_0^j)^2]\left\{C_1(\mu_j(t)) - \sum_{i=1}^{2N}\frac{\tilde{\theta}_{N+1}(\lambda_i,t)}{\lambda_i - \mu_j(t)}\right\}, \qquad (5.8)$$

where $C_1(\mu_j(t)) = \sum_{k=0}^{m}\alpha_{1,k}\mu_j^{m-k}(t)$. The factors $\alpha_{1,k}$, $k = 0, 1, ..., m$ are defined from recursion relations (4.11)-(4.14).

By virtue of the equalities

$$\left\|\theta^j\right\|^2 = a_N\theta_N^j\left(\frac{d\theta_{N+1}^j}{d\lambda}\right)\Big|_{\lambda=\mu_j}, (y_0^j)^2 = \frac{(\theta_0^j)^2}{\left\|\theta^j\right\|^2}, (y_N^j)^2 = \frac{(\theta_N^j)^2}{\left\|\theta^j\right\|^2} \qquad (5.9)$$

we can write (5.8) in the form

$$\dot{\mu}_j(t) = -\frac{2a_0\left(\theta_N^j(\mu_j(t),t) - \frac{1}{\theta_N^j(\mu_j(t),t)}\right)}{(\theta_{N+1}^j)'\Big|_{\lambda=\mu_j(t)}}\left\{C_1(\mu_j(t)) - \sum_{i=1}^{2N}\frac{\tilde{\theta}_{N+1}(\lambda_i,t)}{\lambda_i - \mu_j(t)}\right\}. \qquad (5.10)$$

Using the equality

$$\theta_N(\lambda,t)\varphi_{N+1}(\lambda,t) - \theta_{N+1}(\lambda,t)\varphi_N(\lambda,t) = 1,$$

we obtain

$$\theta_N^j(\mu_j(t),t) - \frac{1}{\theta_N^j(\mu_j(t),t)} = \sigma_j(t)\sqrt{\Delta^2(\mu_j(t)) - 4} \qquad (5.11)$$

where

$$\sigma_j(t) = sign\left(\theta_N^j(\mu_j(t),t) - \frac{1}{\theta_N^j(\mu_j(t),t)}\right), j = 1,\ 2,\ ...,\ N-1.$$

It follows from expansion (3.3) that

$$\theta_{N+1}(\lambda,t) = -a_0\left(\prod_{j=1}^{N}a_j\right)^{-1}\prod_{k=1}^{N-1}(\lambda - \mu_k(t)). \qquad (5.12)$$

Differentiating expansion (5.12) with respect to λ and assuming that $\lambda = \mu_j(t)$, we find

$$\theta'_{N+1}(\lambda)\Big|_{\lambda=\mu_j(t)} = -a_0\left(\prod_{k=1}^{N}a_k\right)^{-1}\prod_{\substack{k=1\\k\neq j}}^{N-1}(\mu_j(t) - \mu_k(t)) \qquad (5.13)$$

Substituting (5.11), (3.2) and (5.13) in (5.10) we obtain equality (5.1).

We now show that $\lambda_k(t)$ independent of t. Let $\left\{g_n^k(t)\right\}$ be the normalized eigenfunction of the operator $L(t)$ corresponding to the eigenvalue- $\lambda_k(t)$, $k=1,\ 2,\ ...,\ 2N$, i.e.

$$a_{n-1}g_{n-1}^k + b_n g_n^k + a_n g_{n+1}^k = \lambda_k g_n^k.$$

By differentiating the last identity with respect to t, multiplying by g_n^k and summing over n we get

$$\frac{d\lambda_k}{dt} = \sum_{n=1}^{N}\left(2\dot{a}_n(t)g_n^k g_{n+1}^k + \dot{b}_n(t)\left(g_n^k\right)^2\right). \tag{5.14}$$

Usining (2.1) we can write equality (5.14) in the following form

$$\frac{d\lambda_k}{dt} = \sum_{n=1}^{N}[2P_m(a_n,\ b_n)g_n^k g_{n+1}^k + Q_m(a_n,\ b_n)(g_n^k)^2] + \sum_{i=1}^{2N}F_i^k(t), \tag{5.15}$$

where

$$F_i^k(t) = \tilde{\theta}_{N+1}(\lambda_i,t)\sum_{n=1}^{N}\left\{2a_n[(f_{n+1}^i)^2-(f_n^i)^2]g_n^k g_{n+1}^k+\right.$$

$$\left.+2a_n(f_n^i f_{n+1}^i)(g_n^k)^2 - 2a_{n-1}(f_{n-1}^i f_n^i)(g_n^k)^2\right\}.$$

As (5.8), taking into account the periodicity of g_n^k, we can easily see that $\dot{\lambda}_k(t)=0$. □

Remark 1. The factors $\alpha_{1,k},\ k=0,\ 1,...,m$ are defined from recursion relations (4.11)–(4.14) and they depend on $a_n(t),\ b_n(t)$. It is easy to see that $a_n(t),\ b_n(t)$ are formulated by λ_k and $\mu_j(t)$ via trace formulas (3.5) and (3.6).

Remark 2. Theorem 1 provides the method for solving problem (2.1)-(2.4).

1. Solving the direct spectral problem for the discrete Hill's equation with $\{a_n^0\}$ and $\{b_n^0\}$ the spectral data $\lambda_i,\ \ i=1,2,...,2N$ and $\mu_j(0),\sigma_j(0)$, $j=1,2,...,N-1$ are obtained.
2. Using the result of Theorem 1, we find the $\mu_j(t),\sigma_j(t)\,,j=1,...,$N-1.
3. Using the trace formulas (3.5) and (3.6), we calculate $a_n(t)$, $b_n(t)$ and hence $f_n^k(t)$, $n\in Z$, $k=1,\ 2,\ ...,\ 2N$.

Corollary 1. *If $N=2p$ and the number p is the period of the initial sequences $\{a_n^0\}$ and $\{b_n^0\}$, then all roots of the equation $\Delta(\lambda)+2=0$ are* *double roots. Because the Lyapunov function corresponding to the coefficients $a_n(t)$ and $b_n(t)$ coincides with $\Delta(\lambda)$, according to the analogue of the Borg inverse theorem for the discrete Hill equation (see [15]), the number p is also the period of the solution $a_n(t)$, $b_n(t)$ with respect to the variable n.*

We now illustrate the use of Theorem 1 to solve the problem (2.1), (2.2) when $m = 2$ and $m = 3$.

Let us consider the following periodic initial value conditions

$$(a_n^0)^2 = \frac{5}{2} - (-1)^n \frac{3}{2}, \; b_n^0 = 0, \; n \in Z,$$

for the periodic Toda lattice hierarchy with a self-consistent source (4.17) $(m = 2)$. In this case,

$$N = 2, \;\; \lambda_1 = -3, \;\; \lambda_2 = -1, \;\; \lambda_3 = 1, \;\; \lambda_4 = 3, \;\; \mu_1(0) = 0, \;\; \sigma_1(0) = 1.$$

Using remark 2, we obtain

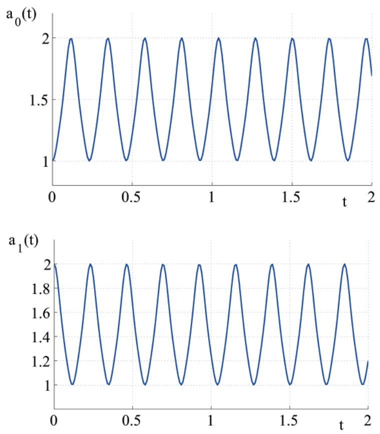
$$a_n(t) = \sqrt{\frac{5}{2} - \frac{1}{2} sn^2 \left(30t, \frac{1}{3}\right) - (-1)^n \frac{3}{2} cn \left(30t, \frac{1}{3}\right) dn \left(30t, \frac{1}{3}\right)},$$

$$b_n(t) = (-1)^n sn \left(30t, \frac{1}{3}\right),$$

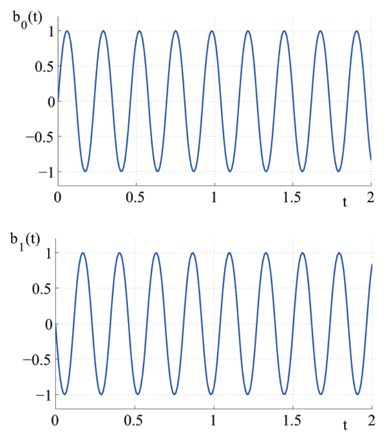
$$f_0^k(t) = \frac{\lambda_k^2 - sn^2\left(30t, \frac{1}{3}\right) - 3cn\left(30t, \frac{1}{3}\right) dn\left(30t, \frac{1}{3}\right)}{2a_0(t)\left[\lambda_k - sn\left(30t, \frac{1}{3}\right)\right]},$$

$$f_1^k(t) = 1, \;\; k = 1, 2, 3, 4,$$

where sn, cn and dn are the Jacobi elliptic functions. The graphs of these functions are listed below.



The graph of $a_0(t)$ and $a_1(t)$ on [0, 2].



The graph of $b_0(t)$ and $b_1(t)$ on [0, 2].

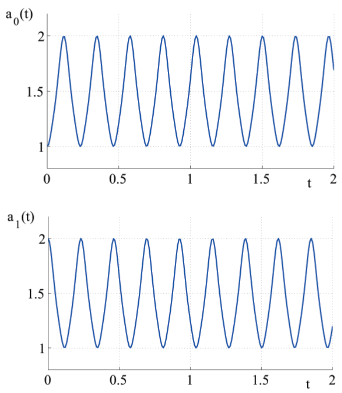
We now illustrate the use of Theorem 1 to solve the problem when $m = 3$. In this case the solution of the problem (2.1), (2.2) has the form

$$a_n(t) = \sqrt{\frac{5}{2} - \frac{1}{2}sn^2\left(-30t, \frac{1}{3}\right) - (-1)^n \frac{3}{2} cn\left(-30t, \frac{1}{3}\right) dn\left(-30t, \frac{1}{3}\right)},$$

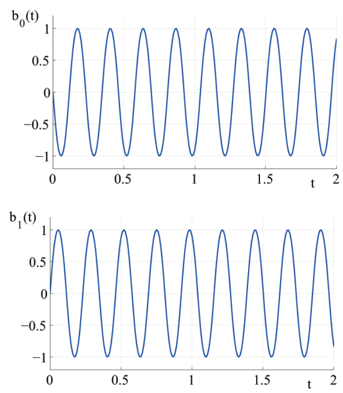
$$b_n(t) = (-1)^n sn\left(-30t, \frac{1}{3}\right),$$

$$f_0^k(t) = \frac{\lambda_k^2 - sn^2\left(-30t, \frac{1}{3}\right) - 3cn\left(-30t, \frac{1}{3}\right) dn\left(-30t, \frac{1}{3}\right)}{2a_0(t)\left[\lambda_k - sn\left(-30t, \frac{1}{3}\right)\right]},$$

$$f_1^k(t) = 1, \; k = 1, 2, 3, 4.$$



The graph of $a_0(t)$ and $a_1(t)$ on [0, 2].



The graph of $b_0(t)$ and $b_1(t)$ on [0, 2].

6. Conclusion

In this paper, we have investigated the hierarchy of the periodic Toda equations with self-consistent sources, where the source is formed by eigenfunctions belonging to the discrete spectrum of the discrete Hill equation. We also presented a new method for constructing a hierarchy for the periodic Toda equations with a self-consistent source. Then the integrable a rich hierarchy for the periodic Toda lattice with a self-consistent source are obtained. We also presented an efficient method for solving the inverse spectral problem for the discrete Hill's equation which is very comfortable for numerical calculation. At the end of the paper, we have illustrated our theory on concrete examples with analytical and numerical results which can be used in some models of special types of electric transmission line.

References

1. Babajanov B.A., Feckan M., Urazbaev G.U. On the periodic Toda Lattice hierarchy with an integral source. *Commun. Sci. Numer. Simul.*, 2017, vol. 52, pp. 110-123. https://doi.org/10.1016/j.cnsns.2017.04.023
2. Babajanov B.A., Feckan M., Urazbaev G.U. On the periodic Toda Lattice with self-consistent source. *Commun. Sci. Numer. Simul.*, 2015, vol. 22, pp. 379-388. http://dx.doi.org/10.1016/j.cnsns.2014.10.013
3. Babajanov B.A. Integration of the Toda-Type Chain with a special self-consistent source. *Algebra, complex analysis and pluripotential theory.* Springer, 2018, vol. 4, pp. 1-13. https://doi.org/10.1007/978-3-030-01144-4
4. Babajanov B.A., Khasanov A.B. Periodic Toda chain an integral source. *Theoret. and Math. Phys.*, 2015, vol. 184, pp. 1114-1128. https://doi.org/10.4213/tmf8859
5. Bulla W., Gesztesy F., Holden H., Teschl G. Algebro-Geometric Quasi-Periodic Finite-Gap Solutions of the Toda and Kac-van Moerbeke Hierarchies. *Memoirs of the Amer. Math. Soc.*, 1998, pp. 135-641. https://doi.org/http://dx.doi.org/10.1090/memo/0641
6. Cabada A., Urazboev G.U. Integration of the Toda lattice with an integral-type source. *Inverse Problems*, 2010, vol. 26, p. 085004. http://dx.doi.org/10.1088/0266-5611/26/8/085004
7. Claude C., Latifi A., Leon J. Nonliear resonant scattering and plasma instability: an integrable model. *J. Math. Phys.*, 1991, vol. 32, pp. 3321-3330. https://doi.org/10.1063/1.529443
8. Chvartatskyi O., Dimakis A. Muller-Hoissen F. Self-Consistent Sources for Integrable Equations Via Deformations of Binary Darboux Transformations. *Lett. Math. Phys.*, 2016, vol. 106, pp. 1139-1179. https://doi.org/10.1007/s11005-016-0859-1
9. Date E., Tanaka S. Analog of inverse scattering theory for discrete Hill‘s equation and exact solutions for the periodic Toda lattice. *Progress Theor. Physics*, 1976, vol. 55, pp. 217-222. https://doi.org/10.1143/PTP.55.457
10. David C., Niels G.J., Bishop A.R., Findikoglu A.T., Reago D. A perturbed Toda lattice model for low loss nonlinear transmission lines. *Phys. D: Nonlinear Phenom.*, 1998, vol. 123, pp. 291-300.
11. Dubrovin B.A., Matveev V.B., Novikov S.P. Nonlinear equations of Korteweg-de Vries type, finite-zone linear operators, and Abelian varieties. *Uspekhi Mat. Nauk,* 1976, vol. 31, no. 1, pp. 55-136. http://dx.doi.org/10.1070/RM1976v031n01ABEH001446
12. Flaschka H., On the Toda lattice. II. *Progress Theor. Physics*, 1974, vol. 51, pp. 703-716. https://doi-org.eres.qnl.qa/10.1143/PTP.51.703
13. Garnier J., Abdullaev F.Kh. Soliton dynamics in a random Toda chain. *Phys. Rev. E.*, 2003, vol. 67, pp. 026609-1. https://doi.org/10.1103/PhysRevE.67.026609
14. Grinevich P.G., Taimanov I.A. Spectral conservation laws for periodic nonlinear equations of the Melnikov type. *Amer. Math. Soc. Transl.* Ser. 2, 2008, vol. 224, pp. 125-138.
15. Hochstadt H. On the theory of Hill's matrices and related inverse spectral problems. *Linear Algebra Appl.*, 1975, vol. 11, pp. 41–52. https://doi.org/10.1016/0024-3795(75)90116-0
16. Hurwitz A., Courant R. *Vorlesungen uber allgemeine Funktionentheorie und elliptische Funktionen.* Springer, Berlin, 1964.
17. Kac M., van Moerbeke P. A complete solution of the periodic Toda problem. *Proc. Nat. Acad. Sci. USA*, 1975, vol. 72, pp. 1627-1629. https://doi.org/10.1073/pnas.72.8.2879

18. Krichever I.M. Algebraic curves and non-linear difference equations. *Uspekhi Mat. Nauk*, 1978, vol. 33, no. 4, pp. 215-216. http://dx.doi.org/10.1070/RM1978v033n04ABEH002503
19. Leon J., Latifi A. Solution of an initial-boundary value problem for coupled nonlinear waves. *J. Phys.A. Math. Gen.*, 1990, vol. 23, pp. 1385-403.
20. Lin R., Du Y. Generalized Darboux Transformation for the Discrete Kadomtsev–Petviashvili Equation with Self-Consistent Sources. *Theor. Math. Phys.,* 2018, vol. 196, pp. 1320-133. https://doi.org/10.1134/S0040577918090064
21. Liu X., Zeng Y. On the Toda lattice equation with self-consistent sources. *J. Phys. A: Math. Gen.*, 2005, vol. 38, pp. 8951-8965. https://doi.org/10.1088/0305-4470/38/41/008
22. Lou S.Y., Tang X.Y. *Method of Nonlinear Mathematical Physics*. Beijing, Science Press, 2006.
23. Mel'nikov V.K. Integration of the Korteweg-de Vries equation with a source. *Inverse Problems*, 1990, vol. 6, pp. 233-246. http://dx.doi.org/10.1088/0266-5611/6/2/007
24. Mel'nikov V.K. A direct method for deriving a multisoliton solution for the problem of interaction of waves on the x,y plane. *Commun. Math. Phys.*, 1987, vol. 112, pp. 639-52. https://doi.org/10.1007/BF01225378
25. Mel'nikov V.K. Integration of the nonlinear Schroedinger equation with a self-consistent source. *Commun. Math. Phys.*, 1991, vol. 137, pp. 359-381. https://doi.org/10.1007/BF02431884
26. Muto V., Scott A.C., Christiansen P.L. Thermally generated solitons in a Toda lattice model of DNA. *Physics Letters A*, 1989, vol. 136, pp. 33-36. https://doi.org/10.1016/0375-9601(89)90671-3.
27. Reyimberganov A.A., Rakhimov I.D. The Soliton Solutions for the Nonlinear Schrodinger Equation with Self-consistent Sources. *The Bulletin of Irkutsk State University. Series Mathematics*, 2021, vol. 35, pp. 84-94. https://doi.org/10.26516/1997-7670.2021.36.84
28. Shchesnovich V.S., Doktorov E.V. Modified Manakov system with self-consistent source. *Phys. Lett .A*, 1996, vol. 213, pp. 23-31. https://doi.org/10.1016/0375-9601(96)00090-4
29. Toda M. Waves in nonlinear lattice. Suppl. *Progress Theor. Physics*, 1970, vol. 45, pp. 74-200. https://doi.org/10.1143/PTPS.45.174.
30. Urazboev G.U. Toda lattice with a special self-consistent source. *Theoret. and Math. Phys.*, 2008, vol. 154, pp. 305-315. https://doi.org/10.4213/tmf6171
31. Urazboev G. Integrating the Toda Lattice with Self-Consistent Source via Inverse Scattering Method // Math. Phys. Anal. Geom., 2012, vol. 15, pp. 401-412. https://doi.org/10.1007/s11040-012-9117-7
32. Yakhshimuratov A.B., Babajanov B.A. Integration of equations of Kaup system kind with self-consistent source in class of periodic functions. *Ufa Mathematical Journal*, 2020, vol. 12, pp. 103-113. https://doi.org/10.13108/2020-12-1-103

Bazar Babajanov, Doctor of Science (Physics and Mathematics), Associate Professor, Urgench State University, 14, Kh. Alimdjan st., Urgench, 220100, Republic of Uzbekistan, tel.:+9(9862)2246700,
email: a.murod@mail.ru
ORCID iD https://orcid.org/0000-0001-6878-791X

Murod Ruzmetov, Senior Lecturer, Urgench State University, 14, Kh. Alimdjan st., Urgench, 220100, Republic of Uzbekistan,

tel.:+9(9862)2246700, email: rmurod2002@gmail.com
ORCID iD https://orcid.org/0000-0002-3572-3482

Received 26.07.2021

Построение и интегрирование иерархической цепочки Тоды с самосогласованным источником

Б. А. Бабаджанов[1], М. М. Рузметов[1]

[1] *Ургенчский государственный университет, Ургенч, Республика Узбекистан*

Аннотация. В статье выведена иерархия для цепочки Тоды с самосогласованным источником в классе периодических функций. Обсуждается полная интегрируемость построенных систем, основанная на преобразовании в спектральные данные связанного дискретного уравнения Хилла с периодическими коэффициентами. В частности, уравнения типа Дубровина выводятся для временной эволюции спектральных данных, соответствующих решениям любой системы в иерархии. Данная теория проиллюстрирована на конкретном примере с аналитическими и численными результатами.

Ключевые слова: периодическая цепочка Тоды, уравнение Хилла, самосогласованный источник, обратная спектральная задача, формулы следов.

Базар Атаджанович Бабаджанов, доктор физико-математических наук, Ургенчский государственный университет, Республика Узбекистан, 220100, г. Ургенч, ул. Х. Алимджана, 14, тел.: +9(9862)2246700, email: a.murod@mail.ru
ORCID iD https://orcid.org/0000-0001-6878-791X

Мурод Марксович Рузметов, старший преподаватель, Ургенчский государственный университет, Республика Узбекистан, 220100, г. Ургенч, ул. Х. Алимджана, 14, тел.: +9(9862)2246700,
email: rmurod2002@gmail.com
ORCID iD https://orcid.org/0000-0002-3572-3482

Поступила в редакцию 26.07.2021