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An Exact Penalty Approach and Conjugate Duality for Generalized Nash Equilibrium Problems with Coupling and Shared Constraints*

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Abstract. Generalized Nash Equilibrium Problems (GNEP) have been attracted by many researchers in the field of game theory, operational research, engineering, economics as well as telecommunication in recent two decades. One of the most important classes of GNEP is a convex GNEP with jointly convex or shared constraints which has been studied extensively. It is considered to be one of the most challenging classes of problems in the field. Moreover, there is a gap in the studies on the GNEP with coupling and shared constraints. The aim of this paper is to investigate the relationship between an exact penalty approach and conjugate duality in convex optimization for the GNEP with coupling and shared constraints. In association with necessary optimality conditions, we obtained the parameterized variational inequality problems. This problem has provided an opportunity to solve many other GNEs. Some numerical results are also presented.

Keywords: generalized Nash equilibrium problems, exact penalty function, conjugate duality, coupling and shared constraints.

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1. Introduction

The GNEP is the extension of the classical Nash equilibrium problem (NEP) in which each player's strategy set depends on the rival player's strategies. Since the mid-1990s many efforts have been devoted to the investigation of GNEP (see [4; 9; 11; 13]), because it has many interesting applications in the fields of economics, operational research and engineering. For instance, Wei and Smeers [15] formulated oligopolistic electricity models as GNEPs. Pang and Fukushima [10] considered a GNEP from multi-leader-follower games.

It is well-known that in the classic NEP to each player corresponds convex programming problem, and it can be reduced to a variational inequality problem (VIP). This formulation provides a powerful theoretical and computational framework [8] for the solution of the classical NEP. On the other hand, the GNEP can be reduced to a quasi-variational inequality problem (QVI) (see [12]). Unfortunately, comparing the VIP, there are few methods available for solving a QVI efficiently.

In recent years, the penalty function methods which are based on eliminating the difficult coupling constraints in the GNEP have been attracted by many researchers. Recently, Fukushima and Pang [10] proposed a sequential penalty approach to GNEP, which is reduced the GNEP to smooth NEP's for values of penalty parameter increasing to infinity. Facchinei and Pang [6] proposed the exact penalization techniques whereby the GNEP is reduced to the solution of a single nonsmooth NEP with a finite value of the penalty parameter. More recently, Facchinei and Lampariello [7] proposed the partial penalization techniques to GNEP with coupling constraints.

On the other hand, from a practical point of view, it is important to find possible many solutions of GNEP (see [1] and [14]).

In this paper, we consider the GNEP with coupling and shared constraints, and aim to show how an exact penalty approach can be related to conjugate duality in convex optimization for GNEP. After penalizing only the coupling constraints, the problem reduces to the penalized GNEP with shared constraints. Analyzing special perturbation function and the formulating optimality conditions for corresponding dual problem, one gets the parameterized variational inequality problem which allows us to find possible many solutions for GNEP.

This paper is organized as follows. The next section deals with an exact penalty approach for GNEP with coupling and shared constraints. In Section 3,4 we obtain the parameterized variational inequality problem based on necessary optimality conditions for primal-dual problems for solving GNEP and show some numerical results.

2. Problem formulation and penalty function approach

The GNEP consists of N players, and each player's strategy set depends on rival players' strategies. Let $x_k \in \mathbb{R}^{n_k},\ n_k \in \mathbb{N}$ be a player k's strategy and $x := (x_1, x_2, ..., x_N) \in \mathbb{R}^n,\ n := \sum_{k=1}^{N} n_k$. By x_{-k} denotes all players' strategies except those of player k, i.e., $x_{-k} := (x_{k'})_{k'=1,k'\neq k}^{N} \in \mathbb{R}^{n_{-k}}$, and $n_{-k} := n - n_k$. Let $\theta_k : \mathbb{R}^n \to \mathbb{R},\ k = \overline{1,N}$, be continuously, differentiable functions such that θ_k is convex with respect to k-th variable. Assume that functions $h_k : \mathbb{R}^n \to \mathbb{R}^{m_k},\ k = \overline{1,N}$, are vector-valued convex with respect to k-th variable, and $g : \mathbb{R}^n \to \mathbb{R}^l$ is a vector-valued convex function. We consider the GNEP which consists in finding $\bar{x} = (\bar{x}_1, ..., \bar{x}_N) \in \mathbb{R}^n$ such that each player's strategy $\bar{x}_k \in \mathbb{R}^{n_k},\ k = \overline{1,N}$ is a solution to the problem

$$\begin{aligned} P_k(\bar{x}_{-k}) \quad & \inf_{x_k} \theta_k(x_k, \bar{x}_{-k}) \\ \text{s.t.} \quad & h_k(x_k, \bar{x}_{-k}) \underset{m_k}{\leq} 0, \\ & g(x_k, \bar{x}_{-k}) \underset{l}{\leq} 0, \end{aligned}$$

which is the GNEP with coupling and shared constraints. For any $x, y \in \mathbb{R}^s,\ x \underset{s}{\leq} y$ means

$$y - x \in \mathbb{R}^s_+ := \{z = (z_1, ..., z_s) \in \mathbb{R}^s |\ z_i \geq 0,\ i = \overline{1,s}\}.$$

Using the notation $l^+ = \max(l, 0)$ for a given function $l : \mathbb{R}^s \to \mathbb{R}$, the exact penalty function is defined by

$$\|c^+(x)\| := \sum_{i=1}^{s} c_i^+(x),$$

where $c : \mathbb{R}^n \to \mathbb{R}^s,\ c(x) = (c_1(x), ..., c_s(x))$, and $x \in \mathbb{R}^n$. Then the GNEP reduces to the problem GNEP$_{pen}$ which consists in finding $\bar{x} = (\bar{x}_1, ..., \bar{x}_N) \in \mathbb{R}^n$ such that each player's strategy $\bar{x}_k \in \mathbb{R}^{n_k},\ k = \overline{1,N}$ is a solution to the penalized problem

$$\begin{aligned} P_{k,p}(\bar{x}_{-k}) \quad & \inf_{x_k} \left[\theta_k(x_k, \bar{x}_{-k}) + \rho_k \cdot \|h_k^+(x_k, \bar{x}_{-k})\|\right] \\ \text{s.t.} \quad & g(x_k, \bar{x}_{-k}) \underset{l}{\leq} 0, \end{aligned}$$

which turns out to the penalized GNEP with shared constraints and nonsmooth cost functions (see [1]), where $\rho_k \geq 0,\ k = \overline{1,N}$, are parameters. Introducing auxiliary variables $y = (y_1, ..., y_N),\ y_k \in \mathbb{R}^{m_k},\ k = \overline{1,N}$, let us consider the problem GNEP$_{aux}$ consists in finding $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m,\ m =$

$m_1 + \cdots + m_N$ such that $(\bar{x}_k, \bar{y}_k) \in \mathbb{R}^{n_k} \times \mathbb{R}^{m_k},\ k = \overline{1, N}$ is a solution to the problem (cf. [3])

$$\begin{array}{rl} \widetilde{P}_{k,p}(\bar{x}_{-k}) & \inf\limits_{(x_k, y_k)} \big[\theta_k(x_k, \bar{x}_{-k}) + \rho_k \sum\limits_{j=1}^{m_k} y_{k,j}\big] \\ \text{s.t.} & g(x_k, \bar{x}_{-k}) \underset{l}{\leq} 0, \\ & h_k(x_k, \bar{x}_{-k}) - y_k \underset{m_k}{\leq} 0, \\ & 0 \underset{m_k}{\leq} y_k, \end{array}$$

where $y_k := (y_{k,1}, ..., y_{k,m_k})^T \in \mathbb{R}^{m_k}$.

Proposition 1. *For a fixed $x_{-k} \in \mathbb{R}^{n_{-k}},\ k = \overline{1, N}$, it holds*

$$v(P_{k,p}(x_{-k})) = v(\widetilde{P}_{k,p}(x_{-k})),$$

where by $v(P)$ we denote the optimal value of the problem (P).

Proof. Let $x_{-k} \in \mathbb{R}^{n_{-k}}$ be fixed and $x_k \in \mathbb{R}^{n_k}$ be feasible to the problem $P_{k,p}(x_{-k})$. Denoting $y_{k,j} = h^+_{k,j}(x_k, x_{-k})$ implies that $y_{k,j} \geq h_{k,j}(x_k, x_{-k})$ and $y_{k,j} \geq 0,\ j = \overline{1, m_k}$. Whence (x_k, y_k) is feasible to the problem $\widetilde{P}_{k,p}(x_{-k})$, where $y_k = (y_{k,1}, ..., y_{k,m_k})$ and

$$h_k(x_k, x_{-k}) = (h_{k,1}(x_k, x_{-k}), ..., h_{k,m_k}(x_k, x_{-k})).$$

Consequently, we have

$$\begin{aligned} & \theta_k(x_k, x_{-k}) + \rho_k \|h^+_k(x_k, x_{-k})\| \\ = \ & \theta_k(x_k, x_{-k}) + \rho_k \sum_{j=1}^{m_k} h^+_{k,j}(x_k, x_{-k}) \\ = \ & \theta_k(x_k, x_{-k}) + \rho_k \sum_{j=1}^{m_k} y_{k,j} \geq v(\widetilde{P}_{k,p}(x_{-k})). \end{aligned}$$

Taking the infimum over all (x_k, y_k) in the left side of the inequality, one gets $v(P_{k,p}(x_{-k})) \geq v(\widetilde{P}_{k,p}(x_{-k}))$.

Conversely, let (x_k, y_k) be feasible to problem $\widetilde{P}_{k,p}(x_{-k})$. Since x_k is feasible to $P_{k,p}(x_{-k})$, we obtain that

$$\begin{aligned} v(P_{k,p}(x_{-k})) \ & \leq \ \theta_k(x_k, x_{-k}) + \rho_k \sum_{j=1}^{m_k} h^+_{k,j}(x_k, x_{-k}) \\ & \leq \ \theta_k(x_k, x_{-k}) + \rho_k \sum_{j=1}^{m_k} y_{k,j}. \end{aligned}$$

Taking the infimum over all (x_k, y_k) in the right side of the inequality, one gets $v(P_{k,p}(x_{-k})) \leq v(\widetilde{P}_{k,j}(x_{-k}))$. In conclusion, we have

$$v(P_{k,p}(x_{-k})) = v(\widetilde{P}_{k,p}(x_{-k})).$$

□

3. Parameterized variational inequality via duality and optimality conditions

Duality theory plays important role in convex optimization. For the excellent comprehensive survey dealing with conjugate duality we refer to [2]. Before go into detail, let us give a short summary about conjugate duality. Let $X \subseteq \mathbb{R}^n$ be a nonempty set and $f : \mathbb{R}^n \to \overline{\mathbb{R}},\ g = (g_1, ..., g_m)^T : \mathbb{R}^n \to \mathbb{R}^m$ be given functions. Let us consider the optimization problem

$$(P) \qquad \inf_{x \in G} f(x), \quad G = \{x \in X|\ g(x) \underset{m}{\leq} 0\}.$$

We consider the function $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ fulfilling $\Phi(x, 0) = f(x)$ for all $x \in \mathbb{R}^n$. The function Φ is the so-called perturbation function of the problem (P). One can obtain so-called perturbed problem

$$(P_y) \qquad \inf_{x \in \mathbb{R}^n} \Phi(x, y).$$

The conjugate dual problem to (P_y) can be now formulated as being

$$(D) \qquad \sup_{p \in \mathbb{R}^m} \Big\{ -\Phi^*(0, p)\Big\},$$

where $\Phi^* : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is the conjugate function of Φ. The conjugate function of a given function $h : \mathbb{R}^l \to \mathbb{R}$ is defined by

$$h^*(p) := \sup_{x \in \mathbb{R}^l} [p^T x - h(x)].$$

Between the primal and the dual problems weak duality always holds, i.e.,

$$-\infty < \sup(D) \leq \inf(P) < +\infty.$$

In [2], different dual problems based on special perturbation functions have been investigated. Moreover, the strong duality results and optimality conditions have been proved.

Let us now consider a dual problem to $P_k(x_{-k})$ for a fixed $x_{-k} \in \mathbb{R}^{n-k},\ k = \overline{1,N}$. In association with problem $\widetilde{P}_{k,p}(x_{-k})$, one can introduce the perturbation function by

$$\Phi_k(x_k, x_{-k}, y_k) = \begin{cases} \theta_k(x_k, x_{-k}) + \rho_k \sum\limits_{j=1}^{m_k} y_{kj}, & \text{if } g(x_k, x_{-k}) \underset{l}{\leq} 0, \\ & h_k(x_k, x_{-k}) - y_k \underset{m_k}{\leq} 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the corresponding conjugate function becomes

$$\Phi_k^*(p_k, x_{-k}, q_k) = \sup_{\substack{x_k \in \mathbb{R}^{n_k} \\ y_k \in \mathbb{R}^{m_k}}} [p_k^T x_k + q_k^T y_k - \Phi(x_k, x_{-k}, y_k)]$$

$$= \sup_{\substack{g(x_k, x_{-k}) \underset{l}{\leq} 0 \\ h_k(x_k, x_{-k}) - y_k \underset{m_k}{\leq} 0}} [p_k^T x_k + q_k^T y_k - \theta_k(x_k, x_{-k}) - \rho_k \sum_{j=1}^{m_k} y_{kj}],$$

where $p_k, q_k \in \mathbb{R}^k$ are the perturbation variables. Taking z_k instead of y_k, by $z_k = h_k(x_k, x_{-k}) - y_k$, and using the notation $e_k = (1, ..., 1)^T \in \mathbb{R}^{m_k}$, we have

$$\begin{aligned} \Phi_k^*(p_k, x_{-k}, q_k) &= \sup_{\substack{g(x_k, x_{-k}) \underset{l}{\leq} 0 \\ z_k \underset{m_k}{\leq} 0}} [p_k^T x_k + {q_k}^T h_k(x_k, x_{-k}) - {q_k}^T z_k \\ &- \theta_k(x_k, x_{-k}) - \rho_k {e_k}^T h_k(x_k, x_{-k}) + \rho_k {e_k}^T z_k] \\ &= \sup_{g(x_k, x_{-k}) \underset{l}{\leq} 0} [p_k^T x_k + {q_k}^T h_k(x_k, x_{-k}) - \theta_k(x_k, x_{-k}) \\ &- \rho_k {e_k}^T h_k(x_k, x_{-k})] + \sup_{z_k \underset{m_k}{\leq} 0} (\rho_k e_k - q_k)^T z_k. \end{aligned}$$

Taking the first variable by zero, it holds

$$\begin{aligned} \Phi_k^*(0, x_{-k}, q_k) &= \sup_{g(x_k, x_{-k}) \underset{l}{\leq} 0} [{q_k}^T h_k(x_k, x_{-k}) \\ &- \theta_k(x_k, x_{-k}) - \rho_k {e_k}^T h_k(x_k, x_{-k})] + \sup_{z_k \underset{m_k}{\leq} 0} (\rho_k e_k - q_k)^T z_k. \end{aligned}$$

Since

$$\sup_{z_k \underset{m_k}{\leq} 0} (\rho_k e_k - q_k)^T z_k = \begin{cases} 0, & \rho_k e_k - q_k \underset{m_k}{\geq} 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

the dual problem for $P_k(x_{-k})$ can be written as

$$\begin{aligned}
D_k(x_{-k}) \qquad & \sup_{q_k\in\mathbb{R}^{m_k}} [-\Phi_k^*(0,x_{-k},q_k)] \\
&= \sup_{\rho_k e_k - q_k \underset{m_k}{\geq} 0} \Big\{ - \sup_{g(x_k,x_{-k}) \underset{l}{\leq} 0} [{q_k}^T h_k(x_k,x_{-k}) - \theta_k(x_k,x_{-k}) \\
&- \rho_k e_k^T h_k(x_k,x_{-k})] \Big\} \\
&= \sup_{\rho_k e_k - q_k \underset{m_k}{\geq} 0} \inf_{g(x_k,x_{-k}) \underset{l}{\leq} 0} [\theta_k(x_k,x_{-k}) + (\rho_k e_k - q_k)^T h_k(x_k,x_{-k})].
\end{aligned}$$

Remark 1. Since the perturbation in the objective function is linear, it turns out to be the Lagrange duality with penalty parameters.

Now we formulate the sufficient condition for the primal-dual pair $P_k(x_{-k})$ and $D_k(x_{-k})$ as follows (cf. [2]):

Proposition 2. *Let $x_{-k} \in \mathbb{R}^{n_{-k}}$, $k = \overline{1,N}$ be fixed. If $\bar{x}_k \in C_k(\bar{x}_{-k}) := \{x_k \in \mathbb{R}^{n_k} |\ g(x_k,\bar{x}_{-k}) \underset{l}{\leq} 0\}$, $(\bar{\rho}_k,\bar{q}_k) \in \mathbb{R}_+ \times \mathbb{R}^{m_k}$ such that $\bar{\rho}_k e_k - \bar{q}_k \underset{m_k}{\geq} 0$ and satisfy conditions*

(i)

$$\begin{aligned}
&\theta_k(\bar{x}_k,\bar{x}_{-k}) + (\bar{\rho}_k e_k - \bar{q}_k)^T h_k(\bar{x}_k,\bar{x}_{-k}) \\
= &\inf_{x_k\in C_k(\bar{x}_{-k})} [\theta_k(x_k,\bar{x}_{-k}) + (\bar{\rho}_k e_k - \bar{q}_k)^T h_k(x_k,\bar{x}_{-k})];
\end{aligned}$$

(ii) $\qquad (\bar{\rho}_k e_k - \bar{q}_k)^T h_k(\bar{x}_k,\bar{x}_{-k}) = 0,$

then $\bar{x}_k$ is an optimal solution to $P_k(x_{-k})$, $\bar{q}_k$ is an optimal solution to $D_k(x_{-k})$, respectively.

Proof. Let $\bar{x}_{-k} \in \mathbb{R}^{n_{-k}}$ be fixed. By (i) and (ii), we obtain that

$$\begin{aligned}
v(D_k(\bar{x}_{-k})) \ &\geq \inf_{x_k\in C_k(\bar{x}_{-k})} [\theta_k(x_k,\bar{x}_{-k}) + (\bar{\rho}_k e_k - \bar{q}_k)^T h_k(x_k,\bar{x}_{-k})] \\
&= \ \theta_k(\bar{x}_k,\bar{x}_{-k}) \geq v(P_k(\bar{x}_{-k}))
\end{aligned}$$

and taking into account the weak duality, the strong duality is fulfilled which leads to the expected conclusion. □

Remark that the condition (i) in Proposition 2 can be rewritten as

$$\begin{aligned}
&\theta_k(\bar{x}_k,\bar{x}_{-k}) + (\bar{\rho}_k e_k - \bar{q}_k)^T h_k(\bar{x}_k,\bar{x}_{-k}) + \delta_{C_k(\bar{x}_{-k})}(\bar{x}_k) \\
= &\inf_{x_k\in\mathbb{R}^{n_k}} [\theta_k(x_k,\bar{x}_{-k}) + (\bar{\rho}_k e_k - \bar{q}_k)^T h_k(x_k,\bar{x}_{-k}) + \delta_{C_k(\bar{x}_{-k})}(x_k)],
\end{aligned}$$

where the indicator function of a set $D \subseteq \mathbb{R}^s$ is defined by

$$\delta_D : \mathbb{R}^s \to \overline{\mathbb{R}}, \delta_D(x) = \begin{cases} 0, & if x \in X \\ +\infty, & otherwise. \end{cases}$$

Let the functions $\theta_k, h_k,\ k \in \{1, ..., N\}$ be differentiable. Then for each $k \in \{1, ..., N\}$ the above condition becomes equivalently

$$\begin{aligned} 0 &\in \partial(\theta_k(\bar{x}_k, \bar{x}_{-k}) + (\bar{\rho}_k e_k - \bar{q}_k)^T h_k(\bar{x}_k, \bar{x}_{-k}) + \delta_{C_k(\bar{x}_{-k})}(\bar{x}_k)) \\ &= \nabla\theta_k(\bar{x}_k, \bar{x}_{-k}) + (\bar{\rho}_k e_k - \bar{q}_k)^T \cdot \nabla h_k(\bar{x}_k, \bar{x}_{-k}) + N_{C_k(\bar{x}_{-k})}(\bar{x}_k), \end{aligned}$$

where the subdifferential of a function $h : \mathbb{R}^s \to \overline{\mathbb{R}}$ at $x \in \mathbb{R}^s$ is the set

$$\partial h(x) = \Big\{ p \in \mathbb{R}^s |\ f(y) - f(x) \geq p^T(y - x),\ \forall y \in \mathbb{R}^s \Big\}$$

and $N_D = \partial\delta_D$ which is called a normal cone of a given set $D \subseteq \mathbb{R}^s$.

On the other hand, the above inclusion reduces into the following variational inequality problem for $k \in \{1, ..., N\}$:

$$(VI^k_{\rho,q}) \langle \nabla\theta_k(\bar{x}_k, \bar{x}_{-k}) + (\bar{\rho}_k e_k - \bar{q}_k)^T \nabla h_k(\bar{x}_k, \bar{x}_{-k}), x_k - \bar{x}_k \rangle \geq 0, \forall x_k \in C_k(\bar{x}_{-k}).$$

Introducing the function $F_{\rho,q} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$F_{\rho,q}(x) := \Big(\nabla\theta_k(\bar{x}_k, \bar{x}_{-k}) + (\bar{\rho}_k e_k - \bar{q}_k)^T \nabla h_k(\bar{x}_k, \bar{x}_{-k}) \Big)_{k=1}^N,$$

we have the following parameterized variational inequality problem of finding $\bar{x} \in G$ such that

$$(VI_{\rho,q}) \qquad \langle F_{\rho,q}(\bar{x}), x - \bar{x} \rangle \geq 0,\ \forall x \in G,$$

where $x \in \mathbb{R}^n$ and $G := \{x \in \mathbb{R}^n |\ g(x_k, x_{-k}) \underset{l}{\leq} 0\}$.

The following assertion deals with relationships between problems $(VI^k_{\rho,q})$ and $(VI_{\rho,q})$.

Proposition 3. *Let $\bar{\rho}_k \geq$ and $\bar{q}_k \in \mathbb{R}^{m_k},\ k = \overline{1, N}$ be fixed. For each $k = \overline{1, N}$ and for a fixed $\bar{x}_{-k} \in \mathbb{R}^{n_{-k}},\ \bar{x}_k$ is a solution to the problem $VI^k_{\bar{\rho},\bar{q}}$ if and only if $\bar{x} = (\bar{x}_1, ..., \bar{x}_N)$ is a solution to $VI_{\bar{\rho},\bar{q}}$.*

Proof. If for each $k = \overline{1, N}$ and for a fixed $\bar{x}_{-k} \in \mathbb{R}^{n_{-k}},\ \bar{x}_k$ is a solution to the problem $VI^k_{\bar{\rho},\bar{q}}$, then summing all variational inequalities, it holds

$$\sum_{k=1}^N \langle \nabla\theta_k(\bar{x}_k, \bar{x}_{-k}) + (\bar{\rho}_k e_k - \bar{q}_k)^T \nabla h_k(\bar{x}_k, \bar{x}_{-k}), x_k - \bar{x}_k \rangle \ \geq \ 0,$$

$$\forall x = (x_1, ..., x_N) \in \prod_{k=1}^N C_k(\bar{x}_{-k}) = G.$$

In other words, $\bar{x} = (\bar{x}_1, ..., \bar{x}_N)$ is a solution of $VI_{\bar{\rho},\bar{q}}$.

Conversely, let $\bar{x}$ be a solution of $VI_{\bar{\rho},\bar{q}}$. Assume that $\exists j \in \{1, ..., N\}$ and $\widetilde{x}_j \in \mathbb{R}^{n_j}$ such that

$$\langle \nabla\theta_j(\bar{x}_j, \bar{x}_{-j}) + (\bar{\rho}_j e_k - \bar{q}_j)^T \nabla h_j(\bar{x}_j, \bar{x}_{-j}), \widetilde{x}_j - \bar{x}_j \rangle < 0.$$

Setting $\widetilde{x} = (\bar{x}_1, ..., \bar{x}_{j-1}, \widetilde{x}_j, \bar{x}_{j+1}, ..., \bar{x}_N)$ in $VI_{\bar{\rho},\bar{q}}$, it holds

$$\langle \nabla\theta_j(\bar{x}_j, \bar{x}_{-j}) + (\bar{\rho}_j e_k - \bar{q}_j)^T \nabla h_j(\bar{x}_j, \bar{x}_{-j}), \widetilde{x}_j - \bar{x}_j \rangle \geq 0,$$

which leads to a contradiction. Therefore for each $k \in \{1, ..., N\}$, $\bar{x}_k$ is a solution of $VI^k_{\bar{\rho},\bar{q}}$.

□

Theorem 1. *Let $\bar{q} = (\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_N) \in \mathbb{R}^m$ be fixed, and choose $\bar{\rho} = (\bar{\rho}_1, ..., \bar{\rho}_N)$ such that $\bar{\rho}_k \geq \max\limits_{j=1,m_k} \bar{q}_{kj}$. If $\bar{x} = (\bar{x}_k, \bar{x}_{-k}) \in \mathbb{R}^n$ is a solution to the problem $(VI_{\bar{\rho},\bar{q}})$ and for each $k = \overline{1, N}$, the conditions*

$$h_k(x_k, x_{-k}) \leq_{q_{m_k}} 0, \text{ and } (\bar{\rho}_k e_k - \bar{q}_k)^T \cdot h_k(\bar{x}_k, \bar{x}_{-k}) = 0, \tag{3.1}$$

are fulfilled, then $\bar{x}$ is a generalized Nash equilibrium point, i.e., for each $k = \overline{1, N}$, $\bar{x}_k$ solves the problem $P_k(\bar{x}_{-k})$.

Proof. According to the choice of $\bar{\rho}_k$, and by assumptions for each $k = \overline{1, N}$, $\bar{x}_k$ and $\bar{q}_k$ are feasible to problems $P_k(\bar{x}_{-k})$ and $D_k(\bar{x}_{-k})$, respectively. On the other hand, since $\bar{x} \in \mathbb{R}^n$ is a solution to the problem $(VI_{\bar{\rho},\bar{q}})$, the conditions $(i) - (ii)$ are fulfilled, and this means that by Proposition 2, for each $k = \overline{1, N}$, $\bar{x}_k$ solves the problem $P_k(\bar{x}_{-k})$. □

Based on the above theorem, let us summarize how to find a solution of GNEP with coupling and shared constraints.

Let $\bar{q} = (\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_N) \in \mathbb{R}^m$ be a fixed parameter.

1) Choose $\bar{\rho} = (\bar{\rho}_1, ..., \bar{\rho}_N)$ such that $\bar{\rho}_k \geq \max\limits_{j=1,m_k} \bar{q}_{kj}$.

2) Solve the variational inequality problem $(VI_{\bar{\rho},\bar{q}})$ and let $\bar{x}$ be the solution of $(VI_{\bar{\rho},\bar{q}})$.

3) If the conditions (3.1) are fulfilled, then $\bar{x}$ is GNE point, otherwise, choose parameter $\bar{q}$ and Goto 1.

Remark 2. If we choose $\bar{\rho} = (\bar{\rho}_1, ..., \bar{\rho}_N)$ such that $\bar{\rho}_k = \max\limits_{j=\overline{1,m_k}} \bar{q}_{kj}$, then it is obvious that the second condition in (3.1) is fulfilled.

4. Numerical Results

In order to solve the variational inequality $(VI_{\bar{\rho},\bar{q}})$, we use the hyperplane projection method and numerical results were tested by using Matlab tools on a Toshiba L305D 2.0Ghz processor with 3.0 GB RAM.

Example 1.([7]) We consider the following two players game, where the first player controls the variable $x \in \mathbb{R}$ and the second $y \in \mathbb{R}$.

$$\begin{aligned} &\min_x (x^2 - 2xy) \\ &\quad 0 \le x \le 1 \\ &x + y - \frac{3}{2} \le 0 \\ &\min_y \left(\frac{1}{2}y^2 + (x-1)y\right) \\ &\qquad 0 \le y \le 1 \end{aligned}$$

We penalize the coupled constraint only for the first player.

$$\begin{aligned} &\min\nolimits_x \left(x^2 - 2xy + \rho_1(x + y - \frac{3}{2})_+\right) \\ &\qquad\qquad 0 \le x \le 1 \\ &\min\nolimits_y \left(\frac{1}{2}y^2 + (x-1)y\right) \\ &\qquad 0 \le y \le 1, \end{aligned}$$

where h_+ is an exact penalty function. Applying results in Section 3, we obtain the following parameterized VIP.

$$\langle F_{\rho,q}(x^*), x - x^* \rangle \ge 0, \ \forall x \in G,$$

where

$$\begin{aligned} F_{\rho q}(x,y) &= \begin{pmatrix} 2x - 2y + \rho_1 - q_1 \\ y + x - 1 \end{pmatrix} \\ G &= \{(x,y) \in \mathbb{R}^2 |\ 0 \le x \le 1, \ 0 \le y \le 1\}. \end{aligned}$$

We choose a penalty parameter as $\rho_1 \ge q_1$, where q_1 is uniformly distributed in $(0,1]$ and the starting point is $(x^0, y^0) = (1,3)^T$.

n	time(sec)	iter	ρ_1/q_1	GNE
1	0.5313	36	0/0	(0.50003, 0.500039)
2	0.5	35	1.09283/0.606843	(0.378536, 0.621536)
3	0.4844	34	1.6534/0.891299	(0.309508, 0.690575)
4	0.4844	35	0.474971/0.456468	(0.49541, 0.504682)
5	0.5000	35	1.26611/0.821407	(0.388856, 0.611217)
6	0.5313	34	1.40737/0.615432	(0.302048, 0.698035)
7	0.5313	34	1.66002/0.921813	(0.315481, 0.684603)
8	0.5156	34	0.581972/0.176266	(0.398608, 0.601471)
9	0.5156	34	1.85237/0.93547	(0.270805, 0.729275)
10	0.5625	33	1.30392/0.41027	(0.276619, 0.723462)

Table 1: Some GNE points by Parameterized VIP.

Example 2. (see [5]) Now we consider the internet switching model. There are N players, each player having a single variable $x_k \in \mathbb{R}$. The utility functions are given by

$$\theta_k(x) = \frac{-x_k}{x_1 + \ldots + x_N}\Big(1 - \frac{x_1 + \ldots + x_N}{B}\Big), \quad k = 1, ..., N$$

for some constant B. The constraints are

$$x_1 + \ldots + x_N \leq B, \; x_k \geq l_k$$

for some lower bounds $l_k \geq 0$. We also take the lower bounds $l_k = 0.01, \; k = 1, ..., N$ and $N = 10, \; B = 1$. We know that in this case the problem has a solution $\bar{x} = \Big(\frac{9}{100}, \cdots, \frac{9}{100}\Big)^T$. For each player, we penalize the constraints $h_k(x) = -x^k + 0.01 \leq 0$ and have the parameterized VIP with

$$\begin{aligned} F_{\rho q}(x) &= (\nabla_{x_k}\theta_k(x) - (\rho_k - q_k))_{k=1}^{10}, \\ G &= \{x | \; x_1 + \ldots + x_{10} \leq 1\}, \end{aligned}$$

where

$$\begin{aligned} \nabla_{x_k}\theta_k(x) &= 1 - \frac{1}{x_1 + \ldots + x_{10}} + \frac{x_k}{(x_1 + \ldots + x_{10})^2}, \\ \rho &= (\rho_1, ..., \rho_{10}), \; q = (q_1, ..., q_{10}), \; \rho_k > q_k \end{aligned}$$

and $q_k, \; k = 1, ..., 10$ are uniformly distributed in $(0, 1]$. We choose the starting point as $x^0 = (0, ..., 0)^T$.

n	time(sec)	iter	ρ/q	GNE
1	0.3750	9	$\rho = (0, ..., 0)^T, q = (0, ..., 0)^T$	$\begin{pmatrix} 0.0899953 \\ 0.0899953 \\ 0.0899953 \\ 0.0899953 \\ 0.0899953 \\ 0.0899953 \\ 0.0899953 \\ 0.0899953 \\ 0.0899953 \\ 0.0899953 \end{pmatrix}$
2	0.4844	14	$\rho = \begin{pmatrix} 1.21352 \\ 0.878855 \\ 0.542837 \\ 0.458195 \\ 0.982805 \\ 0.634084 \\ 1.65256 \\ 1.6656 \\ 0.711841 \\ 1.06865 \end{pmatrix}, \ q = \begin{pmatrix} 0.561196 \\ 0.77268 \\ 0.00107337 \\ 0.00685779 \\ 0.195662 \\ 0.618563 \\ 0.890854 \\ 0.907035 \\ 0.38073 \\ 0.504078 \end{pmatrix}$	$\begin{pmatrix} 0.134652 \\ 0.01 \\ 0.0241183 \\ 0.01 \\ 0.269438 \\ 0.01 \\ 0.244004 \\ 0.24087 \\ 0.01 \\ 0.0469175 \end{pmatrix}$

Table 2: Some GNE points by Parameterized VIP.

5. Conclusion

In this paper, we have examined a generalized Nash equilibrium problem with coupling and shared constraints. Based on the exact penalty and conjugate duality techniques, we reduced this problem to a parameterized variational inequality problem which allows to find many GNEs. Numerical results are given.

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Точные штрафы и сопряженная двойственность для обобщенных задач равновесия Нэша со связанными и общими ограничениями

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Аннотация. Обобщенные задачи равновесия Нэша (GNEP) используются в теории игр, операционных исследованиях, технике, экономике, а также телекоммуникациях в последние два десятилетия. Одним из наиболее важных классов задач GNEP является класс задач с совместно выпуклыми или общими ограничениями, который широко изучается. Эти задачи считаются одними из самых сложных задач в этой области. Кроме того, достаточно мало исследований GNEP с сопряженными и общими ограничениями. Целью данной статьи является исследование взаимосвязи между использованием метода точных штрафов и сопряженной двойственностью в задаче выпуклой оптимизации для GNEP со связанными и общими ограничениями. Авторы статьи с помощью необходимых условий оптимальности получили параметризованные задачи вариационного неравенства. Рассмотренные задачи помогают исследовать многие другие обобщенные задачи равновесия Нэша. В статье также представлены некоторые численные результаты.

Ключевые слова: обобщенные задачи равновесия Нэша, точная функция штрафа, сопряженная двойственность, связанные и общие ограничения.

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