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Controllability of a Singular Hybrid System*

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Abstract. We consider the linear hybrid system with constant coefficients that is not resolved with respect to the derivative of the continuous component of the unknown function. In Russian literature such systems are also called discrete-continuous. Hybrid systems usually appear as mathematical models of a various technical processes. For example, they describe digital control and switching systems, heating and cooling systems, the functioning of a automobile transmissions, dynamical systems with collisions or Coulomb friction, and many others. There are many papers devoted to the qualitative theory of such systems, but most of them deal with nonsingular cases in various directions. The analysis of the note is essentially based on the methodology for studying singular systems of ordinary differential equations and is carried out under the assumptions of the existence of an equivalent structural form. This structural form is equivalent to the nominal system in the sense of solutions, and the operator which transformes the investigated system into the structural form possesses the left inverse operator. The finding of the structural form is constructive and do not use a change of variables. In addition the problem of consistency of the initial data is solved automatically. Necessary and sufficient conditions for R–controllability (controllability in the reachable set) of the hybrid systems are obtained.

Keywords: hybrid systems, differential-algebraic equations, solvability, controllability.

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1. Introduction

Consider the system with a continuous and discrete-time subsystems

$$Ax'(t) = Bx(t) + C_k y_k + U_k u_k(t), \quad t \in T_k = [t_k,\ t_{k+1}), \ \ k = \overline{0,m}, \quad (1.1)$$

$$y_k = D_{k-1}x(t_{k-1}) + \sum_{i=0}^{k-1} G_{k-1,i} y_i + V_{k-1} v_{k-1}, \quad k = \overline{1, m+1}, \quad (1.2)$$

where $A, B, C_k, U_k, D_k, G_{k,i}, V_k$ are given real matrices of size $n \times n, n \times n, n \times s, n \times l, s \times n, s \times s, s \times \lambda$ respectively, $\det A = 0$; $x(t) \in \mathcal{C}(T_k)$ is the continuous and $y_k \in \mathcal{R}^s$ is the discrete component of an unknown function describing the system state; $u_k(t)$ and v_k $(k = \overline{0,m})$ are l- and λ-dimensional vectors of the continuous and discrete control respectively, $t_0 < t_1 < \ldots < t_{m+1}$, $T = [t_0, t_{m+1}]$. The system (1.1), (1.2) is called singular hybrid system.

We introduce the initial-boundary conditions for the system (1.1), (1.2)

$$y_0 = b_0, \ \ x(t_k + 0) = a_k, \quad k = \overline{0,m}, \quad (1.3)$$

where $a_k \in \mathcal{R}^n$ $(k = \overline{0,m}), b_0 \in \mathcal{R}^s$ are some given vectors.

At present, the term “hybrid systems” is used mainly to describe discrete-continuous systems or systems containing logical variables. Strictly speaking, hybrid systems include systems that describe processes or objects with significantly different characteristics, for example, containing in their dynamics continuous and discrete variables, deterministic and random variables or influences, which ultimately determines the nature of these systems. Moreover, in the most nontrivial cases, these aspects of dynamics cannot be effectively separated and must be analyzed simultaneously. Wherein the subsystems of a continuous state can be described by systems of ordinary differential equations (ODE), including singular ones, by partial differential equations or integro-differential equations.

The problem under consideration is relevant due to numerous applications, in particular, in review [1], dedicated to modeling and optimization of hybrid systems, the applied aspect of such research is well presented. It indicates that a continuous state subsystem can be described using ODEs that are not resolved with respect to the derivatives. A physical example that illustrates the usefulness of the problem in the form (1.1), (1.2) is given. In such a form, a model of two rigid bodies rotating on the same axis can be presented, which during rotation switch from the sliding connection mode to the rigid adhesion mode with each other. Also, in the form (1.1), (1.2) it is possible to write down a dynamic intersectoral system based on the model of V.V. Leontiev [8]. Wherein $x(t)$ is the unknown function of gross output in natural terms; y_k is the unknown vector of the total amount of incoming equipment at the moment t_k; A is a nonnegative matrix of capital intensity ratios (while the lines corresponding to non-capital intensive industries are zero); $B = E_n - \tilde{B}$, where $\tilde{B}$ is a productive direct cost matrix; matrices C_k represent the dependence of production on the amount of equipment; D_k reflect the relationship between the volume of products and equipment performance (how much equipment is out of order and how much is working); G_k is a coefficient that determines the amount of equipment required depending on the amount already available at the time t_k; U_k is a control matrix (the amount of wages, the number of products produced, various kinds of investments, etc.); V_k is a control matrix (e.g. investment in equipment). The controllability problem for the system (1.1), (1.2) can be considered as a profit forecasting problem taking into account the control functions $u_k(t)$ and v_k and the initial conditions (1.3). Other classic examples are switching and thermal management systems, described using a finite number of dynamic models, together with a set of rules for switching between these models (q.v. [18]).

Most of the previous works devoted to the controllability issues considered nonsingular cases in various settings (see for example, [2; 9; 14; 19]). This work is in line with the topic of discrete-continuous hybrid systems, but is essentially based on the methodology for studying singular systems of ODE [3–5; 7; 10]. In this article, for the hybrid system (1.1), (1.2) based on the constructed equivalent structural form [12; 13] necessary and sufficient conditions for R-controllability are obtained (controllability in the reachable set). The results of this article are a continuation of the research done in the works [12; 13; 15; 16], devoted to the issues of the controllability and observability for the singular hybrid systems.

2. Equivalent Forms

Consider the system of the ODE that is not resolved with respect to the derivative

$$Ax'(t) = Bx(t) + f(t), \quad t \in I \subset \mathcal{R}, \tag{2.1}$$

where A, B are given $(n\times n)$-matrices, $\det A = 0$; $f(t)$ is some n-dimensional function that is continuous on I; $x(t)$ is n-dimensional unknown function describing the system state. Such systems are called differential-algebraic equations (DAE).

The matrix pencil $\lambda A - B$ of the system (2.1) is called regular if there exists a number λ (generally complex) such that $\det(\lambda A - B) \neq 0$. As shown in the book [6, p. 313] the regularity of the matrix pencil which describe the system, ensures the existence of nonsingular $(n \times n)$-matrices P and S such that by replacing the variable $x(t) = S\chi(t) = S\begin{pmatrix}\chi_1(t)\\ \chi_2(t)\end{pmatrix}$ and left multiplication by a matrix P the system (2.1) reduced to form

$$\chi_1'(t) = J\chi_1(t) + f_1(t),\ N\chi_2'(t) = \chi_2(t) + f_2(t),\ \ t \in I,$$

where J is some $(n-p)\times(n-p)$-matrix, at that $(n-p)$ is the dimension of the solution space of the system (2.1), N is the upper triangular $(p\times p)$-matrix with the l $(0\le l\le p)$ square zero blocks on the diagonal such that $N^l = O$ is the null matrix; $(f_1(t),\ f_2(t)) = Pf(t)$. In the same place [6, p. 340] an algorithm for finding the transforming matrices P and S is shown.

On the other hand with matrix coefficients of the DAEs (2.1) we associate the matrices of size $nr\times nr$, $n(r+1)\times n(r+1)$ and $n(r+1)\times n(r+2)$ respectively:

$$\mathrm{D}_{r,z} = \begin{pmatrix} A & O & \dots & O \\ B & A & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & A \end{pmatrix},\ \ \mathrm{D}_{r,y} = \begin{pmatrix} A & O \\ \begin{pmatrix} B \\ \vdots \\ O \end{pmatrix} & \mathrm{D}_{r,z} \end{pmatrix},\ \ \mathrm{D}_{r,x} = \begin{pmatrix} B_r & \mathrm{D}_{r,y} \end{pmatrix},$$

where $B_r = (B,\ O,\ \dots, O)$.

We assume that for some r $(0 \le r \le n)$ the matrix $\mathrm{D}_{r,x}$ contains nonsingular minor of order $n(r+1)$ consisting of $\rho = \operatorname{rank}\mathrm{D}_{r,z}$ columns of the matrix $\mathrm{D}_{r,z}$ and the first n columns of the matrix $\mathrm{D}_{r,y}$. This minor is said to be *resolving*.

Definition 1. *The smallest integer r for which there is a resolving minor in the matrix* $\mathrm{D}_{r,x}$ *is called index of the DAE (2.1).*

Let $d = nr - \rho$. Permuting columns of $\mathrm{D}_{r,x}$, we obtain the matrix

$$\Gamma_r = \mathrm{D}_{r,x}\ \operatorname{diag}\left\{Q\begin{pmatrix} O \\ E_d \end{pmatrix},\ Q, \dots, Q\right\}, \tag{2.2}$$

where E_d is the identity matrix of order d, Q is the permutation $(n\times n)$-matrix. The matrix Q is constructed by the rule specified in [11, p. 320].

Definition 2. *An n-dimensional vector-valued function $x(t) \in \mathcal{C}^1(I)$ is called a solution to the DAEs (2.1) if (2.1) becomes an identity on I under substitution of x.*

Lemma 1. *We assume that the matrix* $\mathrm{D}_{r,x}$ *contains a resolving minor and condition* $\operatorname{rank}\mathrm{D}_{r+1,y} = \operatorname{rank}\mathrm{D}_{r,y} + n$ *is satisfied. Then there exists an invertible operator on I*

$$\mathcal{R} = R_0 + R_1\frac{d}{dt} + \dots + R_r\left(\frac{d}{dt}\right)^r, \tag{2.3}$$

that reduces (2.1) to the structural form

$$x_1'(t) = J_1 x_1(t) + f_1(t), \tag{2.4}$$

$$x_2(t) = J_2 x_1(t) + f_2(t), \quad t \in I, \tag{2.5}$$

that is equivalent in the sense of coincidence of the sets of solutions.

Herein $(x_1(t), x_2(t)) = Q^{-1}x(t)$, *$Q$ is the permutation matrix from (2.2),* $\begin{pmatrix} R_0 & R_1 & \dots & R_r \end{pmatrix} = \begin{pmatrix} E_n & O & \dots & O \end{pmatrix} \Gamma_r^\top \left(\Gamma_r \Gamma_r^\top\right)^{-1}$, $\begin{pmatrix} J_2 & E_d \\ J_1 & O \end{pmatrix} = R_0 B Q$, $(f_2(t),\ f_1(t)) = \mathcal{R}[f(t)]$.

Lemma 2. *Let the matrix pencil $\lambda A - B$ is regular. Then the systems (2.1) and (2.4), (2.5) are equivalent in the sense of coincidence of the sets of solutions with $r = l$.*

Definition 3. *The system (2.4), (2.5) is called the equivalent form of the DAEs (2.1).*

The proofs of the lemmas 1 and 2 are given in [17, p. 62], [11, p. 325–326].

3. Solvability

We define the vectors $y_1, y_2, ..., y_{m+1}$ from the system (1.1), (1.2)

$$y_k = S_k y_0 + \sum_{i=0}^{k-1} P_{k,i} x(t_i) + \sum_{i=0}^{k-1} L_{k,i} v_i, \quad k = \overline{1, m+1}, \tag{3.1}$$

where the coefficients are determined from the recurrence relations

$$S_0 = E_s, \ \ S_k = \sum_{j=0}^{k-1} G_{k-1,j} S_j, \ \ P_{k,k-1} = D_{k-1}, \ \ L_{k,k-1} = V_{k-1}, \ \ k = \overline{1, m};$$

$$P_{k,i} = \sum_{j=i+1}^{k-1} G_{k-1,j} P_{j,i}, \ \ L_{k,i} = \sum_{j=i+1}^{k-1} G_{k-1,j} L_{j,i}, \ \ k = \overline{2, m}, \ \ i = \overline{0, k-1}.$$

Let $x_k(t) = x(t), \quad t \in T_k \ (k = \overline{0, m})$. Then after substituting the expressions (3.1) into the equation (1.1) we obtain a family of the systems of DAE

$$A x_0'(t) = B x_0(t) + C_0 y_0 + U_0 u_0(t), \quad t \in T_0; \tag{3.2}$$

$$\begin{aligned} A x_k'(t) = B x_k(t) + C_k \Big(S_k y_0 + \sum_{i=0}^{k-1} P_{k,i} x_i(t_i) + \sum_{i=0}^{k-1} L_{k,i} v_i\Big) \\ + U_k u_k(t), \quad t \in T_k, \quad k = \overline{1, m}. \end{aligned} \tag{3.3}$$

After using the operator (2.3) on the DAE (3.2), (3.3) we obtain a system of $2(m+1)$ equations

$$x'_{0,1}(t) = J_1 x_{0,1}(t) + C_{0,1} y_0 + \mathcal{H}_0 \mathbf{d}_r[u_0(t)], \tag{3.4}$$

$$x_{0,2}(t) = J_2 x_{0,1}(t) + C_{0,2} y_0 + \mathcal{K}_0 \mathbf{d}_r[u_0(t)], \quad t \in T_0; \tag{3.5}$$

$$x'_{k,1}(t) = J_1 x_{k,1}(t) + C_{k,1}(S_k y_0 + \sum_{i=0}^{k-1} P_{k,i} x_i(t_i) + \sum_{i=0}^{k-1} L_{k,i} v_i) + \mathcal{H}_k \mathbf{d}_r[u_k(t)], \tag{3.6}$$

$$\begin{aligned} x_{k,2}(t) = J_2 x_{k,1}(t) + C_{k,2}(S_k y_0 + \sum_{i=0}^{k-1} P_{k,i} x_i(t_i) + \sum_{i=0}^{k-1} L_{k,i} v_i) \\ + \mathcal{K}_k \mathbf{d}_r[u_k(t)], \quad t \in T_k, \quad k = \overline{1,m}; \end{aligned} \tag{3.7}$$

where

$$\mathbf{d}_r[f(t)] = (f(t), f'(t), \ldots, f^{(r)}(t)), x_i(t) = Q \begin{pmatrix} x_{i,1}(t) \\ x_{i,2}(t) \end{pmatrix}, \begin{pmatrix} C_{k,2} \\ C_{k,1} \end{pmatrix} = R_0 C_k,$$

$$\begin{pmatrix} \mathcal{K}_k \\ \mathcal{H}_k \end{pmatrix} = \begin{pmatrix} K_{k,0} & K_{k,1} & \ldots & K_{k,r} \\ H_{k,0} & H_{k,1} & \ldots & H_{k,r} \end{pmatrix} = \begin{pmatrix} R_0 U_k & R_1 U_k & \ldots & R_r U_k \end{pmatrix}, k = \overline{0,m}.$$

Here and below the functions $u_k(t)$ are assumed to be sufficiently smooth on the intervals T_k ($k = \overline{0,m}$).

Definition 4. *The systems (3.4), (3.5) and (3.6), (3.7) is called the equivalent forms of the DAEs (3.2) and (3.3) respectively.*

Definition 5. *The set of vectors $y_1, \ldots, y_{m+1}$ and the function $x(t) \in \mathcal{C}^1(T_k)$ ($k = \overline{0,m}$) is called a solution to the system (1.1), (1.2) if (1.1), (1.2) becomes an identity on T under corresponding substitutions.*

Lemma 3. *Let the matrix pencil $\lambda A - B$ is regular or all the assumptions of the lemma 1 hold. The systems (1.1), (1.2) and (3.4)–(3.7) have the same set of solutions on T.*

The proof of this lemma can be carried out according to the scheme from [11, p. 325–326] based on the fact that the system (3.4)–(3.7) and the DAEs (3.2), (3.3) are equivalent in the sense of coincidence of the sets of solutions [17, p. 62].

The conditions (1.3) can be written as

$$y_0 = b_0, \quad x_k(t_k + 0) = a_k, \quad k = \overline{0,m}. \tag{3.8}$$

Lemma 4. *Let all the assumptions of the lemma 3 hold. Then the problem (3.4)–(3.8) has a solution on T if and only if*

$$a_{0,2} = J_2 a_{0,1} + C_{0,2} b_0 + \mathcal{K}_0 \mathbf{d}_r[u_0(t_0)], \tag{3.9}$$

$$a_{k,2} = J_2 a_{k,1} + C_{k,2}(S_k b_0 + \sum_{i=0}^{k-1} P_{k,i} a_i + \sum_{i=0}^{k-1} L_{k,i} v_i) + \mathcal{K}_k \mathbf{d}_r[u_k(t_k)], \quad k = \overline{1,m}, \tag{3.10}$$

where $a_k = Q\,(a_{k,1},\ a_{k,2})$ $(k = \overline{0,m})$. *Moreover, if a solution to the problem (3.4)–(3.8) exists, then it is unique.*

In view of the above, the proof of this lemma is similar to the proof of corollary 2 from [17, p. 9].

Definition 6. *The conditions (3.8) that satisfy the relations (3.9), (3.10) are called consistent with the system (3.4)–(3.7).*

4. Controllability

Definition 7. *The system (3.2), (3.3) is called R-controllable on* T *if for any consistent vectors* $a_k \in \mathcal{R}^n$ $(k = \overline{0,m}), b_0 \in \mathcal{R}^s$ *and any vectors* $\alpha_k \in \mathcal{R}^n$ $(k = \overline{0,m}), \beta \in \mathcal{R}^s$ *from the reachable set* $\mathcal{M}$ *(q.v. [5, p. 24]) there exist vectors* v_k *and sufficiently smooth on* T_k l*-dimensional vector-functions* $u_k(t)$ $(k = \overline{0,m})$ *such that there exists a solution to the system (3.2), (3.3) that satisfies the relations* $x(t_k+0) = a_k,\ x(t_{k+1}-0) = \alpha_k$ $(k = \overline{0,m}),\ y_{m+1} = \beta$.

Theorem 1. *Let all the assumptions of the lemma 3 hold. The system (3.2), (3.3) is R-controllable on* T *if and only if*

$$q^\top \mathcal{Q} \not\equiv 0, \tag{4.1}$$

where $\mathcal{Q} = \begin{pmatrix} \overline{\mathcal{L}} & O \\ \Theta\mathcal{C}_1\mathcal{L} & \Theta_1 \end{pmatrix}$, $\Theta = \operatorname{diag}\{\int_{t_0}^{t_1} X^{-1}(\tau)d\tau, \dots, \int_{t_m}^{t_{m+1}} X^{-1}(\tau)d\tau\}$, $\Theta_1 = \operatorname{diag}\{\int_{t_0}^{t_1} X^{-1}(\tau)\,\mathcal{H}_0\,\mathbf{d}_r[u_0(\tau)]d\tau, \dots, \int_{t_m}^{t_{m+1}} X^{-1}(\tau)\,\mathcal{H}_m\,\mathbf{d}_r[u_m(\tau)]d\tau\}$, $\mathcal{C}_1 = \operatorname{diag}\{C_{0,1}, \dots, C_{m,1}\}$, $X(t)$ *is* $(n-d) \times (n-d)$*-matricant (i.e., a solution to the system* $X'(t) = J_1 X(t),\ X(t) = E_{n-d}$*);* $q \in \mathcal{R}^{s+(m+1)(n-d)}$ *is an arbitrary nonzero vector;*

$$\mathcal{L} = \begin{pmatrix} O & O & \dots & O & O \\ L_{1,0} & O & \dots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ L_{m,0} & L_{m,1} & \dots & L_{m,m-1} & O \end{pmatrix}, \ \overline{\mathcal{L}} = \begin{pmatrix} L_{1,0} & O & \dots & O \\ L_{2,0} & L_{2,1} & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ L_{m+1,0} & L_{m+1,1} & \dots & L_{m+1,m} \end{pmatrix}.$$

Proof. Necessity. Let the DAEs (3.2), (3.3), and hence the system (3.4)–(3.7), are R-controllable on T. By integrating (3.4), (3.6) from t_k to t_{k+1} for each $k = \overline{0,m}$, given the fact that $X(t_k) = E_{n-d}$, we get that

$$x_{0,1}(t_1) - x_{0,1}(t_0) = \int_{t_0}^{t_1} X(\tau)^{-1} \left(C_{0,1} y_0 + \mathcal{H}_0 \mathbf{d}_r[u_0(\tau)]\right) d\tau,$$

$$x_{k,1}(t_{k+1}) - x_{k,1}(t_k) = \int_{t_k}^{t_{k+1}} X(\tau)^{-1}(C_{k,1}(S_k y_0 + \sum_{i=0}^{k-1} P_{k,i} x_i(t_i) + \\ + \sum_{i=0}^{k-1} L_{k,i} v_i)\mathcal{H}_k \mathbf{d}_r[u_k(\tau)])d\tau, k = \overline{1,m}.$$

Let

$$\mathcal{P} = \begin{pmatrix} O & O & \dots & O & O \\ P_{1,0} & O & \dots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{m,0} & P_{m,1} & \dots & P_{m,m-1} & O \end{pmatrix}, \ \overline{\mathcal{P}} = \begin{pmatrix} P_{1,0} & O & \dots & O \\ P_{2,0} & P_{2,1} & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ P_{m+1,0} & P_{m+1,1} & \dots & P_{m+1,m} \end{pmatrix};$$

$$X_1^{(i)} = (x_{0,1}(t_i), \dots, x_{m,1}(t_{i+m})),\ X_2^{(i)} = (x_{0,2}(t_i), \dots, x_{m,2}(t_{i+m})),$$

$$X^{(i)} = (x_0(t_i), \dots, x_m(t_{i+m})), \mathcal{U}_r^{(i)} = (\mathbf{d}_r[u_0(t_i)], \dots, \mathbf{d}_r[u_m(t_{i+m})]), i = \overline{0,1};$$

$$Y = (y_1, \dots, y_{m+1}), V = (v_0, \dots, v_m), S = (s_0, \dots, s_m), \overline{S}_1 = (s_1, \dots, s_{m+1}).$$

Then we have

$$X_1^{(1)} - X_1^{(0)} - \Theta\mathcal{C}_1 S y_0 - \Theta\mathcal{C}_1 \mathcal{P} X^{(0)} = \Theta\mathcal{C}_1 \mathcal{L} V + \Theta_1. \tag{4.2}$$

We get the relation from the equations (3.5), (3.7)

$$X_2^{(i)} = \operatorname{diag}\{J_2, \dots, J_2\} X_1^{(i)} + \mathcal{C}_2 \left(S y_0 + \mathcal{P} X^{(i)} + \mathcal{L} V \right) + \mathcal{K}\mathcal{U}_r^{(i)},$$

whence it follows that

$$\begin{aligned} X_2^{(i)} = ((E - \mathcal{C}_2\mathcal{P}_2)^{-1} ((\operatorname{diag}\{J_2, \dots, J_2\}) + \mathcal{C}_2\mathcal{P}_1) X_1^{(i)} \\ + \mathcal{C}_2 S y_0 + \mathcal{C}_2 \mathcal{L} V + \mathcal{K}\mathcal{U}_r^{(i)}), \quad i = \overline{0,1}, \end{aligned} \tag{4.3}$$

where $\mathcal{C}_2 = \operatorname{diag}\{C_{0,2}, \dots, C_{m,2}\}$, $\mathcal{K} = \operatorname{diag}\{\mathcal{K}_0, \dots, \mathcal{K}_m\}$, $\mathcal{P}Q = (\mathcal{P}_1 \ \ \mathcal{P}_2)$.

We get a representation for Y from (3.1)

$$Y - \overline{S} y_0 - \overline{\mathcal{P}} X^{(0)} = \overline{\mathcal{L}} V. \tag{4.4}$$

We combine (4.2) and (4.4) into one system

$$g = \begin{pmatrix} \overline{\mathcal{L}} & O \\ \Theta\mathcal{C}_1\mathcal{L} & \Theta_1 \end{pmatrix} \begin{pmatrix} V \\ e_{rl(m+1)} \end{pmatrix}, \tag{4.5}$$

where g is the vector consisting of the left-hand sides of the equations (4.2), (4.4) and e_n is the unit vector of dimension n.

Thus, as the R-controllability of the system (3.4)–(3.7) we can understood the existence such v_k and $u_k(t) \in \mathcal{C}^r(T_k)$ $(k = \overline{0,m})$ (for any value of the vector g with corresponding dimension) that ensure equality (4.5) , since $X_2^{(0)}$ and $X_2^{(1)}$ are uniquely determined from the equation (4.3) for known values $X_1^{(0)}$, $X_1^{(1)}$.

Let us show that if in this case the relation

$$q^\top \mathcal{Q}(\tau_0,\tau_1,\dots,\tau_m)=0\ \ \forall \tau_k\in T_k,\ k=\overline{0,m},$$

holds, then it follows that $q=0$, where $q\in\mathcal{R}^{s+(m+1)(n-d)}$. Suppose the opposite, that there exists $q_*\neq 0$ such that

$$q_*^\top \mathcal{Q}(\tau_0,\tau_1,\dots,\tau_m)=0\ \ \forall \tau_k\in T_k,\ k=\overline{0,m}. \tag{4.6}$$

Since equality (4.5) should be performed for any g, then suppose that $g=q_*$ ($q_*\neq 0$) and scalar multiply on the left (4.5) by the vector q_*.Taking into account (4.6) we get

$$0\neq q_*^\top q_* = q_*^\top g = q_*^\top \mathcal{Q}(V,e_{rl(m+1)})=0.$$

We got a contradiction. Therefore, from the R-controllability of the system (3.4)–(3.7) follows that the condition (4.1) holds for any nonzero vector $q\in\mathcal{R}^{s+(m+1)(n-d)}$.

Sufficiency. Let the condition (4.1) holds. In order to prove the sufficiency, it is necessary to show that in this case there exist vectors v_k,u_k ($k=\overline{0,m}$) such that the system (4.5) solvable for any value of the left side.

We take the integrals from Θ_1 $r+1$ times by parts

$$\int_{t_k}^{t_{k+1}} X^{-1}(t)\,\mathcal{H}_k\,\mathbf{d}_r[u_k(t)]dt = I_{k,r}D_k(t_{k+1})$$
$$+(-1)^{r+1}\int_{t_k}^{t_{k+1}}\left(\int_{t_k}^{t}\int_{t_k}^{\tau_r}\dots\int_{t_k}^{\tau_1} X^{-1}(\tau_0)d\tau_0 d\tau_1\dots d\tau_r\,\mathcal{H}_k\right)\mathbf{d}_r^{(r+1)}[u_k(t)]dt; \tag{4.7}$$
$$D_k(t_{k+1})=\left(\mathbf{d}_r[u_k(t_{k+1})],\mathbf{d}_r'[u_k(t_{k+1})],\dots,\mathbf{d}_r^{(r)}[u_k(t_{k+1})]\right),$$
$$I_{k,r}=(\int_{t_k}^{t_{k+1}} X^{-1}(\tau_0)d\tau_0\,\mathcal{H}_k\quad -\int_{t_k}^{t_{k+1}}\int_{t_k}^{\tau_1} X^{-1}(\tau_0)d\tau_0 d\tau_1\,\mathcal{H}_k\ \dots$$
$$\dots\ (-1)^r\int_{t_k}^{t_{k+1}}\int_{t_k}^{\tau_r}\dots\int_{t_k}^{\tau_1} X^{-1}(\tau_0)d\tau_0 d\tau_1\dots d\tau_r\,\mathcal{H}_k),\ \ k=\overline{0,m}.$$

The control functions $u_k(t)$ will be sought in the form of polynomials of degree r

$$u_k(t)=\sum_{j=0}^{r} b_{k,j}t^j,\quad t\in T_k,\ \ k=\overline{0,m}.$$

It is easy to see that in this case $\mathbf{d}_r^{(r+1)}[u_k(t)]\equiv 0$ on T_k, as a result, in (4.7) the last term becomes zero.

Let's use the representation

$$\mathbf{d}_r[u_k(t)]=\Lambda_r(t)(b_{k,0},\ b_{k,1},\dots,b_{k,r}),\ \ t\in T_k,\ \ k=\overline{0,m},$$

where

$$\Lambda_r(t) = \begin{pmatrix} (0!/0!)t^0 E_l & (1!/1!)t^1 E_l & \dots & (r!/r!)t^r E_l \\ O & (1!/0!)t^0 E_l & \dots & (r!/(r-1)!)t^{r-1} E_l \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & (r!/0!)t^0 E_l \end{pmatrix}.$$

Therefore we have

$$\mathbf{d}_r^{(i)}[u_k(t)] = \Lambda_r^{(i)}(t) b_k, \quad t \in T_k, \quad k = \overline{0, m},$$

where $b_k = (b_{k,0},\ b_{k,1},\ \dots,\ b_{k,\sigma-1}), \quad k = \overline{0, m}$.

Then the equality (4.5) takes the form

$$g = \begin{pmatrix} \overline{\mathcal{L}} & O \\ \Theta \mathcal{C}_1 \mathcal{L} & I\Lambda \end{pmatrix} \begin{pmatrix} \overline{V} \\ \operatorname{diag}\{b_0, \dots, b_m\} \end{pmatrix}, \tag{4.8}$$

where

$$\overline{V} = \begin{pmatrix} v_0 & v_0 & \dots & v_0 \\ 0 & v_1 & \dots & v_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & v_m \end{pmatrix}, \ \Lambda = \begin{pmatrix} \Lambda_r(t) \\ \Lambda_r'(t) \\ \vdots \\ \Lambda_r^{(r)}(t) \end{pmatrix}, \ I = \operatorname{diag}\{I_{0,r}, I_{1,r}, \dots, I_{m,r}\}.$$

The system (4.8) is obviously solvable on T_k with respect to the vectors v_k, b_k ($k = \overline{0, m}$) and hence the equation (4.5) is also solvable for any value g. □

The R-controllability condition can be formulated in terms of the controllability matrix. Let $\mathcal{H} = \operatorname{diag}\{\mathcal{H}_0,\ \mathcal{H}_1, \dots, \mathcal{H}_m\}$. Then the controllability matrix of the system (3.4), (3.6) has the form

$$\Phi = (\Phi_0 \ \ \Phi_1 \ \dots \ \Phi_{n-d-1}), \tag{4.9}$$

where $\Phi_0 = \left(\mathcal{C}_1 \mathcal{L} \ \ \mathcal{H} \right)$, $\Phi_i = \operatorname{diag}\{J_1^i,\ \dots\ , J_1^i\}\Phi_0, \ \ i = \overline{1, n-d-1}$.

Theorem 2. *Let all the assumptions of the lemma 3 hold. The system (3.2), (3.3) is R-controllable on T if*

$$\operatorname{rank} \Phi = (n-d)(m+1). \tag{4.10}$$

Proof. Suppose the opposite. Let the assumption (4.10) holds but the system (3.4)–(3.7) and, respectively, DAEs (3.2), (3.3) are not R-controllable on T. Then, according to the theorem 1 there exists a nonzero vector $p_* \in \mathcal{R}^{(n-d)(m+1)}$ such that $p_*^\top \left(\Theta \mathcal{C}_1 \mathcal{L} \ \ \Theta_1 \right) = 0$, whence it follows that $p_*^\top \Theta \mathcal{C}_1 \mathcal{L} = 0, \ p_*^\top \Theta_1 = 0$.

It is easy to see that from the singularity of Θ_1 follows singularity of the matrix $\operatorname{diag}\{X^{-1}(t)\,\mathcal{H}_0\,\mathbf{d}_r[u_0(t)],\dots,X^{-1}(t)\,\mathcal{H}_m\,\mathbf{d}_r[u_m(t)]\}$. In turn, the fundamental matrix $X(t)$ cannot be singular in its structure, which means the fact that expressions

$$p_*^\top \mathcal{C}_1\mathcal{L} = 0,\ p_*^\top \operatorname{diag}\{\mathcal{H}_0\,\mathbf{d}_r[u_0(t)],\dots,\mathcal{H}_m\,\mathbf{d}_r[u_m(t)]\} = 0$$

hold for arbitrary control functions $u_k(t)$ $(k=\overline{0,m})$.

Thus, the matrix $\begin{pmatrix} \mathcal{C}_1\mathcal{L} & \mathcal{H} \end{pmatrix} = 0$, and from the construction of the matrix Φ follows that $\operatorname{rank}\Phi < (n-d)(m+1)$. We got a contradiction. □

Example 1. Consider the hybrid system

$$Ax'(t) = Bx(t) + C_k y_k + U_k u_k(t),\quad t\in T_k = [t_k, t_{k+1}),\ k=0,1,2; \tag{4.11}$$

$$y_1 = D_0 x(t_0) + G_{0,0} y_0 + V_0 v_0, \tag{4.12}$$

$$y_2 = D_1 x(t_1) + G_{1,0} y_0 + G_{1,1} y_1 + V_1 v_1, \tag{4.13}$$

$$y_3 = D_2 x(t_2) + G_{2,0} y_0 + G_{2,1} y_1 + G_{2,2} y_2 + V_2 v_2, \tag{4.14}$$

where

$$A = \begin{pmatrix} 2 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}, B = \begin{pmatrix} -1 & 2 & 0\\ 1 & -1 & 0\\ 1 & 1 & 1\end{pmatrix}, U_0 = \begin{pmatrix} 1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}, U_1 = \begin{pmatrix} 1 & 0\\ 1 & 0\\ 0 & -1\end{pmatrix},$$
$$U_2 = \begin{pmatrix} 0 & 0\\ 1 & 0\\ 0 & 2\end{pmatrix}, C_0 = \begin{pmatrix} 0\\ -1\\ 1\end{pmatrix},\quad C_1 = \begin{pmatrix} 2\\ 0\\ 0\end{pmatrix},\quad C_2 = \begin{pmatrix} 1\\ -1\\ 0\end{pmatrix},$$
$$V_0 = \begin{pmatrix} 2 & -1\end{pmatrix},\quad V_1 = \begin{pmatrix} 0 & 1\end{pmatrix},\quad V_2 = \begin{pmatrix} 1 & 1\end{pmatrix},$$

$$\begin{gathered} G_{0,0}=1,\quad G_{1,0}=0,\quad G_{1,1}=2,\quad G_{2,0}=2,\quad G_{2,1}=-1,\quad G_{2,2}=1,\\ D_0 = \begin{pmatrix} 1 & 0 & 1\end{pmatrix},\quad D_1 = \begin{pmatrix} -1 & 2 & 0\end{pmatrix},\quad D_2 = \begin{pmatrix} 0 & 1 & 1\end{pmatrix}. \end{gathered} \tag{4.15}$$

$u_k(t) \in U$ $(k=0,1,2)$, where U is the set of all piecewise continuous functions from $\mathcal{R}^2$, v_k $(k=0,1,2)$ are scalar control functions.

Let us investigate the system (4.11)–(4.15) for the R-controllability on the interval $T=[t_0,t_3]$. To do this, check the fulfillment of the assumptions of the theorem 2. Let us construct the matrix from the lemma 1:

$$\mathrm{D}_{1,x} = \left(\begin{array}{cc|cccccc|c} -1 & 2 & 0 & 2 & 0 & 1 & 0 & 0 & 0\\ 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & -1 & 2 & 0 & 2 & 0 & 1\\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right).$$

The frame in the matrix $\mathrm{D}_{1,x}$ marks the columns that are included in the resolving minor. It is easy to verify that the condition $\operatorname{rank}\mathrm{D}_{2,y} =$

$\operatorname{rank} \mathrm{D}_{1,y} + 3$ holds. Thus, according to the lemma 1 there exists the operator

$$\mathcal{R} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{d}{dt}. \quad (4.16)$$

Let's construct the controllability matrix Φ from (4.9) using the operator (4.16)

$$\Phi = \begin{pmatrix} O & O & O & M_1 & O & O & O & O & O & O & O & O & M_3 & O & O & O & O & O \\ O & O & O & O & O & M_2 & O & O & O & O & O & O & O & O & M_4 & O & O & O \\ M_5 & M_6 & O & O & O & O & O & M_7 & O & M_5 & M_6 & O & O & O & O & O & M_7 & O \end{pmatrix},$$

where

$$M_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ M_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ M_4 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

$$M_5 = \begin{pmatrix} -4 & 2 \\ 0 & 0 \end{pmatrix}, \ M_6 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \ M_7 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

It is easy to verify that the condition (4.10) from the theorem 2 is satisfied. Thus, the system (4.11)–(4.15) is R-controllable for any consistent initial-boundary conditions on the interval $[t_0, t_3]$.

5. Conclusion

In this paper a class of hybrid stationary systems under the assumptions that ensure the existence of an equivalent structural form is considered. The advantage of this approach is due to the fact that this structural form is equivalent to the nominal system in the sense of the coincidence of the sets of solutions. In addition, the problem of consistent initial data is automatically solved. Moreover, in the construction of an equivalent form the variable substitution is not used. In the future, this methodology can be extended to hybrid systems with variable coefficients. The presented example is only illustrative. Generally, instead of the set U we can consider a smaller class of continuous functions with a finite-dimensional parameter depending on the physical or economic considerations of a particular problem.

References

1. Barton P.I., Lee C.K. Modeling, simulation, sensitivity analysis and optimization of hybrid systems. *ACM Transactions Modeling Comput. Simulation*, 2002, vol. 12, no. 4, pp. 256-289. https://doi.org/10.1109/CACSD.2000.900197

2. Bemborad A., Ferrari-Trecate G., Morari M. Observability and controllability of piecewise affine and hybrid systems. *I*EEE Trans. Automat. Control., 2000, vol. 45, no. 10, pp. 1864-1876. https://doi.org/10.1109/TAC.2000.880987
3. Boyarintsev Ju.E. *Metody resheniya vyrozhdennykh sistem obyknovennykh differentsial'nykh uravneniy* [Methods of solution for degenerate systems of ordinary differential equations]. Novosibirsk, Nauka Publ., 1988, 158 p. (in Russian)
4. Boyarintsev Ju.E. *Regulyarnyye i singulyarnyye sistemy lineynykh obyknovennykh differentsial'nykh uravneniy* [Regular and Singular Systems of Linear Ordinary Differential Equations]. Novosibirsk, Nauka Siberian Branch Publ., 1980, 222 p. (in Russian)
5. Dai L. *S*ingular control system. Lecture notes in control and information sciences, vol. 118. Springer-Verlag, Berlin-Heidelberg, 1989, 332 p. https://doi.org/10.1007/BFb0002475
6. Gantmacher F.R. *Teoriya matrits* [The theory of matrices]. Moscow, Nauka Publ., 1988, 548 p. (in Russian)
7. Kunkel P., Mehrmann W.L. *Differential-algebraic equations: analysis and numerical solutions*. European Mathematical Society, Zurich, Switzerland, 2006, 377 p.
8. Leontyev V.V. *Mezhotraslevaya ekonomika* [Interindustry Economics]. Moscow, Ekonomika Publ., 1990, 415 p. (in Russian)
9. Marchenko V.M., Borkovskaya I.M., Pyzhkova O.N. The stability of hybrid dynamic 2-D-systems [Ustoychivost' gibridnykh dinamicheskikh sistem s mnogomernym vremenem]. *P*roceedings of BSTU, 2016, no. 6, pp. 5-9. (in Russian)
10. Mehrmann V., Stykel T. Descriptor systems: a general mathematical framework for modelling, simulation and control. *A*utomatisierungstechnik, 2006, vol. 54, no. 8, pp. 405-415. https://doi.org/10.1524/auto.2006.54.8.405
11. Petrenko P.S. Differential controllability of linear systems of differential-algebraic equations. *J*ournal of Siberian Federal University. Mathematics & Physics, 2017. vol. 10, no. 3, pp. 320-329. https://doi.org/10.17516/1997-1397-2017-10-3-320-329
12. Petrenko P.S. To the question on controllability of a singular hybrid system [K voprosu ob upravlyayemosti odnoy vyrozhdennoy gibridnoy sistemoy]. *Materialy Mezhdunarodnoy konferentsii "Ustoychivost', upravleniye, differentsial'nyye igry" (SCDG2019)*, Ekaterinburg, Russia, 2019, pp. 251-255. (in Russian)
13. Petrenko P.S. To the question on solvability of a singular hybrid system [K voprosu o razreshimosti vyrozhdennoy gibridnoy sistemy]. *Materialy mezhd. simpoziuma "Dinam. sistemy, optim. upravleniye i matem. modelirovaniye"*, Irkutsk, Russia, 2019, pp. 163-166. (in Russian)
14. Rondepierre A. Piecewise affine systems controllability and hybrid optimal control. *P*roc. Int. Conf. Inform. Control, Automat. Robot (ICINCO), Barselone, Spain, 2005, pp. 294-302.
15. Shcheglova A.A. Duality of the notions of controllability and observability for degenerate linear hybrid systems. *A*utom. Remote Control, 2006, vol. 67, no. 9, pp. 1445–1465. https://doi.org/10.1134/S0005117906090086
16. Shcheglova A.A. Observability of the hybrid linear systems with constant coefficients. *A*utom. Remote Control, 2004, vol. 65, no. 11, pp. 1767-1781. https://doi.org/10.1023/B:AURC.0000047891.80395.ef
17. Shcheglova A.A. The solvability of the initial problem for a degenerate linear hybrid system with variable coefficients. *R*ussian Math. (Iz. VUZ), 2010, vol. 54, no. 9, pp. 49-61. https://doi.org/10.3103/S1066369X10090057
18. Van der Schaft A., Schumacher H. *An introduction to hybrid dynamical systems*. Springer, London, 2000, 174 p.

19. Vidal R., Chiuso A., So atto St., Sastry Sh. Observability of linear hybrid systems. *H*ybrid Systems: Computation and Control, Lecture Notes in Computer Science, Springer, Berlin-Heidelberg, 2003, vol. 2623, pp. 526-539. https://doi.org/10.1007/3-540-36580-X_38

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Управляемость одной вырожденной гибридной системы

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Аннотация. Рассматривается линейная гибридная система с постоянными коэффициентами, неразрешенная относительно производной непрерывной составляющей искомой функции. В литературе подобные системы часто называют дискретно-непрерывными. Такие системы возникают при математическом моделировании ряда технических процессов. С помощью гибридных систем, к примеру, можно описать системы цифрового управления и коммутации, системы нагревания и охлаждения, функционирование коробки передач автомобиля, динамические системы с соударением и кулоновским трением, а также многие другие. Качественной теории такого рода систем посвящено множество работ, однако в большинстве из них рассматриваются невырожденные случаи в различных постановках. Анализ работы существенным образом опирается на методику исследования вырожденных систем обыкновенных дифференциальных уравнений и проводится в предположении существования эквивалентной структурной формы. Данная структурная форма эквивалентна исходной системе в смысле решений, а преобразующий к ней оператор обладает левым обратным. Построение структурной формы носит конструктивный характер и не использует замену переменных, при этом автоматически решается проблема согласования начальных данных. В работе получены необходимые и достаточные условия R-управляемости (управляемости в пределах множества достижимости) исследуемой системы.

Ключевые слова: гибридные системы, дифференциально-алгебраические уравнения, разрешимость, управляемость.

Список литературы

1. Barton P. I., Lee C. K. Modeling, simulation, sensitivity analysis and optimization of hybrid systems // ACM Transactions Modeling Comput. Simulation. 2002. Vol. 12, N 4. P. 256–289. https://doi.org/10.1109/CACSD.2000.900197

2. Bemborad A., Ferrari-Trecate G., Morari M. Observability and controllability of piecewise affine and hybrid systems // IEEE Trans. Automat. Control. 2000. Vol. 45, N 10. P. 1864–1876. https://doi.org/10.1109/TAC.2000.880987
3. Бояринцев Ю. Е. Методы решения вырожденных систем обыкновенных дифференциальных уравнений. Новосибирск : Наука, 1988. 158 с.
4. Бояринцев Ю. Е. Регулярные и сингулярные системы линейных обыкновенных дифференциальных уравнений. Новосибирск : Наука, 1980. 222 с.
5. Dai L. Singular control system. Springer-Verlag Berlin Heidelberg, 1989, 332 p. (Lecture notes in control and information sciences ; vol. 118). https://doi.org/10.1007/BFb0002475
6. Гантмахер Ф. Р. Теория матриц. М. : Наука, 1988. 548 с.
7. Kunkel P., Mehrmann W. L. Differential-algebraic equations: analysis and numerical solutions. Zurich, Switzerland : European Mathematical Society, 2006. 377 p.
8. Леонтьев В. В. Межотраслевая экономика. М. : Экономика, 1997. 479 с.
9. Марченко В. М., Борковская И. М., Пыжкова О. Н. Устойчивость гибридных динамических систем с многомерным временем // Труды Белорусского государственного технологического университета. 2016. № 6. С. 5–9.
10. Mehrmann V., Stykel T. Descriptor systems: a general mathematical framework for modelling, simulation and control // Automatisierungstechnik. 2006. Vol. 54, N 8. P. 405–415. https://doi.org/10.1524/auto.2006.54.8.405
11. Petrenko P. S. Differential controllability of linear systems of differential-algebraic equations // Journal of Siberian Federal University. Mathematics & Physics. 2017. N 10(3). P. 320–329. https://doi.org/10.17516/1997-1397-2017-10-3-320-329
12. Петренко П. С. К вопросу об управляемости одной вырожденной гибридной системой // Устойчивость, управление, дифференциальные игры (SCDG2019) : материалы Междунар. конф. Екатеринбург : Изд-во ИММ УрО РАН им. Н. Н. Красовского, 2019. С. 251–255.
13. Петренко П. С. К вопросу о разрешимости вырожденной гибридной системы // Динамические системы, оптимальное управление и математическое моделирование : материалы Междунар. симп. Иркутск : Изд-во ИГУ, 2019. С. 163–166.
14. Rondepierre A. Piecewise affine systems controllability and hybrid optimal control // Proc. Int. Conf. Inform. Control, Automat. Robot. Barselone. September, 2005. P. 294–302.
15. Щеглова А. А. Двойственность понятий управляемости и наблюдаемости для вырожденных линейных гибридных систем // Автоматика и телемеханика. 2006. № 9. С. 99–119. https://doi.org/10.1134/S0005117906090086
16. Щеглова А. А. Наблюдаемость вырожденных линейных гибридных систем с постоянными коэффициентами // Автоматика и телемеханика. 2004. № 11. С. 86–101. https://doi.org/10.1023/B:AURC.0000047891.80395.ef
17. Щеглова А. А. Существование решения начальной задачи для вырожденной линейной гибридной системы с переменными коэффициентами // Известия вузов. Математика. 2010. № 9. С. 57–70. DOI: 10.3103/S1066369X10090057
18. Van der Schaft A., Schumacher H. An introduction to hybrid dynamical systems. London : Springer, 2000. 174 p.
19. Observability of linear hybrid systems / R. Vidal, A. Chiuso, atto St. So, Sh. Sastry // Hybrid Systems: Computation and Control, Lecture Notes in Computer Science. Vol. 2623. Springer, Berlin, Heidelberg, 2003. P. 526–539. https://doi.org/10.1007/3-540-36580-X_38

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