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On Periodic Shunkov's Groups with Almost Layer-finite Normalizers of Finite Subgroups *

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Abstract. Layer-finite groups first appeared in the work by S. N. Chernikov (1945). Almost layer-finite groups are extensions of layer-finite groups by finite groups. The class of almost layer-finite groups is wider than the class of layer-finite groups; it includes all Chernikov groups, while it is easy to give examples of Chernikov groups that are not layer-finite. The author develops the direction of characterizing well-known and well-studied classes of groups in other classes of groups with some additional (rather weak) finiteness conditions. A Shunkov group is a group G in which for any of its finite subgroups K in the quotient group $N_G(K)/K$ any two conjugate elements of prime order generate a finite subgroup. In this paper, we prove the properties of periodic not almost layer-finite Shunkov groups with condition: the normalizer of any finite nontrivial subgroup is almost layer-finite. Earlier, these properties were proved in various articles of the author, as necessary, sometimes under some conditions, then under others (the minimality conditions for not almost layer-finite subgroups, the absence of second-order elements in the group, the presence of subgroups with certain properties in the group). At the same time, it was necessary to make remarks that this property is proved in almost the same way as in the previous work, but under different conditions. This eliminates the shortcomings in the proofs of many articles by the author, in which these properties are used without proof.

Keywords: periodic group, finitness condition, Shunkov group, almost layer-finite group.

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1. Introduction

For the first time the concept of a layer-finite group appeared in the article by S.N. Chernikov [21]. The group is called *layer-finite*, if the set of its elements of any given order is finite.

In the 70s, interest in layer-finite groups has grown noticeably due to the appearance in a number of works [4; 24; 30] characterizations of almost locally solvable groups with the condition of primary minimality in various classes of groups, where essentially some properties of layer-finite groups are used.

The class of almost layer-finite groups is wider than the class of layer-finite groups; it includes all Chernikov groups, while it is easy to give examples of Chernikov groups that are not layer-finite.

Almost layer-finite groups are extensions of layer-finite by finite groups.

A group is called *Chernikov* if it is either finite or is a finite extension of the direct product of a finite number of quasicyclic groups.

Shunkov group is a group G in which for any of its finite subgroups K in the quotient group $N_G(K)/K$ any two conjugate elements of prime order generate a finite subgroup.

In this paper we prove the properties of a periodic Shunkov group that is not almost layer-finite and satisfies the condition: the normalizer of any finite nontrivial subgroup is almost layer-finite.

The necessity of proving of these properties arose in connection with their use in many articles of the author [5]– [19], in which they were proved sometimes under some conditions, then under others (the minimality condition for not almost layer-finite subgroups, the absence of second-order elements in the group, the presence of subgroups with certain properties in the group). At the same time, many comments had to be made that this property is proved in almost the same way as in the previous work, but under different conditions. Finally, in 2020 in a review of my work it was noted that the author should have published properties of groups with the condition: the normalizer of any finite nontrivial subgroup in G is almost layer-finite, and refer to them, and not use the vague one: " The proof of the lemma is similar to the proof ... " Even in the monograph [9], we had to first prove properties 1, 2, 4 for groups without involutions in §6.2, and then in §6.3 for groups with a strongly embedded subgroup, make a note that the proof of properties for this case is similar to the proof of the properties for groups without involutions. The paper [35] also made many references to lemmas from various articles in which the conditions imposed on the group differ from the conditions of the theorem in the paper.

On the subject of the article, works of other authors [2; 25–29; 36] have been recently published.

2. Properties of periodic Shunkov groups with almost layer-finite normalizers of finite subgroups

In the formulation of the properties, it is assumed that G is a periodic Shunkov group that is not almost layer-finite, with almost layer-finite normalizers of nontrivial finite subgroups.

Property 1. *The group G is not a primary group and Sylow p-subgroups in G are Chernikov.*

Proof. Let P be some p-subgroup of the group G. Among its elementary Abelian subgroups, obviously, there is a maximal subgroup R. The group R cannot be infinite due to the almost layer finiteness of the normalizers of nontrivial finite subgroups. Consequently, any Abelian subgroup of P has a finite lower layer. Then, as is known (see, for example, [22]), Abelian subgroups satisfy the minimality condition and by the main result from [20] P is a Chernikov group. If G is a p-group, then by to what has just been proved, it must be a Chernikov group that impossible since the group G is not almost layer-finite. Contradiction. □

Property 2. *The group G does not have a nontrivial locally finite radical.*

Proof. If a locally finite radical $L(G) \neq 1$, then it is almost layer-finite by conditions of the property and in view of almost layer finiteness of a locally finite subgroup satisfying conditions (see Theorem 1 from [11]). Consequently, it contains a finite characteristic subgroup, whose normalizer is almost layer-finite by conditions. A contradiction with the fact that the group G is not almost layer-finite. □

Property 3. *Any locally finite subgroup of G can be embedded in a maximal almost layer-finite subgroup of G.*

Proof. Among all locally finite subgroups of the group G containing a given subgroup by Zorn's lemma there is maximal one. In view of Shunkov's theorem (Theorem 1 from [11]) it will be almost layer-finite. □

Property 4. *Let F, M be two different infinite maximal almost layer-finite subgroups of the group $G, R(F)$ and $R(M)$ are their layer-finite radicals. Then $R(F) \cap R(M) = 1$.*

Proof. Let $b \in R(F) \cap R(M), b \neq 1$. If $R(F) \cap R(M)$ has finite indices in F and M, then by Proposition 2 from [32] and in view of the almost layer finiteness by Shunkov's theorem (Theorem 1 from [11]) of a locally finite group, satisfying the conditions, we obtain a contradiction with maximality of the subgroup F (by Ditzmann's lemma in the almost layer-finite group there is always a finite characteristic subgroup).

Then let for one of the subgroups, for example, for $M, |M : R(M) \cap R(F)| = \infty$. Subgroup $C = C_G(b)$ due to properties of layer-finite groups intersects F and M by subgroups of finite index. The subgroup C is almost layer-finite by the conditions imposed on the group G. Layer-finite radical $R(C)$ of the group C, obviously, also intersects F and M by subgroups of finite index. Since $|M : R(C) \cap M| < \infty$ and $|M : R(F) \cap M| = \infty$, then $R(C)$ is not contained in F. In view of the layer finiteness of the group $R(C)$, it contains a subgroup B such that $F \cap R(C) = F \cap B, |B : F \cap B| < \infty$. By Proposition 8 from [31] there exists a subgroup $T \leq B \cap F$ in the normalizer of which includes the subgroups F and B. Due to the maximality of the subgroup F we get $G = N_G(T)$. But the group G does not possess a nontrivial locally finite radical by property 2. Contradiction. □

Property 5. *If for an arbitrary maximal almost layer-finite subgroup H of G its nonidentity element has an infinite centralizer in H, then this centralizer itself is contained in G.*

Proof. Let H denote an arbitrary maximal almost layer-finite subgroup of the group G. Suppose that the property statement is false and there is an element $a \in H$ of prime order for which $C_G(a)$ is not contained in H and at the same time $|C_G(a) \cap H| = \infty$. Obviously, the intersection $C_G(a) \cap R(H) = D$ is infinite. Let us include $C_G(a)$ in the maximal almost layer-finite subgroup B (this can be done by property 3). Then $D \subset H \cap B$. Since $|B : R(B)| < \infty$, then the index $|D : R(B) \cap D|$ is finite (recall also that $D < R(H)$). Consequently, the intersection $R(H) \cap R(B)$ is nontrivial and by property 4 $B = H$. Contradiction. □

Property 6. *Let b be an element of prime order and intersections $C_G(b) \cap H$, $C_G(b) \cap H^g$ are infinite, where H is a maximal almost layer-finite subgroup of the group G. Then $H = H^g$.*

Proof. Denote by C the centralizer $C_G(b)$. By property 5 $C \subset H$ and $C \subset H^g$. Therefore, $C \subset H \cap H^g$. As the group C is infinite, then $R(H) \cap C$ is an infinite group and $|C : R(H) \cap C| < \infty$. Similarly, $|C : R(H^g) \cap C| < \infty$. Then the intersection $R(H) \cap R(H^g)$ has a nontrivial element. Using property 4 we obtain the assertion to be proved. □

Property 7. *All involutions in G have infinite centralizers.*

Proof. Suppose that some involution has a finite centralizer in the group G. By the well-known theorem of Shunkov from [33] the group G will be locally finite, which contradicts the almost layer finiteness of a locally finite subgroup satisfying the conditions imposed on the group G (see [11]). □

Property 8. *In a maximal almost layer-finite subgroup V of G all involutions with infinite centralizers in V generate a finite subgroup.*

Proof. Suppose this is not the case and the group generated by involutions from V with infinite centralizers in V is infinite. In view of the structure of the almost layer-finite group and Ditzmann's lemma in this case V contains an involution i with infinite $C_V(i)$ for which the index $|V : C_V(i)|$ is infinite. We denote by $\mathfrak{N}$ the class of involutions from V conjugate to i in V. For an arbitrary element $g \in G\backslash V$ consider the subgroup $V^g = g^{-1}Vg$ and its subset $\mathfrak{M} = \mathfrak{N}^g = g^{-1}\mathfrak{N}g$. Since G is a Shunkov group, any two involutions from the sets $\mathfrak{N}$ and $\mathfrak{M}$ generate finite subgroups. Then for an arbitrary fixed involution x from $\mathfrak{N}$ elements $b_t = xt (t \in \mathfrak{M})$ have finite orders.

If for an infinite subset of $\mathfrak{U}$ from $\mathfrak{M}$ the orders of the elements $b_t, t \in \mathfrak{U}$, are odd, then by properties of dihedral groups in $\langle b_t \rangle$ there is an element c_t with the property $c_t^{-1}tc_t = x$. Since $t \in \mathfrak{U} \leq \mathfrak{M}$, then $t = g^{-1}rg$ for some involution r from $\mathfrak{N}$. Hence we get $c_t^{-1}g^{-1}rgc_t = x$. Denoting $h_t = gc_t$ we see: $x \in h_t^{-1}Vh_t = V_t$. By the definition of the set $\mathfrak{U}$ involutions x, r are conjugate with i in V and, by assumption, have infinite centralizers in V. Hence, the centralizer of the involution x in V_t is also infinite and by property 5 $C_G(x) < V \cap V_t$. Then by property 6 $V = V_t$. Since V is maximal and by the properties of the group G $h_t \in V = N_G(V)$. The element g can be represented as $g = h_tc_t^{-1}(t \in \mathfrak{U})$, then $Vg = Vc_t^{-1}(t \in \mathfrak{U})$.

For two different involutions t_1, t_2 from $\mathfrak{U}$ the corresponding strictly real elements c_{t_1}, c_{t_2} are also different. Otherwise, their coincidence would imply the equality $c_{t_1}^{-1}t_1c_{t_1} = c_{t_2}^{-1}t_2c_{t_2}$, which is impossible for different t_1, t_2. By the properties of dihedral groups the element $j_t = xc_t^{-1}$ from Vg is an involution. The set of such involutions coincides in cardinality with the cardinality of the set $\mathfrak{U}$ and, therefore, is infinite. As representative of the coset Vg, we take the involution $k = xc_t^{-1}$ for some t from $\mathfrak{U}$. Then the involution j_t can be represent in the form $j_t = s_tk(t \in \mathfrak{U})$, where $s_t \in V$ is strictly real with respect to the involution k due to $(s_tk)^2 = (j_t)^2 = 1$ (hence $k^{-1}s_tk = s_t^{-1}$).

Obviously, the group $\langle Z = s_t | t \in \mathfrak{U} \rangle$ is infinite and $Z < V$. The involution k normalizes Z and does not lie in V. Let's include almost layer-finite subgroup $N_G(Z)$ into a maximal almost layer-finite subgroup M of the group G (this can be done by Zorn's lemma in view of almost layer finiteness of locally finite subgroups satisfying the conditions imposed on the group G by Shunkov's theorem (Theorem 1 from [11])). Intersection $V \cap M$ is infinite (it contains the subgroup Z). Hence, by property 4 we obtain the coincidence $V = M$ and the inclusion $k \in V$ contrary to the choice of k.

The resulting contradiction means that for any element $x \in \mathfrak{N}$ there is an infinite subset $\mathfrak{U}_x$ of the set $\mathfrak{M}$ such that the orders of the elements $b_t = xt$ $(t \in \mathfrak{U}_x)$ are even. We denote by $\mathfrak{B}$ the set of involutions of the form $j_t \in \langle b_t \rangle$ $(t \in \mathfrak{U}_x)$. By the properties of dihedral groups and by

Property 5 $\mathfrak{B} \leq V \cap V^g$. Since V is maximal, from infinity of the set $\mathfrak{B}$ by Property 4 follows the coincidence $V = V^g$, which would contradict the choice of the pair V, g. Hence, $\mathfrak{B}$ is a finite set and, without losing the generality of reasoning, we will assume that it consists of one involution j_x. By properties of the dihedral groups $\langle x \rangle, \mathfrak{U}_x \subset C_G(j_x)$ and $\mathfrak{U}_x$ is an infinite set of involutions from V^g. By property 5 $x \in C_G(j_x) \leq V^g$. Hence, in view of the arbitrariness of the choice of the involution x from $\mathfrak{N}$ we obtain $\mathfrak{N} \leq V \cap V^g$. As above, in this situation we come to a contradiction with the choice of the pair V, g. □

Property 9. *In the maximal almost layer-finite subgroup V there is no elementary Abelian subgroup of order 8 from G with almost regular involution in V.*

Proof. Suppose that the statement of the property is false and F is a subgroup of the eighth of order in V, j is its almost regular involution in V.

Since in an infinite locally finite group the quadruple the Klein subgroup has an involution with infinite centralizer (see, for example, [23]), then F contains involution with infinite centralizer in V. Let it be i. Similarly, so as $F = \langle i \rangle \times K$, where K is the dihedral group, we can assume that some involution l is also not almost regular in V. Since by property 8 i belong to a finite normal subgroup V_i in V, and l, respectively, belong to a finite normal subgroup V_l in V, then their product il will also belong to the finite normal subgroup $V_i V_l$ and il also has infinite centralizer in the group V. So, subgroup $L = \langle i \rangle \times \langle l \rangle$ has an infinite centralizer in the group V. Consider maximal almost layer-finite in G a subgroup M containing $C_G(j)$. Obviously $F < C_G(j) \leq M$. As above, we find in F a subgroup L_1 of the fourth order with an infinite centralizer in M. Considering triplet of subgroups L, L_1, F it is easy to see that the intersection $L \cap L_1$ contains some involution whose centralizer lies in $V \cap M$. Since centralizer of any involution in G is infinite, then V, M intersect by infinite subgroup, and hence, by its layer-finite radicals. Contradiction with property 4. □

Property 10. *In an almost layer-finite group V there are only finitely many non-conjugate finite solvable subgroups of a given order.*

Proof. For a Chernikov group the assertion of property follow from the theorem of N.S. Chernikov (see, for example, [34]) Now let V be a non-Chernikov group.

First, suppose that in the group V there are infinitely many elementary Abelian q-subgroups of order k

$$L_1, L_2, ..., L_n, ...$$

We include the group L_n in the Sylow q-subgroup Q_n from V $(n = 1, 2, ...)$. By Property 1 Sylow primary subgroups of V are Chernikov, so we can apply to V the theorem from [1], by which all Sylow q-subgroups $Q_1, Q_2, ...,$ $Q_n, ...$ are conjugate in a locally finite group with Chernikov primary subgroups. Then, since in V Sylow q-subgroups are conjugate, then inside Q_n $(n = 1, 2, ...)$ there is only a finite number of non-conjugate subgroups of order k and for elementary Abelian subgroups the statement of the property is proved. Let now

$$L_1, L_2, ..., L_n, ...$$

be a sequence of solvable subgroups of a given order k. The proof will be carried out by induction by the number k. Since all subgroups of the sequence are solvable, they have normal elementary Abelian subgroups

$$Q = Q_1, Q_2, ..., Q_n, ...$$

respectively. As proved above, among them there are only finitely many such that are not conjugate in V. Without breaking the generality reasoning, we will assume that they are all conjugate with Q, i.e. $Q_n^{c_n} = Q,\ c_n \in V,\ n = 1, 2, ...$

Consider the group $A = N_V(Q)$. Obviously, $L_n^{c_n} \leq A, |L_n^{c_n}/Q| \leq k$. By the properties of almost layer-finite groups A/Q is an almost layer-finite group and, by the inductive hypothesis, among the subgroups

$$L_1^{c_1}/Q, L_2^{c_2}/Q, ..., L_n^{c_n}/Q, ...$$

only a finite number non-conjugate in A/Q. But then the same statement is also true for subgroups of the initial sequence. □

Property 11. *The set of non-conjugate elementary Abelian subgroups from almost layer-finite group V with finite centralizers in V is finite.*

Proof. In view of the fact that in layer-finite group the centralizer of any element has a finite index, it is enough for us to consider only elementary Abelian q-subgroups for $q \in \pi = \pi(V \backslash R(V))$. Insofar as π is a finite set, and the orders of elementary Abelian q-subgroups from V cannot grow indefinitely for each q from π, we have only a finite number of options for orders of such subgroups. Hence by property 8 we obtain the assertion of the property. □

Property 12. *Let V be a maximal almost layer-finite subgroup of G containing involutions. Then*

1) all involutions with infinite centralizers in V conjugate in V;

2) if k is an involution from V and $C_V(k)$ is finite, then k induces an automorphism in some Abelian normal subgroup of finite index from V, which maps each element of this subgroups in reverse.

Proof. Let us prove 1. Let i, k be some involutions from the layer-finite radical $R(V)$ of the group V that are not conjugate in the group V and have infinite centralizers in V. Consider the group $D = \langle i, t\rangle$, where $t = k^g \notin V$. If the order of the element it were odd, then the group D would be a Frobenius group and, contrary to assumption, i, t would be conjugate. Hence, it is an element of an even order. Let us denote by j the involution from $\langle it\rangle$. By properties of dihedral groups j is a central involution in D and, therefore, lies in V due to its infinite isolation (property 5). We denote by S the Sylow 2-subgroup of V, containing i and j. Since in V all Sylow 2-subgroups are conjugated, then we can assume without loss of generality reasoning that k also lies in S, and $i \neq k$, otherwise would contradict the assumption.

The involution j has a finite centralizer in V, since otherwise, due to the infinite isolation of V, the involution t would belong to V with $C_G(j)$.

By property 9 an elementary Abelian 2-subgroup of V containing j is not maybe more than the fourth order. Consider the maximal elementary Abelian subgroup $R = \langle i\rangle \times \langle j\rangle$ from S. Suppose that all involutions from S generate an Abelian group. Then $k \in R$, otherwise there would be an elementary Abelian group of the eighth order in V. Hence by property 8 in view of the structure of an almost layer-finite group we see that j must have an infinite centralizer in V. Contradiction with almost regularity of j in V.

Consequently, involutions from S generate a non-Abelian group. If the involution i does not lie in $Z(S)$, then, since R is maximal, the central involution from S coincides with either j or ij. In the first case j belong to the layer-finite radical of the group V, and in the second case, as above, we obtain a contradiction with almost regularity of j in V. Thus, $i \in Z(S)$.

Consider a maximal almost layer-finite subgroup M in G containing $C_G(t)$. Let now $D_1 = \langle i^{g_1}, t\rangle$ is taken so that $i^{g_1} \notin M$. Consider a Sylow 2-subgroup P from M, containing the involution t and the central involution j_1 from D_1 (D_1 as above is not a Frobenius group). The involution j_1 will belong with $C_G(t)$ to M. We have a situation completely symmetric to the beginning of the proof of the property with the group D. As well as in that case, j_1 is almost regular in M and by property 9 $R_1 = \langle t\rangle \cdot \langle j_1\rangle$ is a maximal elementary Abelian subgroup from P. This immediately entails belonging of the central involutions from P_1 to the subgroup R_1. As before immediately excludes the possibility for involutions j_1 and tj_1 to be central in P due to their almost regularity in M. Thus, $j \in Z(P)$. Now note that $M = V^g$. Indeed, V^g contains the element $t = k^g$ in the layer-finite radical. M also contains t in its layer-finite radical. By property 4 we obtain $M = V^g$. Now, because the Sylow subgroups are conjugate in M, we obtain the conjugacy of the lower layers of the centers of the Sylow subgroups S and P. Using property 9 again, we see that they coincide with $\langle i\rangle$ and $\langle j\rangle$ respectively. This proves Statement 1.

Let us prove assertion 2. Suppose that the centralizer $C_V(k)$ is finite. In this case $k \notin R(V)$. In the case of finiteness of Sylow p-subgroups from $R(V)$ consider the intersection $R(V) \cap C_G(k)$. This intersection is finite and for any element from it by Kargapolov's theorem from [3] there is a normal divisor of finite index in $R(V)$, not containing this element. Then the intersection of all such normal divisors by Poincare's theorem will have a finite index in V (there are only finitely many such normal divisors). If we now take its intersection with $R(V)$, then we obtain a normal subgroup U of finite index in V. Involution k acts on U regularly and, therefore, by Proposition 4.2 from [34], strictly real. Thus, for finite Sylow subgroups Statement 2 is proved.

Now let some of Sylow subgroups of V is infinite. Then, in view of the structure of the almost layer-finite groups the subgroup V has a non-trivial complete part $\tilde{V}$. Consider group $\tilde{V} \lambda \langle k \rangle$. Since k has a finite centralizer in V, then by Proposition 7 from [33] and by the structure of the complete part of the group V, the involution k transforms any element from $\tilde{V}$ by conjugation to the inverse element. In the quotient group $V/\tilde{V}$ all Sylow subgroups are finite. As above for the case of finite Sylow subgroups we find a normal subgroup Q of finite index in $V/\tilde{V}$, consisting of strictly real elements with respect to $k\tilde{V}$. By Proposition 4.2 of [34] Q is an Abelian group. Throwing out from Q Sylow subgroups by prime numbers from $\pi(Q) \cap \pi(\tilde{V})$, we again got Abelian normal subgroup of finite index in $V/\tilde{V}$. By Proposition 4.2 of [34] and since V is a locally finite $(\pi(Q))'$-group, its complete preimage in V will also be an Abelian subgroup normal in V of finite index consisting of strictly real elements with respect to the involution k. □

Property 13. *In a maximal almost layer-finite group V of the group G all involutions with infinite centralizers in V generate an Abelian subgroup of order at most four.*

Proof. By property 8 all involutions with infinite centralizers in H generate a finite subgroup from the layer-finite radical $R(H)$ of the group H. Let the involution $i \in R(H)$. If i is the only involution in $R(H)$, then the property is obvious. Let $R(H)$ contain other involutions $i_1, i_2, ..., i_n$.

Consider elements of the form $a_1 = ji_1, a_2 = ji_2, ..., a_n = ji_n$. Since j is not conjugate in H with any of the involutions $i_1, i_2, ..., i_n$, then by the properties of the dihedral groups the elements $a_1, a_2, ..., a_n$ have even orders. Let t_m $(m = 1, 2, ..., n)$ be an involution from $\langle a_m \rangle$. By Theorem 2 from [31] and property 12 the intersection $\langle j \rangle \times \langle t_m \rangle \cap R(H)$ possesses the involution k_m. Again by the properties of dihedral groups $k_m \in C_G(j)$ and then $\langle i, k_m \rangle \leq R(H) \cap C_G(j)$.

Suppose $i \neq k_m$ for some number m. If the order of the maximal elementary Abelian subgroup in $\langle i, k_m \rangle$ was equal to four, then H would contain an elementary Abelian subgroup of the eighth order, containing j, which is impossible due to property 9. So $\langle i, k_m\rangle = \langle b_m\rangle\lambda\langle i\rangle$, b_m is a non-identity element of odd order.

By property 7 and in view of the conditions imposed on the group G $C_G(j)$ is an infinite almost layer-finite group, and the involution i is almost regular in it, since the involution j is almost regular in H. We also note that in view of property 12 $C_G(j)$ has an Abelian normal subgroup L of finite index, in which i induces an automorphism transforming each element to inverse. If $h \in L$, then $b_m^{-1}hb_m \in L$ for arbitrary $1 \leq m \leq n$, but by property 12 $i^{-1}hi = h^{-1}, i^{-1}(b_m^{-1}hb_m)i = b_m^{-1}h^{-1}b_m$, in addition $i^{-1}b_mi = b_m^{-1}$. Hence $b_m^{-1}h^{-1}b_m = b_mh^{-1}b_m^{-1}$ or $b_m^{-2}h^{-1}b_m^2 = h^{-1}$ for any $h \in L$.

And since b_m is an element of odd order, then $L < C_G(b_m)$. On the other hand, $b_m \in R(H)$ and by property 5 $L < C_G(b_m) \leq H$. But then $C_H(j)$ would be infinite, which contradicts the almost regularity j to H. Therefore, $k_m = i$ and, therefore, $i \in \langle j\rangle \times \langle t_m\rangle$, which implies $t_m \in \langle i\rangle \times \langle j\rangle \quad (m = 1, 2, ..., n)$.

If $t_m = j$ or $t_m = ij$, then $C_H(t_m) < \infty$ and $i, i_m \in C_G(t_m)$, and $i \neq i_m$ by the definition of the sequence $i_1, i_2, ..., i_n$. Similarly to the case $k_m \in C_G(j)$ we obtain a contradiction with the fact that $i \neq i_m$. So $t_m = i$ and $i_m \in C_G(i)$ for any m.

Consider the group $X = \langle i_1, i_2, ..., i_n, i\rangle$. We have shown that $i \in Z(X)$. At the same time X and $Z(X)$ are normal in H. By property 12 all involutions from X are conjugate in H, so X is an elementary Abelian group.

Let us prove that $|X| \leq 4$. If $X = \langle i\rangle$, then the statement is obvious. Let k_1, k_2 be two involutions from X and $k_1, k_2 \notin C_G(j)$. We see that $k_1jk_1j, k_2jk_2j \in X$, so this is involution and equality $(k_1jk_1j)(k_1jk_1j) = e$ implies

$$(k_1jk_1j)^{-1}j(k_1jk_1j) = j.$$

Finally, we get $k_1jk_1j, k_2jk_2j \in X \cap C_G(j)$. By property 9 the center of the group $X\lambda\langle j\rangle$ is the subgroup $\langle i\rangle$, hence $jk_1j = k_1i$ and $jk_2j = k_2i$. From here we get $jk_1k_2j = k_1ik_2i = k_1k_2$. Therefore $k_2 = k_1i$ and $|X| \leq 4$. □

3. Conclusion

In this paper we prove the properties of periodic not almost layer-finite Shunkov groups with the condition: the normalizer of any finite nontrivial subgroup is almost layer-finite. This eliminates the shortcomings in the proofs of many of the author's articles, in which these properties are used without proofs (with references to other works of the author, in which they are proved for groups with very different conditions).

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О периодических группах Шункова с почти слойно конечными нормализаторами конечных подгрупп

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Аннотация. Слойно конечные группы впервые появились без названия в статье С. Н. Черникова (1945). Почти слойно конечные группы являются расширениями слойно конечных групп при помощи конечных групп. Класс почти слойно конечных групп шире, чем класс слойно конечных групп, он включает в себя все группы Черникова, в то время как легко привести примеры групп Черникова, которые не являются слойно конечными. Автор развивает направление характеризации известных хорошо изученных классов групп в других классах групп с некоторыми дополнительными (довольно слабыми) условиями конечности. Группа Шункова — это группа G, в которой для любой ее конечной подгруппы K в факторгруппе $N_G(K)/K$ любые два сопряженных элемента простого порядка порождают конечную подгруппу. В работе доказаны свойства периодических не почти слойно конечных групп Шункова с условием: нормализатор любой конечной неединичной подгруппы почти слойно конечен. Ранее эти свойства доказывались в различных статьях автора по мере необходимости то при одних условиях, то при других (условия минимальности для не почти слойно конечных подгрупп, отсутствие в группе элементов второго порядка, наличие в группе подгрупп с теми или иными свойствами). При этом приходилось делать замечания о том, что данное свойство доказывается практически так же, как и в предыдущей работе, но при других условиях. Тем самым устранены недостатки в доказательствах многих статей автора, в которых эти свойства используются без доказательств.

Ключевые слова: периодическая группа, условие конечности, группа Шункова, почти слойно конечная группа.

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