Серия «Математика»
2021. Т. 37. С. 47—62

Онлайн-доступ к журналу:
http://mathizv.isu.ru

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государственного
университета

УДК 519.63
MSC 65M38
DOI https://doi.org/10.26516/1997-7670.2021.37.47

Analysis of Dual Null Field Methods for Dirichlet Problems of Laplace's Equation in Elliptic Domains with Elliptic Holes: Bypassing Degenerate Scales *

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Abstract. Dual techniques have been used in many engineering papers to deal with singularity and ill-conditioning of the boundary element method (BEM). Our efforts are paid to explore theoretical analysis, including error and stability analysis, to fill up the gap between theory and computation. Our group provides the analysis for Laplace's equation in circular domains with circular holes and in this paper for elliptic domains with elliptic holes. The explicit algebraic equations of the first kind and second kinds of the null field method (NFM) and the interior field method (IFM) have been studied extensively. Traditionally, the first and the second kinds of the NFM are used for the Dirichlet and Neumann problems, respectively. To bypass the degenerate scales of Dirichlet problems, the second and the first kinds of the NFM are used for the exterior and the interior boundaries, simultaneously, called the dual null field method (DNFM) in this paper. Optimal convergence rates and good stability for the DNFM can be achieved from our analysis. This paper is the first part of the study and mostly concerns theoretical aspects; the second part is expected to be devoted to numerical experiments.

Keywords: boundary element method, degenerate scales, elliptic domains, dual null field methods, error analysis, stability analysis.

1. Introduction

Dual techniques have been used in many engineering papers (see [1–3; 11]) to deal with singularity and ill-conditioning of the boundary element method (BEM). However, it seems to be lack of strict analysis, including error and stability analysis. In [6], the analysis for Laplace's equation in circular domains with circular holes is provided by our group, and this paper is a continued study of [6] for Laplace's equation on elliptic domains with elliptic holes ([9;12]) by the dual techniques. When the field nodes are located on the exterior elliptic boundary, the degenerate scales of algorithm singularity occurs at $a + b = 2$ [5], where a and b are two semi-axes of the exterior ellipse. It is too complicated to find all pitfall nodes of the null field method (NFM) causing algorithm singularity, as done in [5]. However, when the field nodes are confined on the same ellipses, the degenerate scales may be bypassed, see Section 2.2.

To guarantee the non-singularity of coefficient matrices obtained, other numerical algorithms should be solicited. In [1], a self-regularized method is proposed in the matrix level to deal with non-unique solutions of the Neumann and Dirichlet problems which contain rigid body mode and degenerate scale, respectively. In [3], they have examined the sufficient and necessary condition of boundary integral formulation for the uniqueness solution of 2D Laplace problem subject to the Dirichlet boundary condition by five regularization techniques, namely hypersingular formulation, method of adding a rigid body mode, rank promotion by adding the boundary flux equilibrium (direct BEM), CHEEF method and the Fichera's method (indirect BEM).

The dual null field method (DNFM) is studied in this paper to avoid the algorithm singularity. More importantly, the error analysis of the DNFM can be made for elliptic domains with one elliptic hole to reach the optimal convergent rates. The bounds of condition numbers of the DNFM of a simple case are derived to display good stability. This paper with [6] may shorten some gap between computation and theory of the dual null field method (DNFM).

This paper is organized as follows. In the next section, for elliptic domains with one elliptic hole, the null field method (NFM) are described, and the degenerate scales are discussed. In Section 3, the dual techniques of the the NFM and the interior field method (IFM) are proposed to remove the degenerate scales. In Section 4, the analysis of errors and stability is explored. In the last section, a few concluding remarks are made.

* The reported study was funded by the Ministry of Science and Technology (MOST), Grant 109-2923-E-216-001-MY3 and RFBR, research project 20-51-S52003.

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2. The Null Field Methods in Elliptic Domains with Elliptic holes

2.1. The First Kind of Null Field Method

The elliptic coordinates are defined in [10] by

$$x = \sigma_0 \cosh\rho\cos\theta, \quad y = \sigma_0 \sinh\rho\sin\theta, \tag{2.1}$$

where $\sigma_0 > 0$ and two coordinates (ρ, θ) have the ranges: $0 \le \rho < \infty$ and $0 \le \theta \le 2\pi$. Denote the large ellipse S_R with $\rho = R$, where the elliptic coordinates (ρ, θ) are given by (2.1) with the origin $(0, 0)$. Also denote a small ellipse $S_{R_1} \subset S_R$ with $\bar\rho = R_1$, where the other (i.e., local) elliptic coordinates $(\bar\rho, \bar\theta)$ are given by

$$\bar x = \sigma_1 \cosh\bar\rho\cos\bar\theta, \quad \bar y = \sigma_1 \sinh\bar\rho\sin\bar\theta, \tag{2.2}$$

where $\sigma_1 > 0$. This Cartesian system $(\bar x, \bar y)$ with the origin (x_1, y_1) is rotated from the axis X, by a counter-clockwise angle Θ as in Figure 1.

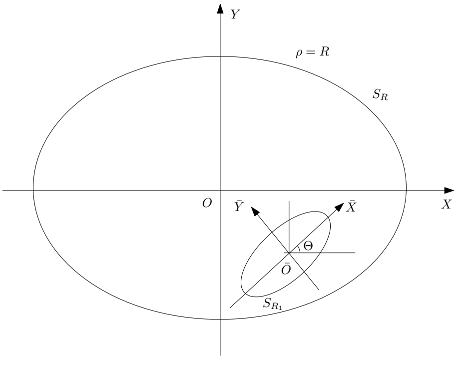

Figure 1. The ellipse S_R with an elliptic hole S_{R_1}.

Denote the annular domain by $S = S_R \setminus S_{R_1}$, and its boundary by $\partial S = \partial S_R \cup \partial S_{R_1}$. In this paper, consider the Dirichlet problem only,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ \text{ in } \ S, \tag{2.3}$$

$$u = f \ \text{ on } \ \partial S_R, \quad u = g \ \text{ on } \ \partial S_{R_1}, \tag{2.4}$$

where f and g are the known functions. On the exterior elliptic boundary ∂S_R, suppose that there exist the approximations of series expansions [9],

$$u = f \approx a_0 + \sum_{k=1}^{M} \{a_k \cos k\theta + b_k \sin k\theta\} \ \text{ on } \ \partial S_R, \tag{2.5}$$

$$\frac{\partial u}{\partial \nu} = f_\nu^* \approx \frac{1}{\sigma_0\tau_0(\theta)}\Big\{p_0 + \sum_{k=1}^{M}\{p_k \cos k\theta + q_k \sin k\theta\}\Big\} \text{ on } \partial S_R, \tag{2.6}$$

where a_k, b_k, p_k and q_k are coefficients, and $\tau_0(\theta) = \sqrt{\sinh^2 R + \sin^2\theta}$. On the interior elliptic boundary ∂S_{R_1}, similarly

$$\bar{u} = g \approx \bar{a}_0 + \sum_{k=1}^{N}\{\bar{a}_k \cos k\bar{\theta} + \bar{b}_k \sin k\bar{\theta}\} \text{ on } \partial S_{R_1} \tag{2.7}$$

$$\frac{\partial \bar{u}}{\partial \bar{\nu}} = -\frac{\partial \bar{u}}{\partial \bar{\rho}} \approx \frac{1}{\sigma_1\tau_1(\bar{\theta})}\left\{\bar{p}_0 + \sum_{k=1}^{N}\{\bar{p}_k \cos k\bar{\theta} + \bar{q}_k \sin k\bar{\theta}\}\right\} \text{ on } \partial S_{R_1}, \tag{2.8}$$

where $\bar{a}_k, \bar{b}_k, \bar{p}_k$ and $\bar{q}_k$ are coefficients, and $\tau_1(\bar{\theta}) = \sqrt{\sinh^2 R_1 + \sin^2\bar{\theta}}$. For the Dirichlet problem, the coefficients a_k and b_k in (2.5) and $\bar{a}_k$ and $\bar{b}_k$ in (2.7) are known, but the coefficients p_k and q_k in (2.1) and $\bar{p}_k$ and $\bar{q}_k$ in (2.8) are unknown to be sought.

In [9], we have derived two explicit algebraic equations of the first kind NFM, $\mathcal{L}_{ext}(\rho,\theta;\bar{\rho},\bar{\theta})$, $\mathcal{L}_{int}(\rho,\theta;\bar{\rho},\bar{\theta})$, and the interior solution $u_{M-N} = u_{M-N}(\rho,\theta;\bar{\rho},\bar{\theta})$ is also given. For the numerical computation of explicit algebraic equations, the coordinate transformations between different elliptic coordinates are needed. In general, the axes of the small ellipse are not along the X and Y axes. The local Cartesian coordinates $X'O'Y'$ are located from the standard Cartesian coordinates XOY by rotating a counter-clockwise angle $\Theta \in [0,\pi)$, see Figure 1. The explicit formulas of the transformations between two different elliptic coordinates can refer to [9].

2.2. Analysis of Degenerate Scales

Denote $\mathcal{L}_{ext}(\rho,\theta;\bar{\rho},\bar{\theta})$ and $\mathcal{L}_{int}(\rho,\theta;\bar{\rho},\bar{\theta})$ simply by

$$\begin{pmatrix} \rho + \ln\frac{\sigma_0}{2} & \bar{\rho} + \ln\frac{\sigma_1}{2} \\ R + \ln\frac{\sigma_0}{2} & R_1 + \ln\frac{\sigma_1}{2}\end{pmatrix}\begin{pmatrix} p_0 \\ \bar{p}_0\end{pmatrix} + \begin{pmatrix} f_0 \\ g_0\end{pmatrix} = \vec{0}, \tag{2.9}$$

where f_0 and g_0 are the remaining terms of algebraic equations without p_0 and $\bar{p}_0$. For the IFM of $\rho = R$ and $\bar{\rho} = R_1$, the matrix singularity occurs when $R + \ln\frac{\sigma_0}{2} = \ln(\frac{a+b}{2}) = 0$, which yields $a+b=2$ of the exterior boundary ∂S_R (see [9]). How about the degenerate scales for $a+b \neq 2$ and $\rho \geq R$ of the NFM? In [5], all pitfall nodes causing algorithm singularity are found for circular domains with one circular hole. For elliptic domains with one elliptic hole, however, it is troublesome and complicated to find all pitfall nodes. Degenerate Case III in [5] is less important in computation, since the filed nodes are not located on the same exterior circular boundary to cause large condition numbers. In applications, it is strongly suggested that the field nodes be located on the same ellipses in [5, Section 4.4]. Hence, the constant ρ is confined in this paper. Denote the ellipse $\partial S_\rho = \{(\rho,\theta)|\rho = \text{constant}, 0 \le \theta \le 2\pi\}$, and all field nodes are located on ∂S_ρ. The global elliptic coordinates (ρ,θ) are defined in (2.1) with focus σ_0, and the local elliptic coordinates $(\bar\rho,\bar\theta)$ in (2.2) with focus σ_1, where x_1, y_1 and Θ are parameters. If $\sigma_1 = \sigma_0, x_1 = y_1 = 0$ and $\Theta = 0$, two elliptic coordinates are identical (i.e., the same as $(\bar\rho,\bar\theta) = (\rho,\theta)$). Otherwise, they are different. For two different elliptic coordinates, $(\rho,\theta) \ne (\bar\rho,\bar\theta)$, we have the following proposition without proof.

Proposition 1. *Suppose that constant ρ $(\ge R)$, $a + b \ne 2$, and two different elliptic coordinates, $(\rho,\theta) \ne (\bar\rho,\bar\theta)$, are used. When $M \ge 2$, there exist no degenerate scales of the NFM.*

When the same elliptic coordinates, i.e., $(\rho,\theta) = (\bar\rho,\bar\theta)$ with $\sigma_1 = \sigma_0$ is used, and suppose that $\rho = R$, the degenerate scales of the IFM do occur at $a^+ + b^+ = a + b = 2$, to coincide with the analysis in [9]. Then the degenerate case with $a + b = 2$ is inevitable to cause the algorithm singularity. In this case, to bypass degenerate scales is essential in computation, and the advanced algorithms and the removal techniques are needed for Dirichlet problems in real application. Our progress has been reported for circular domains in [5; 6]. To deal with Dirichlet problems in elliptic domains, in this paper we explore the application of dual techniques in [1; 2; 4; 11].

3. Dual Techniques

3.1. Second Kind of the NFM

For the dual techniques, we need the second kind Green formula of null field nodes from [12],

$$\frac{\partial}{\partial \nu_{\mathbf{x}}}\Big\{\int_{\partial S_R \cup \partial S_{R_1}} U(\mathbf{x},\mathbf{y})\frac{\partial u(\mathbf{y})}{\partial \nu_{\mathbf{y}}} d\sigma_{\mathbf{y}} - \int_{\partial S_R \cup \partial S_{R_1}} u(\mathbf{y})\frac{\partial U(\mathbf{x},\mathbf{y})}{\partial \nu_{\mathbf{y}}} d\sigma_{\mathbf{y}}\Big\} = 0,$$
$$\mathbf{x} \in S^c, \tag{3.1}$$

where two nodes $\mathbf{x} = Q(x,y)$ and $\mathbf{y} = P(\xi,\eta)$. Denote ν and $\bar\nu$ as the directions of $\theta = const$ and $\bar\theta = const$, respectively, and η and $\bar\eta$ as the angles of ν and $\bar\nu$ from the X axis, respectively. We have from [12]

$$\tan\eta = \frac{\tan\theta}{\tanh\rho}, \quad \bar\eta = \eta^\circ + \Theta, \quad \tan\bar\eta^\circ = \frac{\tan\bar\theta}{\tanh\bar\rho}. \tag{3.2}$$

Based on (3.1), two explicit exterior equations of second kind of the null field method (simply as second kind NFM),

$$\frac{\partial}{\partial\nu}\mathcal{L}_{ext}(\rho,\theta;\bar\rho,\bar\theta) \text{ and } \frac{\partial}{\partial\bar\nu^*}\mathcal{L}_{int}(\rho,\theta;\bar\rho,\bar\theta)$$

are derived from [12], the explicit formulas are not given here.

3.2. Algorithms of Dual Null Field Methods

Traditionally, the first and the second kinds of the NFM are used for the Dirichlet and Neumann problems, respectively, see [9; 12]. The first kind NFM may also be applied to Neumann problems, and the numerical performance is as good as that by the second kind NFM [9]. Hence, we may also apply the second kind NFM for Dirichlet problems. When two kinds of NFMs are applied for exterior and interior boundaries, there are four types, I-I, II-II, I-II and II-I, where I and II denote the first and the second kind NFM, respectively, and their appearances before and behind from "-" denote the exterior and the interior boundaries, respectively. Type I-I is studied in [9] already. For type I-II, $\mathcal{L}_{ext}(\rho,\theta;\bar\rho,\bar\theta)$ and $\frac{\partial}{\partial\bar\nu^*}\mathcal{L}_{int}$ are denoted as

$$\begin{pmatrix} \rho+\ln(\frac{\sigma_0}{2}) & \bar\rho+\ln(\frac{\sigma_1}{2}) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p_0 \\ \bar p_0 \end{pmatrix}+\begin{pmatrix} f_0 \\ g_0 \end{pmatrix}=\vec{0}, \tag{3.3}$$

and for II-II, $\frac{\partial}{\partial\nu}\mathcal{L}_{ext}(\rho,\theta;\bar\rho,\bar\theta)$ and $\frac{\partial}{\partial\bar\nu^*}\mathcal{L}_{int}$ as

$$\frac{\cos(\eta-\bar\eta)}{\sigma_1\tau_1(\bar\rho,\bar\theta)}\begin{pmatrix} \frac{1}{\sigma_0\tau_0(\rho,\theta)} & \frac{\cos(\eta-\bar\eta)}{\sigma_1\tau_1(\bar\rho,\bar\theta)} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p_0 \\ \bar p_0 \end{pmatrix}+\begin{pmatrix} f_0 \\ g_0 \end{pmatrix}=\vec{0}, \tag{3.4}$$

where f_0 and g_0 are the remaining terms of algebraic equations without p_0 and $\bar p_0$. Since there are no leading coefficients p_0 and $\bar p_0$ in $\frac{\partial}{\partial\bar\nu^*}\mathcal{L}_{int}$, the determinants of the matrices of p_0 and $\bar p_0$ in (3.3) and (3.4) are zero, and the algorithm singularity always happens. Only type II-I by using $\frac{\partial}{\partial\nu}\mathcal{L}_{ext}(\rho,\theta;\bar\rho,\bar\theta)$ and $\mathcal{L}_{int}(\rho,\theta;\bar\rho,\bar\theta)$ is worthy to study. $\frac{\partial}{\partial\nu}\mathcal{L}_{ext}(\rho,\theta;\bar\rho,\bar\theta)$ and $\mathcal{L}_{int}(\rho,\theta;\bar\rho,\bar\theta)$ are denoted as

$$\begin{pmatrix} \frac{1}{\sigma_0\tau_0(\rho,\theta)} & \frac{1}{\sigma_1\tau_1(\bar\rho,\bar\theta)}\cos(\eta-\bar\eta) \\ R+\ln\frac{\sigma_0}{2} & R_1+\ln\frac{\sigma_1}{2} \end{pmatrix}\begin{pmatrix} p_0 \\ \bar p_0 \end{pmatrix}+\begin{pmatrix} f_0 \\ \bar g_0 \end{pmatrix}=\vec{0}, \tag{3.5}$$

or can be denoted as

$$\mathcal{D}_{ext}(\rho,\theta;\bar\rho,\bar\theta)=0, \quad \mathcal{L}_{int}(\rho,\theta;\bar\rho,\bar\theta)=0, \tag{3.6}$$

which are called the dual null field method (DNFM) in this paper.

We provide their collocation equations for stability analysis given in Section 4. Choose the uniform nodes on the same ellipses,

$$(\rho,\theta)=(R+\epsilon, j\Delta\theta), \quad j=0,1,...,2M, \tag{3.7}$$

$$(\bar\rho,\bar\theta)=(R_1-\bar\epsilon, j\Delta\bar\theta), \quad j=0,1,...,2N, \tag{3.8}$$

where $\epsilon \geq 0$, $0 < \bar{\epsilon} < R_1$, $\Delta\theta = \frac{2\pi}{2M+1}$ and $\Delta\bar{\theta} = \frac{2\pi}{2N+1}$. We obtain $2(M+N)+2$ collocation equations of the NFM,

$$\frac{\sqrt{w_j}}{M}\mathcal{D}_{ext}(R+\epsilon, j\Delta\theta; \bar{\rho}_j, \bar{\theta}_j) = \frac{\sqrt{w_j}}{M} f(j\Delta\theta), \quad j = 0, 1, ..., 2M, \tag{3.9}$$

$$\sqrt{w_j}\mathcal{L}_{int}(\rho_j, \theta_j; R_1 - \bar{\epsilon}, j\Delta\bar{\theta}) = \sqrt{w_j} g(j\Delta\bar{\theta}), \quad j = 0, 1, ..., 2N, \tag{3.10}$$

where the corresponding coordinates (ρ_j, θ_j) and $(\bar{\rho}_j, \bar{\theta}_j)$ can be evaluated from $(R+\epsilon, j\Delta\theta)$ and $(R_1-\bar{\epsilon}, j\Delta\bar{\theta})$, based on the coordinate transformations in [9]. The wights $w_0 = 1$ and $w_j = 2$ for $j \geq 1$. Eqs. (3.9) and (3.10) are called the collocation Trefftz method (CTM). When $\rho = R$, $\bar{\rho} = R_1$, $\epsilon = \bar{\epsilon} = 0$, and collocation equations of the NFM lead to those of the IFM.

3.3. Removal of Algorithm Singularity

Let us discuss the degenerate scales of the DNFM. We have a proposition, the proof is similar to Proposition 1 and is given next.

Proposition 2. *For Laplace's equation in elliptic domains with one elliptic hole, when $a+b=2$, there do not exist degenerate scales of the DNFM. When $a+b \neq 2$, the statement is true if constant ρ $(\geq R)$ and not small M are chosen.*

Proof. We have the zero determinant from (3.5),

$$|Dual| = \begin{vmatrix} \frac{1}{\sigma_0\tau_0(\rho,\theta)} & \frac{1}{\sigma_1\tau_1(\bar{\rho},\bar{\theta})}\cos(\eta-\bar{\eta}) \\ R+\ln\frac{\sigma_0}{2} & R_1+\ln\frac{\sigma_1}{2} \end{vmatrix} = \frac{R_1+\ln\frac{\sigma_1}{2}}{\sigma_0\tau_0(\rho,\theta)} - \tag{3.11}$$

$$-\frac{R+\ln\frac{\sigma_0}{2}}{\sigma_1\tau_1(\bar{\rho},\bar{\theta})}\cos(\eta-\bar{\eta}) = \frac{\ln\frac{\bar{a}+\bar{b}}{2}}{\sigma_0\tau_0(\rho,\theta)} - \frac{\ln\frac{a+b}{2}}{\sigma_1\tau_1(\bar{\rho},\bar{\theta})}\cos(\eta-\bar{\eta}) = 0.$$

When $a+b=2$, we have

$$|Dual| = \frac{\ln\frac{\bar{a}+\bar{b}}{2}}{\sigma_0\tau_0(\rho,\theta)} < 0, \tag{3.12}$$

since $\bar{a}+\bar{b} < a+b = 2$. Hence, the DNFM may remove algorithm singularity at $a+b=2$ of the IFM. This confirms the first statement.

Next, for $a+b \neq 2$, the exterior ellipse ∂S_ρ with constant $\rho(\geq R)$ is fixed. Then from (3.11), we have a nonlinear equation with respect to $\theta \in [0, 2\pi]$,

$$\Phi(\theta) = \frac{\sigma_0\tau_0(\rho,\theta)}{\sigma_1\tau_1(\bar{\rho},\bar{\theta})}\cos(\eta-\bar{\eta}) = \frac{\ln\frac{\bar{a}+\bar{b}}{2}}{\ln\frac{a+b}{2}}, \quad a+b \neq 2, \tag{3.13}$$

For the given constant ρ, the coordinates $(\bar{\rho}, \bar{\theta})$ via the transforation in [9] are dependent only on variable θ, and so is $(\eta - \bar{\eta})$. The solutions θ from

(3.13) are the roots of a nonlinear equation. In $[0, 2\pi]$, since the sign changes of derivatives $\Phi'(\theta)$ are finite, only a few roots exist. When M is not small (or even large), not all $\theta = \theta_j (j = 0, 1, ..., 2M + 1)$ are just equal to the roots of (3.13). Then Eq. (3.11) does not always hold, to imply no algorithm singularity of the DNFM. □

Note that the degenerate case, $a + b = 2$ of the IFM, disappears in the DNFM. Not only is the algorithm singularity bypassed, but also the optimal stability as Cond $= O(M)$ can be achieved, see Section 4.2.

4. Analysis of Errors and Stability

4.1. Error Bounds

For simplicity, we only explore the analysis for elliptic domains with one elliptic hole. The other mixed types of elliptic and circular boundaries with circular and elliptic holes are similar. We also choose $\rho = R$ and $\bar{\rho} = R_1$, and the original NFM is equivalent to the interior field method (IFM), see [9, Section 4].

Since the DNFM $\mathcal{D}_{ext}(\rho, \theta; \bar{\rho}, \bar{\theta})$ (or $\frac{\partial}{\partial \nu}\mathcal{L}_{ext}(\rho, \theta; \bar{\rho}, \bar{\theta})$) at $\rho = R$ and $\mathcal{L}_{int}(\rho, \theta; \bar{\rho}, \bar{\theta})$ at $\bar{\rho} = R_1$ can be classified to the Trefftz methods, we will follow the outlines of analysis in [9]. Define the energy

$$I(v) = \omega^2 \int_{\partial S_R} (v_\nu - f_\nu^*)^2 ds + \int_{\partial S_{R_1}} (v - g)^2 ds, \tag{4.1}$$

where $v = u_{M-N}$ is given in [9]. The function g is approximated in (2.7) with known coefficients a_k and $\bar{a}_k$, but the function f_ν^* in (2.1) is still unknown yet. The weight $\omega = \frac{1}{M}$ is used to seek optimal convergence for the mixed problems of the Dirichlet and the Neumann problems [7]. For the Dirichlet problems, the coefficients $p_k, q_k, \bar{p}_k$ and $\bar{q}_k$ are unknowns, and the total number is $2(M + N + 1)$. Denote the set of $u^*_{M-N}(\rho, \theta; \bar{\rho}, \bar{\theta})$ [1]as V_{M-N}. The Trefftz method reads: To seek u_{M-N} such that

$$I(u_{M-N}) = \min_{v \in V_{M-N}} I(v). \tag{4.2}$$

When there exists the numerical integration, Eq. (4.2) gives

$$\hat{I}(u_{M-N}) = \min_{v \in V_{M-N}} \hat{I}(v), \tag{4.3}$$

where

$$\hat{I}(v) = \omega^2 \widehat{\int}_{\partial S_R} (v_\nu - f_\nu^*)^2 \, ds + \widehat{\int}_{\partial S_{R_1}} (v - g)^2 \, ds, \tag{4.4}$$

[1] u^*_{M-N} is the interior solution with true Fourier coefficients

where $\widehat{\int}_{\partial S_R}$ and $\widehat{\int}_{\partial S_{R_1}}$ are the approximations by the rules of numerical integrals. For the DNFM, the collocation equations in Eqs. (3.9) and (3.10) at $\epsilon = \bar{\epsilon} = 0$ can be described as (4.3) with the trapezoidal rule.

From the solution, $u^*_{M-N}(\rho, \theta; \bar{\rho}, \bar{\theta})$, we have the derivatives and is given in [8], the explicit formula is not given here.

Then, the remainders of solution derivatives on the exterior boundary ∂S_R are given as

$$
\begin{aligned}
&\frac{\partial}{\partial \nu}(u - u^*_{M-N}) = \frac{\partial}{\partial \nu}(u(R, \theta; \bar{\rho}, \bar{\theta}) - u^*_{M-N}(R, \theta; \bar{\rho}, \bar{\theta})) \qquad (4.5)\\
&= \frac{1}{\sigma_0 \tau_0(R, \theta)} \Big\{ \sum_{k=M+1}^{\infty} k e^{-kR} \{ a_k \sinh kR \cos k\theta + b_k \cosh kR \sin k\theta \} \\
&+ \sum_{k=M+1}^{\infty} e^{-kR} \{ p_k \sinh kR \cos k\theta + q_k \cosh kR \sin k\theta \} \Big\} \\
&- \frac{1}{\sigma_1 \tau_1(\bar{\rho}, \bar{\theta})} \Big\{ \sum_{k=N+1}^{\infty} k e^{-k\bar{\rho}} \{ \bar{a}_k \sinh kR_1 \cos[k\bar{\theta} - \eta + \bar{\eta}] \\
&+ \bar{b}_k \cosh kR_1 \sin[k\bar{\theta} + \eta - \bar{\eta}] \} \\
&+ \sum_{k=N+1}^{\infty} e^{-k\bar{\rho}} \{ \bar{p}_k \cosh kR_1 \cos[k\bar{\theta} - \eta + \bar{\eta}] + \bar{q}_k \sinh kR_1 \sin[k\bar{\theta} + \eta - \bar{\eta}] \} \Big\}.
\end{aligned}
$$

We cite the following lemma from [8].

Lemma 1. *Suppose $u \in H^p(\partial S_R)$, $u_\nu \in H^{p-1}(\partial S_R)$ $(p \geq 2)$, $u \in H^\sigma(\partial S_{R_1})$ and $u_\nu \in H^{\sigma-1}(\partial S_{R_1})$ $(\sigma \geq 2)$. Then there exist the bounds of the remainders of,*

$$
\begin{aligned}
\Big\| \frac{\partial}{\partial \nu}(u - u^*_{M-N}) \Big\|_{0, \partial S_R} \leq C \Big\{ &\frac{1}{M^{(p-1)}} (\|u\|_{p, \partial S_R} + \|u_\nu\|_{p-1, \partial S_R}) \\
+ &\frac{1}{N^{(\sigma-1)}} (\|u\|_{\sigma, \partial S_{R_1}} + \|u_\nu\|_{\sigma-1, \partial S_{R_1}}) \Big\}, \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
\|u - u^*_{M-N}\|_{0, \partial S_{R_1}} \leq C \Big\{ &\frac{1}{M^p} (\|u\|_{p, \partial S_R} + \|u_\nu\|_{p-1, \partial S_R}) \\
+ &\frac{1}{N^\sigma} (\|u\|_{\sigma, \partial S_{R_1}} + \|u_\nu\|_{\sigma-1, \partial S_{R_1}}) \Big\}, \qquad (4.7)
\end{aligned}
$$

where all coefficients in u^*_{M-N} and $\frac{\partial}{\partial \nu} u^*_{M-N}$ are the true Fourier coefficients, and C is a constant independent of M and N.

Define the norm $\|v\|^*_0 = \sqrt{\omega^2 \int_{\partial S_R} v_\nu^2 ds + \int_{\partial S_{R_1}} v^2 ds}$, we have the following theorem.

Theorem 1. *Let the conditions in Lemma 1 hold, and the exact coefficients of the Dirichlet conditions in (2.5) and (2.7) be given. Then the solutions from the DNFM* $\frac{\partial}{\partial \nu}\mathcal{L}_{ext}(\rho,\theta;\bar{\rho},\bar{\theta})$ *and* $\mathcal{L}_{int}(\rho,\theta;\bar{\rho},\bar{\theta})$ *have the following bound,*

$$\|u-u_{M-N}\|^*_{0,\Gamma} \le C\Big\{\frac{1}{M^p}(\|u\|_{p,\partial S_R}+\|u_\nu\|_{p-1,\partial S_R}) \tag{4.8}$$
$$+\frac{1}{N^\sigma}(\|u\|_{\sigma,\partial S_{R_1}}+\|u_{\bar{\nu}}\|_{\sigma-1,\partial S_{R_1}})\Big\}.$$

Proof. For the exterior boundary condition (2.1), denote

$$\hat{D}u_M(\partial S_R)=\frac{1}{\sigma_0\tau_0(\theta)}\left\{p_0+\sum_{k=1}^{M}\{p_k\cos k\theta+q_k\sin k\theta\}\right\}, \tag{4.9}$$
$$\hat{D}u_\infty(\partial S_R)=\frac{1}{\sigma_0\tau_0(\theta)}\left\{p_0+\sum_{k=1}^{\infty}\{p_k\cos k\theta+q_k\sin k\theta\}\right\}, \tag{4.10}$$

where coefficients p_k and q_k are the true Fourier coefficients. The remainder is given by

$$\hat{D}Ru_M=\hat{D}u_\infty(\partial S_R)-\hat{D}u_M(\partial S_R)=\frac{1}{\sigma_0\tau_0(\theta)}\sum_{k=M+1}^{\infty}\{p_k\cos k\theta+q_k\sin k\theta\}.$$

From [8] we have

$$\|\hat{D}Ru_M\|_{0,\partial S_R}=\|(\hat{D}u_\infty-\hat{D}u_M)\|_{0,\partial S_R}\le C\frac{1}{M^{p-1}}\|u_\nu\|_{p-1,\partial S_R}.$$

Since $g=\hat{D}u_M$ and $u_\nu=f^*_\nu=\hat{D}u_\infty$, we have from (4.1),

$$\|u-v\|^*_0=\Big\{\omega^2\int_{\partial S_R}(u_\nu-v_\nu-(\hat{D}u_\infty-\hat{D}u_M))^2ds+\int_{\partial S_{R_1}}(u-v)^2ds\Big\}^{\frac{1}{2}}.$$

From (4.2) we have

$$\|u-u_{M-N}\|^*_0\le\inf_{v\in V_{M-N}}\Big\{\omega\|u_\nu-v_\nu-(\hat{D}u_\infty-\hat{D}u_M)\|_{0,\partial S_R}+\|u-v\|_{0,\partial S_{R_1}}\Big\}.$$

Let $u=u^*_{M-N}$ and $\omega=\frac{1}{M}$, where u^*_{M-N} is the interior solution given in [9] with true Fourier coefficients. We have

$$\|u-u_{M-N}\|^*_0\le C\Big\{\frac{1}{M}\{\|\hat{D}Ru_M\|_{0,\partial S_R}+\Big\|\frac{\partial}{\partial\nu}(u-u^*_{M-N})\Big\|_{0,\partial S_R}\}$$
$$+\|u-u^*_{M-N}\|_{0,\partial S_{R_1}}\Big\}. \tag{4.11}$$

Eq. (4.8) follows from Lemma 1 and (4.11). □

4.2. Condition Numbers

We choose $\rho = R$ and $\bar{\rho} = R_1$. For simplicity, consider the simple case: (1) the symmetric cases $q_k = \bar{q}_k = 0$ and $M = N$, and (2) the same elliptic coordinates with $(\rho, \theta) = (\bar{\rho}, \bar{\theta})$ are used, i.e., $\sigma_0 = \sigma_1, x_1 = y_1 = 0, \Theta = 0$ and $\bar{\eta} = \eta$. We obtain from $\frac{\partial}{\partial \nu}\mathcal{L}_{ext}(\rho, \theta; \bar{\rho}, \bar{\theta})$ at $\rho = R$,

$$\mathcal{D}_{ext}(\rho, \theta) = \frac{1}{\sigma_0 \tau_0(R, \theta)} \Big\{ -(p_0 + \bar{p}_0) - \sum_{k=1}^{M} p_k e^{-k\rho} \cosh kR \cos k\theta$$
$$- \sum_{k=1}^{N} \bar{p}_k e^{-kR} \cosh kR_1 \cos k\theta\} + \sum_{k=1}^{M} a_k k e^{-kR} \sinh kR \cos k\theta$$
$$- \sum_{k=1}^{N} \bar{a}_k k e^{-kR} \sinh kR_1 \cos k\theta \Big\} = 0, \tag{4.12}$$

and $\mathcal{L}_{int}(\rho, \theta; \bar{\rho}, \bar{\theta})$ at $\bar{\rho} = R_1$,

$$\mathcal{L}_{int}(\rho, \theta) = -[R + \ln(\frac{\sigma_0}{2})]p_0 + \sum_{k=1}^{M} \frac{p_k}{k} e^{-kR} \cosh kR_1 \cos k\theta \tag{4.13}$$
$$-[R_1 + \ln(\frac{\sigma_0}{2})]\bar{p}_0 + \sum_{k=1}^{N} \frac{\bar{p}_k}{k} e^{-kR_1} \cosh kR_1 \cos k\theta$$
$$+a_0 - \bar{a}_0 + \sum_{k=1}^{M} a_k e^{-kR} \cosh kR_1 \cos k\theta - \sum_{k=1}^{N} \bar{a}_k e^{-kR_1} \cosh k\rho \cos k\theta = 0.$$

Eqs. (4.12) and (4.13) lead to

$$p_0 + \bar{p}_0 + \sum_{k=1}^{M} (p_k e^{-kR} \cosh kR + \bar{p}_k e^{-kR} \cosh kR_1) \cos k\theta = f_1(\theta), \tag{4.14}$$
$$-[R + \ln(\frac{\sigma_0}{2})]p_0 - [R_1 + \ln(\frac{\sigma_0}{2})]\bar{p}_0 \tag{4.15}$$
$$+ \sum_{k=1}^{M} (\frac{p_k}{k} e^{-kR} \cosh kR_1 + \frac{\bar{p}_k}{k} e^{-kR_1} \cosh kR_1) \cos k\theta = g_1(\theta),$$

where f_1 and g_1 are the remaining terms of algebraic equations without p_k and $\bar{p}_k$.

Below, we give the stability analysis. For the collocation equations in (3.9) and (3.10) at $\epsilon = \bar{\epsilon} = 0$, define the matrices $\mathbf{B}_k \in \mathbb{R}^{2 \times 2}$ such that

$$\mathbf{B}_0 = \begin{pmatrix} \frac{1}{M} & \frac{1}{M} \\ -[R + \ln \frac{\sigma_0}{2}] & -[R_1 + \ln \frac{\sigma_0}{2}] \end{pmatrix}, \tag{4.16}$$

$$\mathbf{B}_k = \begin{pmatrix} \frac{1}{M}e^{-kR}\cosh kR & \frac{1}{M}e^{-kR}\cosh kR_1 \\ \frac{1}{k}e^{-kR}\cosh kR_1 & \frac{1}{k}e^{-kR_1}\cosh kR_1 \end{pmatrix}, \quad k = 1, 2, ..., M. \quad (4.17)$$

Lemma 2. *For the symmetric matrix* $\mathbf{B}_k$ $(k \geq 1)$ *in (4.17), two singular values* $\sigma_k^{\pm}$ *have the bounds,*

$$\sigma_k^+ \leq C\frac{1}{k}, \quad \sigma_k^- \geq c_0\frac{1}{M}, \quad (4.18)$$

where C *and* c_0 (> 0) *are two constants independent of* M*.*

Proof. The determinant of (4.17) is given by

$$\text{Det}(\mathbf{B}_k) = \frac{t_k}{kM}, \quad (4.19)$$

where

$$t_k = e^{-kR}\cosh kR_1(e^{-kR_1}\cosh kR - e^{-kR}\cosh kR_1) > 0, k = 1, 2, ..., M.$$

Since matrices $\mathbf{B}_k$ are symmetric, we may seek their eigenvalues. Two eigenvalues satisfy

$$\lambda_k^+ + \lambda_k^- = \frac{1}{M}e^{-kR}\cosh kR + \frac{1}{k}e^{-kR_1}\cosh kR_1 > 0 \quad (4.20)$$

$$\lambda_k^+\lambda_k^- = \text{Det}(\mathbf{B}_k) > 0. \quad (4.21)$$

We conclude that $\lambda_k^{\pm} > 0$, and that the symmetric matrices $\mathbf{B}_k$ are also positive definite. Hence, we have from (4.20)

$$\lambda_k^+ < \lambda_k^+ + \lambda_k^- = \frac{1}{M}e^{-kR}\cosh kR + \frac{1}{k}e^{-kR_1}\cosh kR_1 \leq C\frac{1}{k}, \quad (4.22)$$

and then from (4.21)

$$\lambda_k^- = \frac{t_k}{kM\lambda_k^+} \geq c_0\frac{1}{M}. \quad (4.23)$$

Since the symmetric matrices $\mathbf{B}_k$ are also positive definite, their eigenvalues and singular values are the same. The desired results (4.18) are obtained from (4.22) and (4.23). □

Lemma 3. *For matrix* $\mathbf{B}_0$ *in (4.16), two singular values* $\sigma_0^{\pm}$ *have the bounds,*

$$\sigma_0^+ \leq C, \quad \sigma_0^- \geq c_0\frac{1}{M}. \quad (4.24)$$

Denote the matrix $\mathbf{B} = \text{Diag}\{\mathbf{B}_0, \mathbf{B}_1, ..., \mathbf{B}_M\} \in \mathbb{R}^{n\times n}$ with $n = 2M + 2$. By following the arguments in [9], we have the following lemma.

Lemma 4. *There exist the bounds,*

$$\sigma_{\max}(\mathbf{A}) \le C\sqrt{M}\sigma_{\max}(\mathbf{B}), \quad \sigma_{\min}(\mathbf{A}) \ge c_0\sqrt{M}\sigma_{\min}(\mathbf{B}).$$

Theorem 2. *Under the simple case of elliptic domains with one elliptic hole, for the DNFM (3.9) and (3.10) at $\epsilon = \bar{\epsilon} = 0$, there exist the bounds,*

$$Cond(\mathbf{A}) = O(M). \tag{4.25}$$

Proof. From Lemmas 2-4, we have

$$\sigma_{\max}(\mathbf{A}) \le C\sqrt{M}\sigma_{\max}(\mathbf{B}) \le C\sqrt{M},$$
$$\sigma_{\min}(\mathbf{A}) \ge c_0\sqrt{M}\sigma_{\min}(\mathbf{B}) \ge c_0\sqrt{M}\lambda_M^- \ge c_0\sqrt{M}\frac{1}{M} = c_0\frac{1}{\sqrt{M}}.$$

The desired result (4.25) follows from $\text{Cond}(\mathbf{A}) = \frac{\sigma_{\max}(\mathbf{A})}{\sigma_{\min}(\mathbf{A})}$. □

5. Concluding Remarks

Let us give a few remarks, to address the novelties of this paper.

1. Although the dual null field method (DNFM) have been widely used in engineering computation to deal with degenerate scales (see [2;4;11]), so far there exists no strict analysis. The second and the first kind NFM are used for the exterior and the interior boundaries, respectively, called the DNFM in this paper. The DNFM for Laplace's equation in circular domains with circular holes was first proposed in [6]; but this paper is devoted to the DNFM for Laplace's equation in elliptic domains with elliptic holes. This paper and [6] may establish a theoretical foundation to fill up some gap between theory and computation.

2. For the DNFM, the error bounds are derived in Theorem 1, to achieve the optimal convergence rates. The stability analysis is explored for the simple case in Theorem 2, to reach good stability with Cond $= O(M)$.

3. Numerical experiments will be carried in the second part of the paper to support the theoretical analysis made here. Moreover, the collocation Trefftz methods (CTM) will also be used for comparisons. Both the CTM and the DNFM can offer the excellent numerical performance.

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Received 29.06.2021

Анализ методов двойного нулевого поля в задаче Дирихле для уравнения Лапласа в эллиптических областях с эллиптическими отверстиями: проблема алгоритмической сингулярности

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Аннотация. Двойственные методы часто используются для решения проблемы сингулярности и плохой обусловленности метода граничных элементов (МГЭ). В статье усилия авторов направлены на изучение теоретических аспектов данной проблемы, включая анализ ошибок и исследование устойчивости, чтобы заполнить пробел между теорией и вычислительным экспериментом. Ранее авторами выполнен анализ уравнения Лапласа в круговых областях с круговыми отверстиями, а в настоящей статье рассматриваются эллиптические области с эллиптическими отверстиями. Получены явные алгебраические уравнения первого и второго вида метода нулевого поля (МНП) и метода внутреннего поля (МВП). Традиционно первый и второй виды МНП используются соответственно для задач Дирихле и Неймана. Чтобы преодолеть алгоритмическую сингулярность в задаче Дирихле, второй и первый виды МНП используются для внешних и внутренних границ одновременно. Такой подход называется методом двойственного нулевого поля (ДМНП). В результате проведенного исследования достигнуты быстрая сходимость и хорошая устойчивость ДМНП. Данная статья является первой частью исследования и касается теоретических аспектов, вторая часть будет посвящена вычислительным экспериментам.

Ключевые слова: метод граничных элементов, вырожденные шкалы, эллиптические области, метод двойственного нулевого поля, анализ ошибок, анализ устойчивости.

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Поступила в редакцию 29.06.2021