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Antiperiodic Boundary Value Problem for a Semilinear Differential Equation of Fractional Order*

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Abstract. The present paper is concerned with an antiperiodic boundary value problem for a semilinear differential equation with Caputo fractional derivative of order $q \in (1, 2)$ considered in a separable Banach space. To prove the existence of a solution to our problem, we construct the Green's function corresponding to the problem employing the theory of fractional analysis and properties of the Mittag-Leffler function . Then, we reduce the original problem to the problem on existence of fixed points of a resolving integral operator. To prove the existence of fixed points of this operator we investigate its properties based on topological degree theory for condensing mappings and use a generalized B.N. Sadovskii-type fixed point theorem.

Keywords: Caputo fractional derivative, semilinear differential equation, boundary value problem, fixed point, condensing mapping, measure of noncompactness.

1. Introduction

Recent years have seen a wide spread of the theory of fractional analysis and differential equations of fractional order in modern mathematics. The increasing interest in this subject is explained by numerous applications of the theory in various branches of applied mathematics, physics, engineering, biology, economics ant others (cf., e.g., the monographs [8; 9; 13; 15; 17]). Several different approaches for solving differential equations and their

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boundary value problems in the case of fractional order $q \in (0,1)$ have been introduced in the literature (see [1; 11; 12] and references therein). Differential equations of the fractional order $q > 1$ have received a particular attention in the past few years.

At the same time, along with periodic problems, antiperiodic boundary value problems are being presently studied intensively in view of their applications in physics and interpolation problems (see, e.g., [5], [6], [16] and references therein). Next, we shortly describe some of the existing results in this research direction. In the work [3], invoking Leray-Schauder degree theory and Green's functions method, the authors prove the existence of solutions for the antiperiodic boundary value problem

$$^{C}D^{q}x(t) = f(t, x(t)),\ t \in [0, T],$$

$$x(0) = -x(T),\, x'(0) = -x'(T),$$

where $^{C}D^{q}$ stands for the Caputo fractional derivative of order $q \in (1,2)$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In the paper [2], applying Green's functions method and Krasnoselskii-Krein fixed point theorem the authors solve the following boundary value problem

$$^{C}D^{q}x(t) = f(t, x(t)),\ t \in [0, T],$$

$$x(0) = -x(T),\, x'(0) = -x'(T),\, x''(0) = -x''(T),\, x'''(0) = -x'''(T),$$

for the case of fractional order $q \in (3,4)$ and a continuous function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$.

Our problem is considered in the same vein. Namely, in the present work we investigate the solvability of the following boundary value problem for semilinear differential equation of fractional order

$$^{C}D^{q}x(t) = \lambda x(t) + f(t, x(t)),\ t \in [0, T], \tag{1.1}$$

with the antiperiodic boundary condition

$$x(0) = -x(T),\, x'(0) = -x'(T) \tag{1.2}$$

in a separable Banach space E, where $^{C}D^{q}$ is the Caputo fractional derivative of order $q \in (1,2)$, $\lambda > 0$, $f\colon [0,T] \times E \to E$ is a nonlinear mapping.

The equation (1.1) has important applications in theoretical physics. For instance, when the space has a finite dimension it generalizes the Ginzburg-Landau equation of fractional order (see [17]):

$$^{C}D^{q}\psi(t) = a\psi(t) + b\|\psi(t)\|^{2}\psi(t), \tag{1.3}$$

where a, b are some constants. The equation (1.3) is used to describe the behaviour of a superconductor in media with dispersion in the absence of an external magnetic field as well as in the study of the phenomena of superfluidity and propagation of nonlinear waves.

2. Preliminaries

2.1. Fractional integral and fractional derivative.

First, we introduce some notions and notation from fractional mathematical analysis necessary for our study (cf. the monographs [13; 15]).

Definition 1. A factional integral of order $q > 0$ of a function $g : [0, T] \to \mathbb{R}$ is the function $I^q g$ of the form:

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s)\, ds,$$

where Γ is the Euler Gamma function.

We note that the following property holds for the Euler Gamma function (see, e.g., [15]):

$$\frac{1}{\Gamma(q)} = 0 \text{ for } q = 0, -1, -2, ... \tag{2.1}$$

Definition 2. A factional derivative of order $q \geq 0$ of a function $g \in C^n([0, T])$ is the function ${}^C D^q g$ of the form:

$$^C D^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)\, ds, \quad n = [q] + 1,$$

provided that the right-hand side is correctly defined.

Definition 3. The function of the form

$$E_{q,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(qn + \beta)}, \quad q > 0,\, \beta \in \mathbb{C},\, z \in \mathbb{C},$$

is called the Mittag-Leffler function.

As a rule, the function $E_{q,1}$ is simply denoted by E_q. The Mittag-Leffler function is of great importance in fractional calculus. Consider the following Cauchy problem for the scalar differential equation of fractional order

$$^C D^q x(t) = \lambda x(t) + f(t), \quad t \in [0, T],\, 1 < q < 2, \tag{2.2}$$

$$x(0) = c_1, x'(0) = c_2, \tag{2.3}$$

where $\lambda \in \mathbb{R}$, $f : [0,T] \to \mathbb{R}$ a function for which there exists the fractional integral of order q. It is known (cf. [13]) that the unique solution to this problem is the function

$$x(t) = c_1 E_q(\lambda t^q) + c_2 t E_{q,2}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s)\, ds. \quad (2.4)$$

In what follows, we will use the following relations and statements (see [8])

$$E_{q,\beta}(t) = \frac{1}{\Gamma(\beta)} + t E_{q,\beta+q}(t), \quad (2.5)$$

$$\left(\frac{d}{dt}\right)^n (t^{\beta-1} E_{q,\beta}(\lambda t^q)) = t^{\beta-n-1} E_{q,\beta-n}(\lambda t^q), \quad (2.6)$$

$$\int_0^z t^{\beta-1} E_{q,\beta}(\lambda t^q) dt = z^\beta E_{q,\beta+1}(\lambda z^q), \quad (2.7)$$

Lemma 1. *For a function $f \in C([0,T]; E)$ and $1 < q < 2$ we have*

$$\left(\int_0^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s)\, ds\right)'_t = \int_0^t (t-s)^{q-2} E_{q,q-1}(\lambda(t-s)^q) f(s)\, ds. \quad (2.8)$$

In order to establish a similar result for a function $f \in L^\infty([0,T]; E)$ we need the following results.

Lemma 2. *For any function $f \in L^\infty([0,T]; E)$ there exists a sequence $\{f_n\} \subset C([0,T]; E)$ such that $f_n(t) \to f(t)$ at all Lebesgue points of the function f from $[0,T]$ and $\|f_n\|_{C([0,T];E)} \leq \|f\|_{L^\infty([0,T];E)}$.*

Such a sequence can be constructed using the Steklov projector as follows

$$f_n(t) = \begin{cases} \frac{1}{2n} \int_{t-\frac{1}{n}}^{t+\frac{1}{n}} \hat{f}(s) ds, t \in [0,T]; \\ 0, t \notin [0,T], \end{cases}$$

$$\hat{f}(t) = \begin{cases} f(t), t \in [0,T]; \\ 0, t \notin [0,T]. \end{cases}$$

Lemma 3. *(see [4]) For any function $f \in L^\infty([0,T]; E)$ the set of its Lebesgue points is a set of full measure for $[0,T]$.*

Lemma 4. *(see [7]) Let all the functions $\{f_n\}$ be differentiable on the interval $[0,T]$ and the sequence of derivatives $\{f'_n\}$ converge uniformly with respect to $t \in [0,T]$ in the entire interval. If the sequence $\{f_n\}$ converges at least at one point from $[0,T]$, then the sequence $\{f_n\}$ converges uniformly in the entire interval and the limit function f is differentiable with $f'(t) = \lim_{n\to\infty} f'_n(t)$.*

The following statement is a consequence of the last three lemmas and Lebesgue's theorem on the passage to the limit under the integral sign.

Lemma 5. *For a function $f \in L^\infty([0,T];E)$ and $1 < q < 2$ we have*

$$\Big(\int_0^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s)\, ds\Big)'_t = \int_0^t (t-s)^{q-2} E_{q,q-1}(\lambda(t-s)^q) f(s)\, ds. \tag{2.9}$$

2.2. Measures of noncompactness and condensing mappings.

Let $\mathcal{E}$ be a Banach space. We introduce the following notation:

– $P(\mathcal{E}) = \{A \subseteq \mathcal{E} : A \neq \varnothing\}$;
– $Pb(\mathcal{E}) = \{A \in P(\mathcal{E}) : A \text{ is bounded}\}$;
– $Pv(\mathcal{E}) = \{A \in P(\mathcal{E}) : A \text{ is convex}\}$;
– $K(\mathcal{E}) = \{A \in Pb(\mathcal{E}) : A \text{ is compact}\}$;
– $Kv(\mathcal{E}) = Pv(\mathcal{E}) \cap K(\mathcal{E})$.

Definition 4. *(see, e.g., [10; 14]).* Let $(\mathcal{A}, \geq)$ be a partially ordered set. A function $\beta : Pb(\mathcal{E}) \to \mathcal{A}$ is called a measure of noncompactness (MNC) in $\mathcal{E}$, if for every $\Omega \in Pb(\mathcal{E})$ we have $\beta(\overline{\mathrm{co}}\,\Omega) = \beta(\Omega)$, where $\overline{\mathrm{co}}\,\Omega$ stands for the closure of the convex hull of Ω.

The Hausdorff MNC $\chi(\Omega) = \inf\{\varepsilon > 0$, for which Ω has a finite ε-net in $\mathcal{E}\ \}$ is an example of a real MNC which is monotone, nonsingular, regular, algebraically semiadditive, and semi-homogeneous (cf., e.g., [10]).

The following notion and statement can be found in the monographs [10; 14].

Definition 5. Let X be a closed subset of $\mathcal{E}$ and β be a MNK in $\mathcal{E}$. A mapping $f : X \to \mathcal{E}$ is called condensing relative to β (or β-condensing), if for every not relatively compact $\Omega \in Pb(X)$ we have $\beta(f(\Omega)) \ngeq \beta(\Omega)$.

Theorem 1. *(See [10]). Let $\mathcal{M}$ be a convex bounded closed subset of $\mathcal{E}$ and $f : \mathcal{M} \to \mathcal{M}$ be a continuous β-condensing mapping, where β is a nonsingular MNK in $\mathcal{E}$. Then, the set of fixed points* $\mathrm{Fix}\, f = \{x : x = f(x)\}$ *is nonempty.*

3. Construction of Green's function

In a separable Banach space E consider the boundary value problem (2.2)–(2.3):

$$^{C}D^q x(t) = \lambda x(t) + f(t), \quad t \in [0,T], 1 < q < 2,$$

$$x(0) = c_1, x'(0) = c_2,$$

where $f : [0,T] \to E$.

Definition 6. *A solution of the boundary value problem (2.2)–(2.3) is a function $x \in C([0,T];E)$ which satisfies*

$$x(t) = c_1 E_q(\lambda t^q) + c_2 t E_{q,2}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) f(s)\, ds.$$

Lemma 6. *Let $f \in C([0,T];E)$ and*

$$(1 + E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q) \neq 0. \tag{3.1}$$

Then, the boundary value problem (2.2), (1.2) has a unique solution

$$x(t) = \int_0^T G(t,s) f(s) ds,$$

where Green's function $G(t,s)$ has the form

$$G(t,s) = \Bigg(\frac{-(1 + E_q(\lambda T^q))(T-s)^{q-1} E_{q,q}(\lambda (T-s)^q)}{(1 + E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} + \\ + \frac{T E_{q,2}(\lambda T^q)(T-s)^{q-2} E_{q,q-1}(\lambda (T-s)^q)}{(1 + E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} \Bigg) E_q(\lambda t^q) + \\ \Bigg(\frac{-(1 + E_q(\lambda T^q))(T-s)^{q-2} E_{q,q-1}(\lambda (T-s)^q)}{(1 + E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} + \\ + \frac{T^{-1} E_{q,0}(\lambda T^q))(T-s)^{q-1} E_{q,q}(\lambda (T-s)^q)}{(1 + E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} \Bigg) t E_{q,2}(\lambda t^q) + \\ + (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q), \\ if \;\; 0 \le s \le t < T,$$

$$G(t,s) = \Bigg(\frac{-(1 + E_q(\lambda T^q))(T-s)^{q-1} E_{q,q}(\lambda (T-s)^q)}{(1 + E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} + \\ + \frac{T E_{q,2}(\lambda T^q)(T-s)^{q-2} E_{q,q-1}(\lambda (T-s)^q)}{(1 + E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} \Bigg) E_q(\lambda t^q) +$$

$$\Bigg(\frac{-(1 + E_q(\lambda T^q))(T-s)^{q-2} E_{q,q-1}(\lambda (T-s)^q)}{(1 + E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} + \\ + \frac{T^{-1} E_{q,0}(\lambda T^q))(T-s)^{q-1} E_{q,q}(\lambda (T-s)^q)}{(1 + E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} \Bigg) t E_{q,2}(\lambda t^q), \\ if \;\; 0 \le t < s < T.$$

Proof. A solution of the boundary value problem (2.2), (2.3) in the Banach space E has the following form

$$x(t) = c_1 E_q(\lambda t^q) + c_2 t E_{q,2}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) f(s)\, ds.$$

Applying the formula (2.6) and Lemma 1, we can find the derivative

$$x'(t) = c_1 t^{-1} E_{q,0}(\lambda t^q) + c_2 E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-2} E_{q,q-1}(\lambda(t-s)^q) f(s)\, ds.$$

Note that in view of the property $\frac{1}{\Gamma(0)} = 0$, for the function $E_{q,0}(\lambda t^q)$ we have

$$E_{q,0}(\lambda t^q) = \sum_{n=0}^{\infty} \frac{(\lambda t^q)^n}{\Gamma(qn)} = \frac{1}{\Gamma(0)} + \sum_{n=1}^{\infty} \frac{(\lambda t^q)^n}{\Gamma(qn)} = \sum_{n=1}^{\infty} \frac{(\lambda t^q)^n}{\Gamma(qn)},$$

consequently

$$t^{-1} E_{q,0}(\lambda t^q) = \sum_{n=1}^{\infty} \frac{\lambda^n t^{qn-1}}{\Gamma(qn)}.$$

By virtue of the last formula we obtain $x(0) = c_1$, $x'(0) = c_2$. Now, using (1.2) we have the system

$$\begin{cases} -c_1 = c_1 E_q(\lambda T^q) + c_2 T E_{q,2}(\lambda T^q) + \int_0^T (T-s)^{q-1} E_{q,q}(\lambda(T-s)^q) f(s)\, ds, \\ -c_2 = c_1 T^{-1} E_{q,0}(\lambda T^q) + c_2 E_{q,1}(\lambda T^q) + \int_0^T (T-s)^{q-2} E_{q,q-1}(\lambda(T-s)^q) f(s)\, ds. \end{cases}$$

Solving the last system by Cramer's rule we have

$$c_1 = \frac{-(1+E_q(\lambda T^q)) \int_0^T (T-s)^{q-1} E_{q,q}(\lambda(T-s)^q) f(s)\, ds}{(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} +$$

$$\frac{T E_{q,2}(\lambda T^q) \int_0^T (T-s)^{q-2} E_{q,q-1}(\lambda(T-s)^q) f(s)\, ds}{(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)},$$

$$c_2 = \frac{-(1+E_q(\lambda T^q)) \int_0^T (T-s)^{q-2} E_{q,q-1}(\lambda(T-s)^q) f(s)\, ds}{(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} +$$

$$\frac{T^{-1} E_{q,0}(\lambda T^q) \int_0^T (T-s)^{q-1} E_{q,q}(\lambda(T-s)^q) f(s)\, ds}{(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)}.$$

Inserting the coefficients that we have found into the solution's formula we obtain

$$x(t) = \frac{-(1+E_q(\lambda T^q)) \int_0^T (T-s)^{q-1} E_{q,q}(\lambda(T-s)^q) f(s)\, ds}{(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} E_q(\lambda t^q) +$$

$$\frac{T E_{q,2}(\lambda T^q) \int_0^T (T-s)^{q-2} E_{q,q-1}(\lambda(T-s)^q) f(s)\, ds}{(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} E_q(\lambda t^q) +$$

$$\frac{-(1+E_q(\lambda T^q)) \int_0^T (T-s)^{q-2} E_{q,q-1}(\lambda(T-s)^q) f(s)\, ds}{(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q) E_{q,2}(\lambda T^q)} t E_{q,2}(\lambda t^q) +$$

$$\frac{T^{-1}E_{q,0}(\lambda T^q)\int_0^T(T-s)^{q-1}E_{q,q}(\lambda(T-s)^q)f(s)\,ds}{(1+E_q(\lambda T^q))^2-E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)}tE_{q,2}(\lambda t^q)+$$

$$\int_0^t(t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)f(s)\,ds=\int_0^T G(t,s)f(s)ds.$$

□

Reasoning as in deriving Lemma 5, we can construct Green's function for the boundary value problem (2.2), (1.2) having the same form as in the last lemma, but under the assumption that $f\in L^\infty([0,T];E)$.

4. Existence of a solution

We assume that the function $f:[0,T]\times E\to E$ from the problem (1.1) - (1.2) has the following properties:

$(f1)$ for all $x\in E$ the function $f(\cdot,x):[0,T]\to E$ is measurable;

$(f2)$ for a.e. $t\in[0,T]$ the function $f(t,\cdot):E\to E$ is continuous;

$(f3)$ for every $r>0$ there exists a function $\omega_r\in L^\infty_+([0,T])$ such that for any $x\in E$ with $\|x\|_E<r$ we have: $\|f(t,x)\|_E\le\omega_r(t)$;

$(f4)$ there exists a function $\mu\in L^\infty_+([0,T])$ such that for any bounded set $\Omega\subset E$ we have: $\chi(f(t,\Omega))\le\mu(t)\chi(\Omega)$, for a.e. $t\in[0,T]$, where χ is the Hausdorff MNC in E.

The example of a function f satisfying the above properties in case of the infinite dimensional space $E=L^2[0,T]$ is given next:

$$f:[0,T]\times L^2[0,T]\to L^2[0,T],$$

$$f(t,x(t))=f(t,\xi_t)=h\Big(\int_0^T\xi_t^2(s)ds\Big)\xi_t+g(t),$$

where $h:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous bounded function, and $g\in C[0,T]$. Since h and g are continuous bounded functions, the conditions $(f1)-(f3)$ hold for the mapping f. We show that the condition $(f4)$ is valid as well. Let $\Omega\subset L^2[0,T]$ be a bounded set and for $\varepsilon>0$ the set $K=\{y_1,y_2,...,y_n\}$ be a finite ε-net for Ω. Then, for any $\xi\in\Omega$ there is $y_i\in K$ such that $\|\xi-y_i\|_{L^2}\le\varepsilon$. Using the last inequality, for a.e. $t\in[0,T]$ we have:

$$\|f(t,\xi)-f(t,y_i)\|_{L^2}\le\mu\|\xi-y_i\|_{L^2}\le\mu\varepsilon,$$

where $\mu=\sup_{s\in\mathbb{R}_+}h(s)$. From this estimate it follows that for a.e. $t\in[0,T]$ the relatively compact set $\cup_{i=1}^n f(t,y_i)$ is a $\mu\varepsilon$-set for $f(t,\Omega)$, hence

$$\chi_{L^2}(f(t,\Omega))\le\mu\chi_{L^2}(\Omega).$$

Consider the operator F defined as follows:

$$Fx(t) = \int_0^T G(t,s)f(s,x(s))ds.$$

From the conditions $(f1)$–$(f4)$ it follows that for a function $x \in C([0,T];E)$ the function $f(\cdot,x(\cdot)) \in L^\infty([0,T];E)$. In this case, from the definition of Green's function we infer that for any $t \in [0,T]$ and $1 < q < 2 : G(\cdot,s) \in L^p([0,T]), p > 1$, and Green's function has a singularity only at the point $s = T$, hence $F : C([0,T];E) \to C([0,T];E)$. It is obvious that if a function $x \in C([0,T];E)$ is a solution to the problem (1.1) – (1.2), then it is a fixed point of the operator F. Therefore, in what follows, we prove the existence of fixed points of the operator F. To this aim, we consider the operator $S : L^\infty([0,T];E) \to C([0,T];E)$ of the form

$$S(f)(t) = \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)f(s)\,ds.$$

We have the following statement (see [11]).

Lemma 7. *For every compact set $K \subset E$ and a bounded sequence $\{\eta_n\} \subset L^\infty([0,T];E)$ such that $\{\eta_n(t)\} \subset K$ for a.e. $t \in [0,T]$, the weak convergence $\eta_n \rightharpoonup \eta_0$ in $L^1([0,T];E)$ implies the convergence $S(\eta_n) \to S(\eta_0)$ in $C([0,T];E)$.*

To prove that the operator F is condensing, consider the cone $\mathbb{R}^2_+ = \{\zeta = (\zeta_1,\zeta_2) : \zeta_1 \geq 0, \zeta_2 \geq 0\}$ with the natural partial order, and introduce in the space $C([0,T];E)$ the vector measure of noncompactness $\nu : P(C([0,T];E)) \to \mathbb{R}^2_+$ defined as $\nu(\Omega) = (\varphi(\Omega), mod_C(\Omega))$, where $\varphi(\Omega)$ is the modulus of fiber noncompactness

$$\varphi(\Omega) = \sup_{t\in[0,T]} \chi(\{y(t) : y \in \Omega\}),$$

and the second component is the modulus of equicontinuity

$$mod_C(\Omega) = \lim_{\delta\to 0} \sup_{y\in\Omega} \max_{|t_1-t_2|\leq\delta} \|y(t_1) - y(t_2)\|.$$

Theorem 2. *Let the conditions $(f1) - (f4)$, (3.1) and*

$$L\frac{\|\mu\|_\infty}{\lambda} < 1, \tag{4.1}$$

hold, where

$$L = \frac{E_q^2(\lambda T^q) - 1 + 3E_{q,0}(\lambda T^q)E_q(\lambda T^q)E_{q,2}(\lambda T^q)}{|(1 + E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|} + E_q(\lambda T^q) - 1, \tag{4.2}$$

$\mu(\cdot)$ is the function from $(f4)$, then the operator F is ν-condensing.

Proof. Let $\Omega \subset C([0,T];E)$ be a nonempty bounded set such that

$$\nu(F(\Omega)) \geq \nu(\Omega). \tag{4.3}$$

We prove that Ω is a relatively compact set.

From the inequality (4.3) it follows that

$$\varphi(F(\Omega)) \geq \varphi(\Omega). \tag{4.4}$$

Using the condition $(f4)$ along with the monotonicity, algebraically semi-additivity, and semi-homogeneity of the Hausdorff MNK for $t \in [0,T]$, we obtain the following estimates

$$\chi\left(F(\Omega)(t)\right) \leq$$

$$\frac{(1+E_q(\lambda T^q))\int_0^T (T-s)^{q-1}E_{q,q}(\lambda(T-s)^q)\chi(\Omega(s))\,ds}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\left\|\mu\right\|_\infty E_q(\lambda t^q)+$$

$$\frac{TE_{q,2}(\lambda T^q)\int_0^T (T-s)^{q-2}E_{q,q-1}(\lambda(T-s)^q)\chi(\Omega(s))\,ds}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\left\|\mu\right\|_\infty E_q(\lambda t^q)+$$

$$\frac{(1+E_q(\lambda T^q))\int_0^T (T-s)^{q-2}E_{q,q-1}(\lambda(T-s)^q)\chi(\Omega(s))\,ds}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\left\|\mu\right\|_\infty tE_{q,2}(\lambda t^q)+$$

$$\frac{T^{-1}E_{q,0}(\lambda T^q)\int_0^T (T-s)^{q-1}E_{q,q}(\lambda(T-s)^q)\chi(\Omega(s))\,ds}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\left\|\mu\right\|_\infty tE_{q,2}(\lambda t^q)+$$

$$\left\|\mu\right\|_\infty \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)\chi(\Omega(s))\,ds \leq$$

$$\frac{(1+E_q(\lambda T^q))\int_0^T (T-s)^{q-1}E_{q,q}(\lambda(T-s)^q)\,ds}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\left\|\mu\right\|_\infty E_q(\lambda t^q)\varphi(\Omega)+$$

$$\frac{TE_{q,2}(\lambda T^q)\int_0^T (T-s)^{q-2}E_{q,q-1}(\lambda(T-s)^q)\,ds}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\left\|\mu\right\|_\infty E_q(\lambda t^q)\varphi(\Omega)+$$

$$\frac{(1+E_q(\lambda T^q))\int_0^T (T-s)^{q-2}E_{q,q-1}(\lambda(T-s)^q)\,ds}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\left\|\mu\right\|_\infty tE_{q,2}(\lambda t^q)\varphi(\Omega)+$$

$$\frac{T^{-1}E_{q,0}(\lambda T^q)\int_0^T (T-s)^{q-1}E_{q,q}(\lambda(T-s)^q)\,ds}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\left\|\mu\right\|_\infty tE_{q,2}(\lambda t^q)\varphi(\Omega)+$$

$$\left\|\mu\right\|_\infty \varphi(\Omega)\int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)\,ds.$$

To further estimate $\chi\left(F(\Omega)(t)\right)$ we calculate the integrals in the last expression with the help of the formula (2.7):

$$\int_0^T (T-s)^{q-1}E_{q,q}(\lambda(T-s)^q)ds = -\int_0^T (T-s)^{q-1}E_{q,q}(\lambda(T-s)^q)d(T-s) =$$

$$\int_0^T y^{q-1}E_{q,q}(\lambda y^q)dy = T^q E_{q,q+1}(\lambda T^q).$$

Similarly, we have

$$\int_0^T (T-s)^{q-2}E_{q,q-1}(\lambda(T-s)^q)\,ds = T^{q-1}E_{q,q}(\lambda T^q),$$

$$\int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)ds = t^q E_{q,q+1}(\lambda t^q).$$

Now, we note that if we take $\beta = 1$ in the formula (2.5), then we obtain

$$E_q(\lambda T^q) = \frac{1}{\Gamma(1)} + \lambda T^q E_{q,q+1}(\lambda T^q) = 1 + \lambda T^q E_{q,q+1}(\lambda T^q),$$

$$E_q(\lambda t^q) = \frac{1}{\Gamma(1)} + \lambda t^q E_{q,q+1}(\lambda t^q) = 1 + \lambda t^q E_{q,q+1}(\lambda t^q).$$

Making use of the property (2.1), if we take $\beta = 0$ in the formula (2.5), then we have

$$E_{q,0}(\lambda T^q) = \frac{1}{\Gamma(0)} + \lambda T^q E_{q,q}(\lambda T^q) = \lambda T^q E_{q,q}(\lambda T^q).$$

Therefore, we obtain the following equalities

$$\int_0^T (T-s)^{q-1}E_{q,q}(\lambda(T-s)^q)ds = T^q \frac{1}{\lambda T^q}\left(E_q(\lambda T^q) - 1\right) = \frac{1}{\lambda}\left(E_q(\lambda T^q) - 1\right),$$

$$\int_0^T (T-s)^{q-2}E_{q,q-1}(\lambda(T-s)^q)\,ds = \frac{1}{\lambda T}E_{q,0}(\lambda T^q),$$

$$\int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)ds = \frac{1}{\lambda}\left(E_q(\lambda t^q) - 1\right).$$

Using the last equalities, we can continue to estimate $\chi\left(F(\Omega)(t)\right)$ for $t \in [0,T]$ as follows

$$\chi\left(F(\Omega)(t)\right) \le$$

$$\frac{(1+E_q(\lambda T^q))\left(E_q(\lambda T^q)-1\right)+TE_{q,2}(\lambda T^q)\frac{1}{T}E_{q,0}(\lambda T^q)}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\frac{\|\mu\|_\infty}{\lambda}E_q(\lambda t^q)\varphi(\Omega)+$$

$$\frac{(1+E_q(\lambda T^q))\frac{1}{T}E_{q,0}(\lambda T^q)+T^{-1}E_{q,0}(\lambda T^q)\left(E_q(\lambda T^q)-1\right)}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\frac{\|\mu\|_\infty}{\lambda}tE_{q,2}(\lambda t^q)\varphi(\Omega)$$

$$+\varphi(\Omega)\frac{\|\mu\|_\infty}{\lambda}\left(E_q(\lambda t^q) - 1\right) \le$$

$$\frac{E_q^2(\lambda T^q) - 1 + E_{q,2}(\lambda T^q)E_{q,0}(\lambda T^q)}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\frac{\|\mu\|_\infty}{\lambda}E_q(\lambda T^q)\varphi(\Omega)+$$

$$\frac{2T^{-1}E_{q,0}(\lambda T^q)E_q(\lambda T^q)}{|(1+E_q(\lambda T^q))^2-E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\frac{\|\mu\|_\infty}{\lambda}TE_{q,2}(\lambda T^q)\varphi(\Omega)+$$

$$+\varphi(\Omega)\frac{\|\mu\|_\infty}{\lambda}\left(E_q(\lambda T^q)-1\right)=$$

$$\Big(\frac{E_q^2(\lambda T^q)-1+3E_{q,0}(\lambda T^q)E_q(\lambda T^q)E_{q,2}(\lambda T^q)}{|(1+E_q(\lambda T^q))^2-E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}+E_q(\lambda T^q)-1\Big)\times$$

$$\frac{\|\mu\|_\infty}{\lambda}\varphi(\Omega)=L\frac{\|\mu\|_\infty}{\lambda}\varphi(\Omega).$$

From the last estimate we see that $\sup_{t\in[0,T]}\chi\left(F(\Omega)(t)\right)\leq L\frac{\|\mu\|_\infty}{\lambda}\varphi(\Omega)$, or, which is the same, that $\varphi\left(F(\Omega)\right)\leq L\frac{\|\mu\|_\infty}{\lambda}\varphi(\Omega)$. Taking into account the conditions (4.1) and (4.4) together with the last inequality, we obtain $\varphi(\Omega)=0$.

In the work [11] it was proved that the set of functions

$$M=\left\{S(f)(t)=\int_0^t(t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)f(s,x(s))ds : x\in\Omega\right\}$$

is equicontinuous, consequently $mod_C\left(\Omega\right)=0$, hence $\nu(\Omega)=(0,0)$, which proves the relative compactness of the set Ω. □

Now, we are in a position to prove the main result of our work.

Theorem 3. *Let the conditions* $(f1),(f2),(f4)$, (3.1) *hold. In addition, assume that instead of the condition* $(f3)$ *we have* $(f3')$: *there exists a function* $\alpha\in L_+^\infty([0,T])$ *such that* $\|f(t,\xi)\|_E\leq\alpha(t)(1+\|\xi\|_E)$. *If* $Lk/\lambda<1$, *where* $k=\max\left\{\|\alpha\|_\infty,\|\mu\|_\infty\right\}$, μ *is the function from the condition* $(f4)$, L *is the constant defined by the formula* (4.2), *then the problem* (1.1)–(1.2) *has a solution.*

Proof. Take an arbitrary $x\in\mathcal{C}=C([0,T];E)$, then for $t\in[0,T]$ we have the following estimate:

$$\|Fx(t)\|_E\leq$$

$$\frac{(1+E_q(\lambda T^q))\int_0^T(T-s)^{q-1}E_{q,q}(\lambda(T-s)^q)\,ds}{|(1+E_q(\lambda T^q))^2-E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\|\alpha\|_\infty(1+\|x\|_{\mathcal{C}})E_q(\lambda t^q)+$$

$$\frac{TE_{q,2}(\lambda T^q)\int_0^T(T-s)^{q-2}E_{q,q-1}(\lambda(T-s)^q)\,ds}{|(1+E_q(\lambda T^q))^2-E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\|\alpha\|_\infty(1+\|x\|_{\mathcal{C}})E_q(\lambda t^q)+$$

$$\frac{(1+E_q(\lambda T^q))\int_0^T(T-s)^{q-2}E_{q,q-1}(\lambda(T-s)^q)\,ds}{|(1+E_q(\lambda T^q))^2-E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\|\alpha\|_\infty(1+\|x\|_{\mathcal{C}})tE_{q,2}(\lambda t^q)+$$

$$\frac{T^{-1}E_{q,0}(\lambda T^q)\int_0^T(T-s)^{q-1}E_{q,q}(\lambda(T-s)^q)\,ds}{|(1+E_q(\lambda T^q))^2-E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\|\alpha\|_\infty(1+\|x\|_{\mathcal{C}})tE_{q,2}(\lambda t^q)+$$

$$\|\alpha\|_\infty (1+\|x\|_{\mathcal{C}}) \int_0^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q)\, ds =$$

$$\frac{(1+E_q(\lambda T^q))\,(E_q(\lambda T^q)-1) + T E_{q,2}(\lambda T^q)\frac{1}{T}E_{q,0}(\lambda T^q)}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\times$$

$$\frac{\|\alpha\|_\infty}{\lambda}(1+\|x\|_{\mathcal{C}})E_q(\lambda t^q)+$$

$$\frac{(1+E_q(\lambda T^q))\frac{1}{T}E_{q,0}(\lambda T^q) + T^{-1}E_{q,0}(\lambda T^q)\,(E_q(\lambda T^q)-1)}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\times$$

$$\frac{\|\alpha\|_\infty}{\lambda}(1+\|x\|_{\mathcal{C}})tE_{q,2}(\lambda t^q) + \frac{\|\alpha\|_\infty}{\lambda}(1+\|x\|_{\mathcal{C}})\,(E_q(\lambda t^q)-1) \le$$

$$\frac{E_q^2(\lambda T^q) - 1 + E_{q,2}(\lambda T^q)E_{q,0}(\lambda T^q)}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\frac{\|\alpha\|_\infty}{\lambda}(1+\|x\|_{\mathcal{C}})E_q(\lambda T^q)+$$

$$\frac{2T^{-1}E_{q,0}(\lambda T^q)E_q(\lambda T^q)}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|}\frac{\|\alpha\|_\infty}{\lambda}(1+\|x\|_{\mathcal{C}})TE_{q,2}(\lambda T^q)+$$

$$\frac{\|\alpha\|_\infty}{\lambda}(1+\|x\|_{\mathcal{C}})\,(E_q(\lambda T^q)-1) =$$

$$\Big(\frac{E_q^2(\lambda T^q) - 1 + 3E_{q,0}(\lambda T^q)E_q(\lambda T^q)E_{q,2}(\lambda T^q)}{|(1+E_q(\lambda T^q))^2 - E_{q,0}(\lambda T^q)E_{q,2}(\lambda T^q)|} + E_q(\lambda T^q) - 1\Big)\times$$

$$\frac{\|\alpha\|_\infty}{\lambda}(1+\|x\|_{\mathcal{C}}) \le L\frac{k}{\lambda}(1+\|x\|_{\mathcal{C}}).$$

Now, if we take $R \ge \frac{Lk\lambda^{-1}}{1-Lk\lambda^{-1}}$, then the inequality $\|x\|_{\mathcal{C}} \le R$ implies that $\|Fx\|_{\mathcal{C}} \le R$. Hence, the operator F maps the closed ball $B_R(0) \subset \mathcal{C}$ into itself. Therefore, the operator F satisfies all the assumptions of the theorem 1, and thus F has fixed points, and the problem (1.1)–(1.2) has a solution. □

5. Conclusion

In the paper, the existence of a solution to an antiperiodic boundary value problem for a semilinear differential equation of fractional order $q \in (1,2)$ was considered in a separable Banach space. The original problem was reduced to the problem on existence of fixed points of the corresponding resolving integral operator. Using the topological degree theory for condensing mappings and a generalized B.N. Sadovskii-type fixed point theorem, conditions which guarantee the existence of fixed points for the resolving operator were obtained.

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Антипериодическая задача для полулинейного дифференциального уравнения дробного порядка

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Аннотация. Рассматривается антипериодическая краевая задача для полулинейного дифференциального уравнения с дробной производной Капуто порядка $q \in (1, 2)$ в сепарабельном банаховом пространстве. Для разрешения поставленной задачи мы конструируем, используя теорию дробного анализа и свойства функции Миттаг-Леффлера, соответствующую задаче функцию Грина. Затем исходная задача сводится к задаче о существовании неподвижных точек разрешающего интегрального оператора. Для доказательства существования неподвижных точек разрешающего оператора мы исследуем его свойства на основе теории топологической степени для уплотняющих отображений и используем обобщенную теорему типа Б. Н. Садовского о неподвижной точке.

Ключевые слова: дробная производная Капуто, полулинейное дифференциальное уравнение, краевая задача, неподвижная точка, уплотняющее отображение, мера некомпактности.

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