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On Invariant Sets for the Equations of Motion of a Rigid Body in the Hess-Appelrot Case

V. D. Irtegov[1], T. N. Titorenko[1]

[1] *Matrosov Institute for System Dynamics and Control Theory of SB of RAS, Irkutsk, Russian Federation*

Abstract. We consider the problem of motion of a rigid body in the Hess-Appelrot case when the equations of motion have three first integrals as well as the invariant manifold of Hess. On the basis of the Routh-Lyapunov method and its generalizations, the qualitative analysis of the above equations written on the manifold is done. Stationary invariant sets for the equations are found and their Lyapunov stability is investigated. By stationary sets, we mean sets which consist of the trajectories of the equations of motion and possess the extremal property: the necessary extremum conditions for the elements of the algebra of problem's first integrals are satisfied on them. In this paper, an extension of the technique for finding such sets is proposed: obtaining new sets from previously known ones and by means of "the inverse Lagrange method". Applying these techniques, we have found a family of invariant manifolds for the differential equations on the invariant manifold of Hess. From this family, several invariant manifolds of greater dimension than those of the family have been obtained, and an analysis of differential equations on one of them was done. Equilibrium positions and families of permanent rotations of the body have been found. For a number of the solutions, sufficient stability conditions have been derived, including with respect to part of variables.

Keywords: Hess's case, invariant sets, stability.

1. Introduction

By now there are many publications devoted to studying the classical problems of rigid body dynamics and their generalizations (see, e.g., [3] and references therein). Nevertheless, such problems draw the attention of researchers so far. These often are a source of new ideas, extend the area of application for the methods developed [10]. On this way it is possible to obtain interesting results in the classical problems themselves.

By the analysis of the Hess-Appelrot problem [11], the paper discusses additional possibilities of the Routh-Lyapunov method [8] for finding and the study of stationary invariant sets of differential equations possessing smooth first integrals.

By stationary sets, we mean sets of any finite dimension on which the necessary extremum conditions for the elements of the algebra of first integrals in the problem under study are satisfied. Zero dimension sets having this property are traditionally called stationary solutions, while nonzero dimension sets are named stationary invariant manifolds (IMs).

The problem of finding stationary sets of differential equations by the Routh-Lyapunov method is reduced to solving the stationary equations for the family of first integrals of the problem. In the case of systems with polynomial first integrals, it consists in seeking solutions of a system of algebraic equations. The non-degenerate system allows one to obtain stationary solutions, while the degenerate one gives IMs and their families [5]. The integrals and their families taking a stationary value on the found solutions are used to derive the sufficient conditions of stability of the solutions.

In the present work, within the framework of the Routh-Lyapunov method, we propose an extension of technique for seeking stationary sets: obtaining new IMs by elimination of a family parameter from a previously known family of IMs; using "the inverse Lagrange method", when first we find (constant) solutions from the equations of motion and then obtain a combination of the integrals which assumes a stationary value on the solutions. These techniques are applied to seeking stationary sets for the equations of motion of a rigid body in the case under consideration that gives a possibility to obtain new results. The qualitative analysis for the equations of motion on found IMs is done. Other approaches to the analysis of the problem in question were applied, e.g., in [1; 2; 4; 7].

2. Formulation of the problem

Consider the problem of motion of a rigid body about a fixed point in a constant gravity field. The equations of motion of the body in the

Euler-Poisson form have the form:

$$A\dot{p} = (B - C)qr + z_0\gamma_2 - y_0\gamma_3,\ \dot{\gamma_1} = r\gamma_2 - q\gamma_3.$$

The rest of them are derived from the above ones by circular permutation $(ABC, p\,q\,r,\ \gamma_1\gamma_2\gamma_3,\ x_0y_0z_0)$.

Let the following constraints be imposed on the parameters characterizing the geometry of mass of the body

$$y_0 = 0,\ \ x_0^2A(B - C) - z_0^2C(A - B) = 0 \tag{2.1}$$

or

$$B = AC(x_0^2 + z_0^2)(Ax_0^2 + Cz_0^2)^{-1}. \tag{2.2}$$

Having eliminated B and y_0 from the Euler equations with (2.1), we have the differential equations describing the motion of the body in the Hess-Appelrot case:

$$\begin{aligned}
&A(Ax_0^2 + Cz_0^2)\dot{p} = Cz_0^2(A - C)qr + z_0(Ax_0^2 + Cz_0^2)\gamma_2,\\
&AC(x_0^2 + z_0^2)\dot{q} = (Ax_0^2 + Cz_0^2)[(C - A)pr - z_0\gamma_1 + x_0\gamma_3],\\
&C(Ax_0^2 + Cz_0^2)\dot{r} = Ax_0^2(A - C)qp - (Ax_0^2 + Cz_0^2)x_0\gamma_2,\\
&\dot{\gamma_1} = r\gamma_2 - q\gamma_3,\ \dot{\gamma_2} = p\gamma_3 - r\gamma_1,\ \dot{\gamma_3} = q\gamma_1 - p\gamma_2.
\end{aligned} \tag{2.3}$$

Equations (2.3) admit the first integrals

$$\begin{aligned}
&2H = Ap^2 + \frac{AC(x_0^2 + z_0^2)}{Ax_0^2 + Cz_0^2}\,q^2 + Cr^2 + 2(x_0\gamma_1 + z_0\gamma_3) = 2h,\\
&V_1 = Ap\gamma_1 + \frac{AC(x_0^2 + z_0^2)}{Ax_0^2 + Cz_0^2}\,q\gamma_2 + Cr\gamma_3 = m,\\
&V_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1
\end{aligned}$$

and the invariant manifold (the particular integral of Hess):

$$V_3 = Ax_0p + Cz_0r = 0. \tag{2.4}$$

To write differential equations (2.3) on IM (2.4), we eliminate r from them with the help of (2.4):

$$\begin{aligned}
&A(Ax_0^2 + Cz_0^2)\dot{p} = Ax_0z_0\,(C - A)\,p\,q + z_0(Ax_0^2 + Cz_0^2)\gamma_2,\\
&AC^2z_0(x_0^2 + z_0^2)\dot{q} = -[Ax_0(C - A)p^2 + Cz_0(z_0\gamma_1 - x_0\gamma_3)](Ax_0^2 + Cz_0^2),\\
&\dot{\gamma_1} = -\frac{Ax_0}{Cz_0}\,p\gamma_2 - q\gamma_3,\ \dot{\gamma_2} = p\left(\frac{Ax_0}{Cz_0}\gamma_1 + \gamma_3\right),\ \dot{\gamma_3} = q\gamma_1 - p\gamma_2.
\end{aligned} \tag{2.5}$$

System (2.5) has the following first integrals:

$$\begin{aligned}
&2\tilde{H} = A\Big(\frac{Ax_0^2}{Cz_0^2} + 1\Big)p^2 + \frac{AC(x_0^2 + z_0^2)}{Ax_0^2 + Cz_0^2}\,q^2 + 2(x_0\gamma_1 + z_0\gamma_3) = 2h,\\
&\tilde{V_1} = Ap\left(\gamma_1 - \frac{x_0}{z_0}\gamma_3\right) + \frac{AC(x_0^2 + z_0^2)}{Ax_0^2 + Cz_0^2}\,q\gamma_2 = m,\\
&V_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.
\end{aligned} \tag{2.6}$$

Let us set the problem of the qualitative analysis of the above conservative system according to Poincaré [9]. In other words, peculiar sets of the differential equations shall be found and their vicinity (in some cases) will be investigated. These sets and their properties mostly determine the structure of the phase space and the qualitative behavior of solutions of differential equations. Within the framework of the chosen method, stationary solutions and IMs are considered as peculiar ones.

3. Finding invariant manifolds

3.1. Using degenerate stationary conditions

We compose the complete linear combination of integrals (2.6)

$$K = \tilde{H} - \lambda_1 \tilde{V}_1 - \frac{1}{2}\lambda_2 V_2 \tag{3.1}$$

and write the necessary extremum conditions for K with respect to the phase variables:

$$\begin{aligned}
&\frac{\partial K}{\partial p} = A\Big[p\left(\frac{Ax_0^2}{Cz_0^2}+1\right) - \lambda_1\Big(\gamma_1 - \frac{x_0}{z_0}\gamma_3\Big)\Big] = 0,\\
&\frac{\partial K}{\partial q} = \frac{AC(x_0^2+z_0^2)}{Ax_0^2+Cz_0^2}\,(q-\lambda_1\gamma_2) = 0,\\
&\frac{\partial K}{\partial \gamma_2} = -\lambda_1\,\frac{AC(x_0^2+z_0^2)}{Ax_0^2+Cz_0^2}\,q - \lambda_2\gamma_2 = 0,\\
&\frac{\partial K}{\partial \gamma_1} = x_0 - \lambda_1 Ap - \lambda_2\gamma_1 = 0,\ \ \frac{\partial K}{\partial \gamma_3} = z_0 + \lambda_1\frac{Ax_0}{z_0}\,p - \lambda_2\gamma_3 = 0.
\end{aligned} \tag{3.2}$$

Under the condition

$$AC(x_0^2+z_0^2)\lambda_1^2 = -(Ax_0^2+Cz_0^2)\lambda_2, \tag{3.3}$$

system (3.2) is degenerate and has the family of solutions with the parameter λ_1:

$$\begin{aligned}
&x_0 - \lambda_1 Ap + \lambda_1^2 B\,\gamma_1 = 0,\\
&z_0 + \lambda_1\frac{Ax_0}{z_0}\,p + \lambda_1^2 B\,\gamma_3 = 0,\ \ q - \lambda_1\gamma_2 = 0.
\end{aligned} \tag{3.4}$$

From now and further B is expression (2.2).

Equations (3.4) define a family of stationary IMs of codimension 3 for system (2.5) on the Hess IM.

The equations of motion on the elements of the family of IMs (3.4) can be written as:

$$A\dot{p} = \frac{q}{\lambda_1 C z_0}[\lambda_1 A(C-B)x_0\, p + C z_0^2],$$
$$B\dot{q} = \frac{Ap}{\lambda_1 B C z_0}[\lambda_1 B(A-C)x_0\, p - C(x_0^2+z_0^2)].$$

The energy integral for these equations takes the form:

$$2\tilde{H} = A\Big(\frac{Ax_0^2}{Cz_0^2}+1\Big)p^2 + Bq^2 - \frac{2(x_0^2+z_0^2)}{\lambda_1^2 B}.$$

When condition (3.3) does not hold, system (3.2) has two solutions

$$p = q = \gamma_2 = 0,\ \gamma_1 = \pm x_0(x_0^2+z_0^2)^{-1/2},\ \gamma_3 = \pm z_0(x_0^2+z_0^2)^{-1/2} \quad (3.5)$$

which correspond to equilibria of system (2.5). These solutions are stationary: integral K (3.1) assumes a stationary value on them. The 2nd equilibrium position belongs to the elements of the family of IMs (3.4) corresponding to $\lambda_1^4 = (x_0^2+z_0^2)B^{-2}$.

3.2. Elimination of a family parameter from a family of IMs

Eliminate, e.g., the parameter $\lambda_1 = q\gamma_2^{-1}$ with the help of the last equation of (3.4) from the rest of the equations. As a result, we have two equations

$$(x_0\gamma_2 - Ap\,q)\,\gamma_2 + Bq^2\gamma_1 = 0,$$
$$(z_0^2\gamma_2 + Ax_0 p\,q)\,\gamma_2 + Bz_0 q^2\gamma_3 = 0 \quad (3.6)$$

which define an IM of codimension 2 of the system under consideration. As can be seen from the method for its finding, it stratifies into the IMs of codimension 3 (IMs (3.4)).

The equations of motion on IM (3.6) (after elimination γ_1, γ_3 from (2.5) with (3.6)) are given by

$$\dot{p} = \frac{(C-A)x_0 z_0}{Ax_0^2 + Cz_0^2}\, pq + \frac{z_0}{A}\,\gamma_2,$$
$$\dot{q} = \frac{Ap}{B^2 C z_0\, q}\,[B\,(A-C)\,x_0\, pq - C(x_0^2+z_0^2)\,\gamma_2],$$
$$\dot{\gamma_2} = \frac{p\gamma_2}{BCz_0\, q^2}\,[A\,(A-C)x_0 pq - (Ax_0^2+Cz_0^2)\gamma_2]. \quad (3.7)$$

It is not difficult to verify that expression $\lambda_1 = q\gamma_2^{-1}$ is the first integral of equations (3.7). Note that IM (3.6) and the first integral $\lambda_1 = q\gamma_2^{-1}$ can be obtained as a solution of stationary equations (3.2) with respect to the phase variables γ_1, γ_3 and the parameters λ_1, λ_2 [6].

The above IMs can be "lifted" into the original phase space. To do this, it is enough to add the Hess integral to their equations.

Substitute expressions (3.5) into equations (3.6). These turn into identity. Whence we conclude, solutions (3.5) belong to IM (3.6).

Now eliminate the parameter λ_1 from equations (3.4) with the help of the first of them. We obtain two following IMs of system (2.5):

$$\begin{aligned} &\frac{1}{2B\gamma_1}[2Bq\gamma_1 - (Ap \pm \alpha)\gamma_2] = 0, \\ &\frac{1}{4Bz_0\gamma_1^2}\left[4Bz_0^2\gamma_1^2 + (Ap \pm \alpha)(2Ax_0p\gamma_1 + (Ap \pm \alpha)\, z_0\gamma_3)\right] = 0, \end{aligned} \tag{3.8}$$

where $\alpha = \sqrt{A^2p^2 - 4Bx_0\gamma_1}$. As can be verified, these IMs belong to IM (3.6).

Analogously, having eliminated the parameter λ_1 with the help of the 2nd equation of (3.4), we have two more IMs of differential equations (2.5):

$$\begin{aligned} &\frac{1}{2Bz_0q\gamma_3}[2Bz_0q\gamma_3 \pm (\beta \pm Apx_0)\gamma_2] = 0, \\ &\frac{1}{4Bz_0^2\gamma_3^2}\left[4Bx_0z_0^2\gamma_3^2 + (\beta \pm Apx_0)((\beta \pm Apx_0)\,\gamma_1 \pm 2Az_0p\gamma_3)\right] = 0. \end{aligned} \tag{3.9}$$

Here $\beta = \sqrt{x_0^2A^2p^2 - 4z_0^3B\gamma_3}$. These also belong to IM (3.6).

The relations

$$\bar{\lambda}_{1(1,2)} = (Ap \pm \alpha)(2B\gamma_1)^{-1} \text{ and } \tilde{\lambda}_{1_{(1,2)}} = -(Ax_0p \pm \beta)(2z_0B\gamma_3)^{-1} \text{ are}$$

the first integrals of differential equations on IM (3.8) and IM (3.9), respectively.

So, it is true

Proposition 1. *Elimination of a family parameter from a family of IMs allows one to obtain new IM of the equations of motion, and an expression for the parameter which appears in the process of its elimination is the first integral of differential equations on the found IM.*

3.3. The invariant manifolds of 2nd level

Let us state the problem of seeking IMs on previously found IMs. Such IMs we call the 2nd level IMs.

Consider differential equations (3.7) on IM (3.6). Apart from the integral $\lambda_1 = q\gamma_2^{-1}$, these equations admit the following first integrals

$$
\begin{aligned}
&2\bar{H} = A\Big(\frac{Ax_0^2}{Cz_0^2}+1\Big)p^2 + Bq^2 - \frac{2(x_0^2+z_0^2)}{Bq^2}\,\gamma_2^2 = 2h,\\
&\bar{V}_1 = \frac{\gamma_2}{Bq}\Big[A^2\Big(\frac{x_0^2}{z_0^2}+1\Big)p^2 + B^2q^2\Big] = m,\\
&\bar{V}_2 = \frac{\gamma_2^2}{B^2q^4}\Big[A^2\Big(\frac{x_0^2}{z_0^2}+1\Big)p^2q^2 + B^2q^4 + (x_0^2+z_0^2)\gamma_2^2\Big] = 1 \qquad (3.10)
\end{aligned}
$$

which are derived from integrals (2.6) after eliminating $\gamma_1,\ \gamma_3$ from them with (3.6).

Integrals (3.10) are dependent. These can be represented, e.g., as follows:

$$
\begin{aligned}
&2B\bar{H} = \tilde{V}_{11} - 2(x_0^2+z_0^2)B^2\tilde{V}_{12}^2,\\
&\bar{V}_1 = \tilde{V}_{11}\tilde{V}_{12},\ \bar{V}_2 = \tilde{V}_{12}^2\,[(x_0^2+z_0^2)B^2\tilde{V}_{12}^2 + \tilde{V}_{13}],
\end{aligned}
$$

where $\tilde{V}_{11} = [(x_0^2+z_0^2)A^2p^2 + z_0^2B^2q^2]\,z_0^{-2} = m_1$, $\tilde{V}_{12} = \gamma_2(Bq)^{-1} = m_2$,
$\tilde{V}_{13} = (x_0^2+z_0^2)A^2p^2 + z_0^2B^2q^2 = m_3$.

To seek IMs of equations (3.7), we compose the linear combination of the integrals $\bar{H}$ and $\tilde{V}_{11}$

$$
\begin{aligned}
\tilde{K} = \bar{H} - \frac{\mu}{2}\tilde{V}_{11} &= \frac{1}{2}A\Big(\frac{Ax_0^2}{Cz_0^2}+1\Big)p^2 + \frac{1}{2}Bq^2 - \frac{(x_0^2+z_0^2)}{Bq^2}\,\gamma_2^2\\
&-\frac{\mu}{2}[A^2\Big(\frac{x_0^2}{z_0^2}+1\Big)p^2 + B^2q^2] \quad (\mu = const)
\end{aligned}
$$

and write the necessary extremum conditions for the integral $\tilde{K}$ with respect to the phase variables:

$$
\begin{aligned}
&\frac{\partial\tilde{K}}{\partial p} = Ap\,\Big[\Big(\frac{Ax_0^2}{Cz_0^2}+1\Big) - \mu A\Big(\frac{x_0^2}{z_0^2}+1\Big)\Big] = 0,\\
&\frac{\partial\tilde{K}}{\partial q} = Bq + \frac{2(x_0^2+z_0^2)\gamma_2^2}{Bq^3} - \mu B^2 q = 0,\ \frac{\partial\tilde{K}}{\partial\gamma_2} = \frac{2}{q^2}(x_0^2+z_0^2)\gamma_2 = 0.
\end{aligned}
$$

Under the condition $\mu = B^{-1}$, the above system has the family of solutions:

$$\gamma_2 = 0,\ p, q \text{ are arbitrary.}$$

The differential equations on this IM can be written as:

$$(Ax_0^2 + Cz_0^2)\dot{p} = (C-A)x_0z_0p\,q,\ CB\dot{q} = A(A-C)x_0p^2. \qquad (3.11)$$

These possess the first integral:

$$\check{V}_3 = CBz_0^2q^2 + (Ax_0^2 + Cz_0^2)p^2 = const.$$

Obviously, the integral $\check{V}_3$ assumes a stationary value on the solution $p = q = 0$ of differential equations (3.11). The solution defines an equilibrium position, and it is stable.

Remark 1. The process of seeking new IMs can be continued by considering manifolds of the 3rd and higher level. It gives us a possibility to classify IMs according to their embedding and degree of their degeneration [5].

4. Stationary solutions

In sect. 3, stationary solutions were obtained from the non-degenerate stationary conditions for the family of first integrals in the problem under study. Let us consider another technique. We shall solve in some sense the "inverse" problem. First, we find constant solutions directly from the differential equations, and then we derive families of the integrals which take a stationary value on these solutions. In some cases, this approach allows one to obtain both IMs and families of stationary solutions.

Equate the right-hand sides of differential equations (2.5) to zero

$$\begin{aligned}
&A(C-A)x_0 z_0 p\, q + z_0(Ax_0^2 + Cz_0^2)\gamma_2 = 0,\\
&-[Ax_0(C-A)p^2 + Cz_0(z_0\gamma_1 - x_0\gamma_3)](Ax_0^2 + Cz_0^2) = 0,\\
&-\frac{Apx_0}{Cz_0}\gamma_2 - q\gamma_3 = 0,\ p\left(\frac{Ax_0}{Cz_0}\gamma_1 + \gamma_3\right) = 0,\ q\gamma_1 - p\gamma_2 = 0. \qquad (4.1)
\end{aligned}$$

and construct a lexicographical Gröbner basis with respect to some part of the phase variables, e.g., p, γ_1, γ_3, for the polynomials of system (4.1). As a result, the initial system is transformed to the form:

$$\begin{aligned}
&C(C-A)\, z_0 q^2 \gamma_3 - (Ax_0^2 + Cz_0^2)\, \gamma_2^2 = 0,\\
&A(C-A)\, x_0 q^2 \gamma_1 + (Ax_0^2 + Cz_0^2)\, \gamma_2^2 = 0,\\
&A(C-A)\, x_0 pq + (Ax_0^2 + Cz_0^2)\, \gamma_2 = 0. \qquad (4.2)
\end{aligned}$$

It is easy to verify by IM definition that equations (4.2) define an IM of codimension 3 of differential equations (2.5).

Next, resolve equations (4.2) with respect to p, γ_1, γ_3 and substitute the obtained expressions into (3.6). These turn into identity. Thus, IM (4.2) is a submanifold of IM (3.6).

Differential equations on IM (4.2) are given by

$$\dot{q} = 0,\ \dot{\gamma}_2 = 0$$

and have the following family of solutions:

$$q = q^0 = const,\ \gamma_2 = \gamma_2^0 = const. \qquad (4.3)$$

Equations (4.2) together with (4.3) and relation $V_2 = \sum_{i=1}^{3} \gamma_i^2 = 1$ (2.6) define 4 families of solutions of differential equations (2.5):

$$p = \pm\sqrt{\frac{C(Ax_0^2 + Cz_0^2)\, z_0}{A(A-C)\, x_0}}\,\kappa^{-1},\ q = \pm\sqrt{\frac{Ax_0^2 + Cz_0^2}{AC(A-C)\, x_0 z_0}}\,\gamma_2^0\kappa,$$
$$\gamma_1 = Cz_0\kappa^{-2},\ \gamma_2 = \gamma_2^0,\ \gamma_3 = -Ax_0\kappa^{-2}; \tag{4.4}$$
$$p = \pm\sqrt{\frac{C(Ax_0^2 + Cz_0^2)\, z_0}{A(C-A)\, x_0}}\,\kappa^{-1},\ q = \mp\sqrt{\frac{Ax_0^2 + Cz_0^2}{AC(C-A)\, x_0 z_0}}\,\gamma_2^0\kappa,$$
$$\gamma_1 = -Cz_0\kappa^{-2},\ \gamma_2 = \gamma_2^0,\ \gamma_3 = Ax_0\kappa^{-2}, \tag{4.5}$$

where $\kappa = ((A^2x_0^2 + C^2z_0^2)/(1 - {\gamma_2^0}^2))^{1/4}$, γ_2^0 is the parameter of families.

The conditions for solutions (4.4) to be real are:
$|\gamma_2^0| < 1$ and ($x_0 > 0, z_0 < 0$ ($x_0 < 0, z_0 > 0$) when $0 < A < C$) and ($x_0 > 0$, $z_0 > 0$ ($x_0 < 0, z_0 < 0$) when $A > C$). Correspondingly, the conditions for solutions (4.5) have the form: $|\gamma_2^0| < 1$ and ($x_0 > 0, z_0 > 0$ ($x_0 < 0, z_0 < 0$) when $0 < A < C$) and ($x_0 > 0, z_0 < 0$ ($x_0 < 0, z_0 > 0$) when $A > C$).

From a mechanical viewpoint, the elements of the above families of solutions correspond to permanent rotations of the body on the IM of Hess.

Now we find families of the integrals taking a stationary value on the above solutions. First, consider the family of solutions (4.4).

From equations (3.2) we find constraints on λ_1, λ_2 under which solutions (4.4) satisfy these equations:

$$\lambda_1 = \pm\sqrt{\frac{Ax_0^2 + Cz_0^2}{AC(A-C)\, x_0 z_0}}\,\kappa,\ \lambda_2 = -\frac{(x_0^2 + z_0^2)\,\kappa^2}{(A-C)\, x_0 z_0}.$$

The latter expressions are substituted into (3.1) give

$$K_{1,2} = \tilde{H} \mp \sqrt{\frac{Ax_0^2 + Cz_0^2}{AC(A-C)\, x_0 z_0}}\,\kappa\tilde{V}_1 + \frac{(x_0^2 + z_0^2)\,\kappa^2}{2(A-C)\, x_0 z_0}\, V_2. \tag{4.6}$$

As can be verified by simple computation, the integrals K_1 (K_2) assume a stationary value on the elements of the families of solutions (4.4). Similarly, we find families of the integrals that take a stationary value on solutions (4.5):

$$K_{3,4} = \tilde{H} \mp \sqrt{\frac{Ax_0^2 + Cz_0^2}{AC(A-C)\, x_0 z_0}}\,\kappa\tilde{V}_1 - \frac{(x_0^2 + z_0^2)\,\kappa^2}{2(C-A)\, x_0 z_0}\, V_2.$$

Thus, one can formulate

Proposition 2. *The "inverse Lagrange method", when we first find (constant) solutions from the equations of motion, and then, from the necessary* *extremum conditions for a family of problem's first integrals, the values of the family parameters are derived under which the given family of integrals takes a stationary value on the solutions, allows one to obtain both families of stationary solutions, and the IMs which these solutions belong to.*

5. Stability

Now we investigate the stability of the above found solutions. For this purpose, the integrals taking a stationary value on these solutions are used.

5.1. On stability of invariant manifolds

Let us analyze stability for the elements of the family of IMs (3.4).
Introduce the deviations from the IMs

$$y_1 = x_0 - \lambda_1 Ap + B\lambda_1^2\gamma_1,\ y_2 = z_0 + \lambda_1 \frac{x_0}{z_0} Ap + B\lambda_1^2\gamma_3,\ y_3 = q - \lambda_1\gamma_2$$

and eliminate $\gamma_1,\ \gamma_2,\ \gamma_3$ with the help of the above expressions from integral K (3.1). As a result, we have the variation of the integral in the vicinity of the elements of the family under consideration:

$$2\Delta\tilde{K} = \frac{1}{B\lambda_1^2}(y_1^2 + y_2^2) + By_3^2.$$

Since the quadratic form $\Delta\tilde{K}$ is sign definite at $\lambda_1 \neq 0$, the elements of the family of IMs (3.4) are stable under the above condition. It should be paid attention to the fact that the stability condition imposes a constraint on the parameter of the IMs family, isolating thereby a subfamily whose elements are stable.

5.2. On stability of equilibria

Consider the question of stability for one of equilibrium positions (3.5):

$$p = q = \gamma_2 = 0,\ \gamma_1 = -x_0(x_0^2 + z_0^2)^{-1/2},\ \gamma_3 = -z_0(x_0^2 + z_0^2)^{-1/2}.$$

Introduce the deviations in the perturbed motion from the given solution:

$$p = \xi_1,\ q = \xi_2,\ \gamma_2 = \eta_2,\ \gamma_1 = \eta_1 + x_0(x_0^2+z_0^2)^{-1/2},\ \gamma_3 = \eta_3 + z_0(x_0^2+z_0^2)^{-1/2}.$$

The 2nd variation of the family of integrals K (3.1) on the manifold

$$\delta\tilde{V}_1 = x_0\eta_1 + z_0\eta_3 = 0$$

is written as:

$$2\delta^2\tilde{K} = A\Big(\frac{Ax_0^2}{Cz_0^2}+1\Big)\,\xi_1^2 + B\xi_2^2 - 2\lambda_1\Big[A\Big(\frac{x_0^2}{z_0^2}+1\Big)\xi_1\eta_1 + B\xi_2\eta_2\Big]$$
$$+(x_0^2+z_0^2)^{1/2}\Big[\Big(\frac{x_0^2}{z_0^2}+1\Big)\,\eta_1^2+\eta_2^2\Big].$$

The elements of the above family of quadratic forms are sign definite under the conditions:

$$B>0,\ A\Big(\frac{Ax_0^2}{Cz_0^2}+1\Big)>0,\ \sqrt{x_0^2+z_0^2}\,(Ax_0^2+Cz_0^2)-AC(x_0^2+z_0^2)\,\lambda_1^2>0.$$

These are reduced only to the constraints on the parameter of the family of integrals:

$$\lambda_1^2<\sqrt{x_0^2+z_0^2}\,B^{-1}.$$

Hence, the equilibrium position under investigation is stable.

It was proved that the 2nd equilibrium position is unstable.

5.3. On stability of permanent rotations

Investigate the stability of permanent rotations (4.4).

The variation of the integral K_1 (K_2) in the deviations

$$y_1=\gamma_1-Cz_0\kappa^{-2},\ y_2=\gamma_2-\gamma_2^0,\ y_3=\gamma_3+Ax_0\kappa^{-2},$$
$$y_4=p\mp\sqrt{\frac{C(Ax_0^2+Cz_0^2)\,z_0}{A(A-C)\,x_0}}\,\kappa^{-1},\ y_5=q\mp\sqrt{\frac{Ax_0^2+Cz_0^2}{AC(A-C)\,x_0z_0}}\,\gamma_2^0\kappa$$

from the unperturbed motion is written as:

$$2\Delta K_{1,2}=(x_0^2+z_0^2)\Big(\frac{\kappa y_2}{\sqrt{(A-C)x_0z_0}}\mp\sqrt{\frac{AC}{Ax_0^2+Cz_0^2}}\,y_5\Big)^2$$
$$+\Big(\sqrt{\frac{x_0^2+z_0^2}{(A-C)\,x_0z_0}}\,\kappa y_1\mp AB^{-1/2}\,y_4\Big)^2$$
$$+\Big(\sqrt{\frac{x_0^2+z_0^2}{(A-C)\,x_0z_0}}\,\kappa y_3\pm A\sqrt{\frac{x_0}{z_0B}}\,y_4\Big)^2.$$

In the variables

$$z_1=\frac{\kappa y_2}{\sqrt{(A-C)x_0z_0}}\mp\sqrt{\frac{AC}{Ax_0^2+Cz_0^2}}\,y_5,\ \ z_2=\sqrt{\frac{x_0^2+z_0^2}{(A-C)\,x_0z_0}}\,\kappa y_1$$
$$\mp AB^{-1/2}\,y_4,\ \ z_3=\sqrt{\frac{x_0^2+z_0^2}{(A-C)\,x_0z_0}}\,\kappa y_3\pm A\sqrt{\frac{x_0}{z_0B}}\,y_4,$$

the $\Delta K_{1,2}$ takes the form: $2\Delta K_{1,2} = (x_0^2 + z_0^2)z_1^2 + z_2^2 + z_3^2$.

Since the latter quadratic form is sign definite with respect to its variables, the families of solutions under study are stable with respect to the variables $\varrho_1\gamma_2 \pm \varrho_2 q$, $\varrho_3\gamma_1 \pm \varrho_4 p$, $\varrho_3\gamma_3 \pm \varrho_5 p$,
where $\varrho_1 = \kappa\,[(A-C)\,x_0 z_0]^{-1/2}$, $\varrho_2 = (AC)^{1/2}(Ax_0^2 + Cz_0^2)^{-1/2}$, $\varrho_3 = (x_0^2 + z_0^2)^{1/2}[(A-C)\,x_0 z_0]^{-1/2}$, $\varrho_4 = AB^{-1/2}$, $\varrho_5 = x_0^{1/2}(z_0 B)^{-1/2}$.

For solutions (4.5), we have the analogous result.

Remark 2. Using the families of integrals assuming a stationary value on the found solutions, we have obtained sufficient conditions for the stability of the solutions. As one can see, in some cases, these conditions are reduced to constraints on the parameters λ_i of the families of integrals, isolating thereby from these families some subfamilies whose elements give a possibility to prove the stability of the solutions.

6. Conclusion

We have presented some results of the qualitative analysis for the equations of motion of a rigid body in the Hess-Appelrot case. The analysis was rested on some generalizations of the Routh-Lyapunov method. The family of IMs has been found on the Hess manifold as a solution of degenerate stationary conditions for the family of problem's first integrals. It was shown that elimination of a family parameter from this family allows one to obtain new IMs of greater dimension than the initial ones. An analysis for the equations of motion on one of the IMs obtained in such a way was done. By the Gröbner basis method, the families of solutions whose elements correspond to fixed points of the phase space in the problem under study have been found. The linear combinations of the first integrals of the problem that assume a stationary value on the solutions have been derived. These were used to investigate the stability of the given solutions. Their stability with respect to part of variables was proved. A way for "lifting" the solutions obtained on the Hess IM into the original phase space was given.

References

1. Belyaev A.V. On the general solution of the problem of the motion of a heavy rigid body in the Hess case. *Sbornik: Mathematics*, 2015, vol. 206, no. 5, pp. 621–649. http://dx.doi.org/10.1070/SM2015v206n05ABEH004473
2. Belyaev A.V. Representation of solutions to the problem of the motion of a heavy rigid body in the Kovalevskaya case in terms of Weierstrass ζ- and $\wp$-functions and nonintegrability of the Hess case by quadratures. *Sbornik: Mathematics*, 2016, vol. 207, no. 7, pp. 889-914. http://dx.doi.org/10.1070/SM8552

3. Borisov A.V., Mamaev I.S. *Dynamics of a rigid body. Hamiltonian methods, integrability, chaos.* Moscow, Izhevsk, Institute of Computer Science Publ., 2005, 576 p. (in Russian)
4. Golubev V.V. *Lectures on integration of the equations of the motion of a heavy rigid body about a fixed point.* Moscow, GIT-TL Publ., 1953, 287 p. (in Russian)
5. Irtegov V.D. *Invariant manifolds of stationary motions and their stability.* Novosibirsk, Nauka Publ., 1985, 144 p. (in Russian)
6. Irtegov V.D., Titorenko T.N. The invariant manifolds of systems with first integrals. *J. Appl. Math. Mech.*, 2009, vol. 73, pp. 379–384. http://dx.doi.org/10.1016/j.jappmathmech.2009.08.014
7. Kovalev A.M. On motion of a rigid body in the Hess case. *Mechanics of solid body*, Ukraine Acad. Sci., Institute of Applied Mathematics and Mechanics, 1969, no. 1, pp. 12-27. (in Russian)
8. Lyapunov A.M. *On permanent helical motions of a rigid body in fluid. Collected Works.* Moscow, Leningrad, USSR Acad. Sci. Publ., 1954, vol. 1, pp. 276-319. (in Russian)
9. Poincaré H. Mémoire sur les courbes définies par une équation différentielle. *Journal de Mathématiques Pures et Appliquées*, 1881, vol. 7, pp. 375-422.
10. Kharlamov M.P. Topological analysis and Boolean functions. I. Methods and application to classical systems. *Rus. J. Nonlin. Dyn.*, 2010, vol. 6, no. 4, pp. 769-805. http://nd.ics.org.ru/nd1004005/
11. Hess W. Über die Euler'schen Bewegungsgleichungen und über eine neue partikuläre Lösung des Problems der Bewegung eines starren Körpers um einen festen Punkt. *Math. Ann.*, 1890, vol. 37, no. 2, pp. 153-181.

Valentin Irtegov, Doctor of Sciences (Physics and Mathematics), Matrosov Institute for System Dynamics and Control Theory SB RAS, 134, Lermontov st., Irkutsk, 664033, Russian Federation, tel.: (3952)453092, email: irteg@icc.ru.

Tatiana Titorenko, Candidate of Sciences (Technical Sciences), Matrosov Institute for System Dynamics and Control Theory SB RAS, 134, Lermontov st., Irkutsk, 664033, Russian Federation, tel.: (3952)453012, email: titor@icc.ru.

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Об инвариантных множествах уравнений движения твердого тела в случае Гесса – Аппельрота

В. Д. Иртегов[1], Т. Н. Титоренко[1]

[1] *Институт динамики систем и теории управления им. В. М. Матросова СО РАН, Иркутск, Российская Федерация*

Аннотация. Рассматривается задача о движении твердого тела в случае Гесса – Аппельрота, когда уравнения движения кроме трех первых интегралов имеют инвариантное многообразие Гесса. На основе метода Рауса – Ляпунова и его обобщений проводится качественный анализ дифференциальных уравнений на этом многообразии. Выделяются стационарные инвариантные множества указанных уравнений и исследуется их устойчивость по Ляпунову. Под стационарными понимаются множества, состоящие из траекторий уравнений движения и обладающие экстремальным свойством: на этих множествах удовлетворяются необходимые условия экстремума элементов алгебры первых интегралов задачи. В статье предлагается некоторое расширение методики нахождения таких множеств: получение новых множеств из ранее известных, применение “обратного метода Лагранжа“. С использованием этих способов для дифференциальных уравнений на многообразии Гесса найдено семейство инвариантных многообразий, из которого получено несколько инвариантных многообразий более высокой размерности, чем многообразия семейства, и проведен анализ дифференциальных уравнений на одном из них. Найдены положения равновесия и семейства перманентных вращений тела. Для ряда решений получены достаточные условия устойчивости.

Ключевые слова: случай Гесса, инвариантные множества, устойчивость.

Список литературы

1. Беляев А. В. Об общем решении задачи о движении тяжелого твердого тела в случае Гесса // Математический сборник. 2015. Т. 206, № 5. С. 5–34. https://doi.org/10.4213/sm8335
2. Беляев А. В. О представлении решений задачи о движении тяжелого твердого тела в случае Ковалевской в ζ- и ϱ- функциях Вейерштрасса и неинтегрируемости в квадратурах случая Гесса // Математический сборник. 2016. Т. 207, № 7. С. 3–28. https://doi.org/10.4213/sm8552
3. Борисов А. В., Мамаев И. С. Динамика твердого тела. Гамильтоновы методы, интегрируемость, хаос. Москва ; Ижевск : Ин-т компьютерных исследований, 2005. 576 с.
4. Голубев В. В. Лекции по интегрированию уравнений движения тяжелого твердого тела около неподвижной точки. М. : ГИТ-ТЛ, 1953. 287 с.
5. Иртегов В. Д. Инвариантные многообразия стационарных движений и их устойчивость. Новосибирск : Наука, 1985. 144 с.
6. Иртегов В. Д., Титоренко Т. Н. Об инвариантных многообразиях систем с первыми интегралами // Прикладная математика и механика. 2009. Т. 73, № 4. С. 531–537.
7. Ковалев А. М. О движении тела в случае Гесса // Механика твердого тела. 1969. № 1. С. 12–27.
8. Ляпунов А. М. О постоянных винтовых движениях твердого тела в Жидкости // Собрание сочинений. М. : АН СССР, 1954. Т. 1. С.276–319.
9. Пуанкаре А. О кривых, определяемых дифференциальными уравнениями. М. ; Л. : ОГИЗ, 1947. 392 с.
10. Харламов М. П. Топологический анализ и булевы функции. I. Методы и приложения к классическим системам // Нелинейная динамика. 2010. Т. 6, № 4. С. 769–805.
11. Hess W. Über die Euler'schen Bewegungsgleichungen und über eine neue partikuläre Lösung des Problems der Bewegung eines starren Körpers um einen festen Punkt // Math. Ann. 1890. Vol. 37, N 2. P. 153–181.

Валентин Дмитриевич Иртегов, доктор физико-математических наук, старший научный сотрудник, Институт динамики систем и теории управления им. В. М. Матросова СО РАН, Российская Федерация,

664033, г. Иркутск, ул. Лермонтова, 134, тел.: (3952) 453092,
e-mail: `irteg@icc.ru`.

Татьяна Николаевна Титоренко, кандидат технических наук, старший научный сотрудник, Институт динамики систем и теории управления им. В. М. Матросова СО РАН, Российская Федерация, 664033, г. Иркутск, ул. Лермонтова, 134, тел.: (3952) 453012,
e-mail: `titor@icc.ru`.

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