On the Stability of Tubes of Discontinuous Solutions of Bilinear Systems with Delay

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Abstract. The paper considers the stability property of tubes of discontinuous solutions of a bilinear system with a generalized action on the right-hand side and delay. A feature of the system under consideration is that a generalized (impulsive) effect is possible non-unique reaction of the system. As a result, the unique generalized action gives rise to a certain set of discontinuous solutions, which in the work will be called the tube of discontinuous solutions. The concept of stability of discontinuous solutions tubes is formalized. Two versions of sufficient conditions for asymptotic stability are obtained. In the first case, the stability of the system is ensured by the stability property of a homogeneous system without delay; in the second case, the stability property is ensured by the stability property of a homogeneous system with delay. These results generalized the similar results for systems without delay.

Keywords: differential equations with delay, impulsive disturbance, stability.

1. Problem Statement

Consider the bilinear system of the differential equations with delay

$$\dot{x}(t) = \left(A(t) + \sum_{j=1}^{m-1} D_j(t)\dot{v}_j(t) \right) x(t) + g(t)\dot{v}_m(t) + A_\tau(t)x(t-\tau) +$$

$$+ \int_{-\tau}^{0} G(t,s)x(t+s)ds + f(t), \quad t \geqslant t_0 \tag{1.1}$$

Here $A(t)$, $A_\tau(t)$, $G(t,s)$, $D_j(t)$, $j \in \overline{1,m}$ are continuous of the bounded matrix functions of dimension $n \times n$, $v_j(t)$ $(j \in \overline{1,m})$ are components bounded variation vector functions $v(t) = (v_1(t), v_2(t), \ldots, v_m(t))^T$, $\tau > 0$ is constant delay, $\varphi(t)$ — is initial function, which is n-dimensional vector function of bounded variation, defined on $[t_0 - \tau, t_0]$, $f(t)$ is n-dimensional vector function with the integrated elements, $g(t)$ — is continuous n – dimensional vector function.

Characteristic of the system (1.1) is that its right part contains an incorrect multiplication operation discontinuous function to generalized one. This is due to the next fact. If the function $v(t)$ is discontinuous at some moment time, then the system is subjected to impulse action at this moment. Therefore, the function $x(t)$ appears breaking at the same moment, and in the term $\sum\limits_{j=1}^{m} D_j(t)\dot{v}_i(t)\,x(t)$ is an incorrect operation of multiplying a generalized function by a discontinuous functions, which leads to the problem of formalizing the concept of decisions.

The formalization of the concept of a solution to the system (1.1) has been considered by various authors. The monograph [10] provides a fairly complete overview of possible approaches. Note that various formalizations of the concept of a solution lead to different trajectories. In this work we will take an approach, which is based on the approximation of generalized actions by smooth approximations and the determination of a solution based on the closure of the set of continuous trajectories resulting from the approximation of generalized actions by summable functions. For more information, see [2;4;10]. N.N. Krasovskii in the monograph [3] noted, that such definition is natural from the point of view of control theory. In the case where the matrices D_j are mutually commutative, for any admissible $t \geq t_0$, any sequence $v_k(t)$, which converges pointwise to $v(t)$, generates a sequence of solutions of the equation (1.1), which will have a single limit and will not depend on the method of approximating the function of the function of bounded variation $v(t)$. This case for systems without delay was considered in [2; 4; 10], and for systems with delay was considered in [7].

In the case when the sequence of smooth solutions is not convergent, in [6; 10] it is proposed to take all partial pointwise limits of such sequence. As in [6; 10] we will say that the sequence $v_k(t)$ V – converges to $v(t)$, if $v_k(t)$ converges pointwise to $v(t)$ and $\operatorname*{var}\limits_{[t_0,\,t]} v_k(\cdot)$ converges pointwise to $V(t) \in BV[t_0, \vartheta]$. For this convergence we will use the symbol $v_k(t) \xrightarrow{V} v(t)$.

Definition 1. *Any partial pointwise sequence limit $x_k(t)$, $k = 1, 2, \ldots,$ generated by arbitrary V-convergent sequence of absolutely continuous functions $v_k(t)$, $k = 1, \ldots,$ we will call V - solution of the system (1.1), which satisfies the initial condition $x(t) = \varphi(t)$, $t \in [t_0 - \tau, t_0]$.*

Let $z(0) = x(\bar{t})$, $\mu(0) = v(\bar{t})$ are the initial conditions of the system

$$\dot{z}(\xi) = \sum_{i=1}^{m-1} D_i(\bar{t}) z(\xi) \eta_i(\xi) + f(\bar{t}) \eta_m(\xi) \ \ \dot{\mu}(\xi) = \eta(\xi). \qquad (1.2)$$

According to [10] all $V-$ solutions of the equation (1.1) will satisfy the following integral inclusion:

$$x(t) \in \varphi(t_0) + \int\limits_{t_0}^{t} A(\xi) x(\xi) \, d\xi + \sum_{i=1}^{m-1} \int\limits_{t_0}^{t} D_j(\xi) x(\xi) \, dv_i^c(\xi) + \int\limits_{t_0}^{t} g(\xi) \, dv_m^c(\xi) +$$

$$+ \int\limits_{t_0}^{t} f(\xi) \, d\xi + \int\limits_{t_0}^{t} C(\xi) x(\xi - \tau) \, d\xi + \int\limits_{t_0}^{t} \int\limits_{-\tau}^{0} G(\xi, s) x(\xi + s) d\xi ds +$$

$$+ \sum_{t_i \leq t, \, t_i \in \Omega_-} S(t_i, x(t_i - 0), \Delta v(t_i - 0), V(t_i - 0), \Delta V(t_i - 0)) +$$

$$+ \sum_{t_i < t, \, t_i \in \Omega_+} S(t_i, x(t_i), \Delta v(t_i + 0), V(t_i), \Delta V(t_i + 0)) \qquad (1.3)$$

where $v^c(t)$ - is the continuous component of the vector of the function of bounded variation $v(t)$.

In (1.3) set $S(\bar{t}, x(\bar{t}), \Delta v(\bar{t}), V(\bar{t}), \Delta V(\bar{t}))$ (where $\bar{t} = t_i - 0$ $t_i \in \Omega_-$ and $\bar{t} = t_i$ if $t_i \in \Omega_+$) defined as a sectional shift $(\mu(\Delta V(\bar{t})) = v(t_i)$ if $\bar{t}_i \in \Omega_-$ and $\mu(\Delta V \bar{t}) = v(t_i + 0)$, if $\bar{t}_i \in \Omega_+$) are system reachability sets (1.2) at a size $-x(t)$ at the moment $\xi = \Delta V(\bar{t})$, where the control $\eta(\xi)$ satisfies the constraint $\|\eta(\xi)\| \leq 1, \|\eta(\xi)\| = \sum_{i=1}^{m} |\eta_i(\xi)|$.

Thus, to each discontinuity point (left or right) of the function $v(t)$ and every possible jump in the trajectory of the system (1.1) at moment $\bar{t}$ the function $\eta^{(\bar{t})}(\xi)$, defined on the segment $[0, \Delta V(\bar{t})]$, which, by solving the system of equations (1.2) will determine the jump value of $\Delta x(\bar{t})$ of the trajectory at the time $\bar{t}$.

Definition 2. *Continuing solutions of integral inclusion (1.3) for $[t_0, \infty)$ will be called the solution of the equation (1.1) on the interval $[t_0, \infty)$.*

We denote the tube section of the solutions of the integral inclusion (1.3) by $X(t, \varphi_0(\cdot), v(\cdot), V(\cdot))$ which is generated by the initial condition $\varphi_0(\cdot)$ and a pair of functions $v(\cdot)$, $V(\cdot)$

Definition 3. *We say that the solution tube for the integral inclusion $X(t, \varphi_0(\cdot), v(\cdot), V(\cdot))$ is stable if $\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0$, what if*

$$\sup_{\xi \in [t_0 - \tau, t_0]} |\varphi_0(\xi) - \varphi_1(\xi)| < \delta,$$

that

$$\rho(X(t, \varphi_0(\cdot), v(\cdot),\, V(\cdot)), X(t, \varphi_1(\xi), v(\cdot),\, V(\cdot))) < \varepsilon,$$

for any $t > t_0$, where $\rho(A, B)$ is the Hausdorff distance between the sets A and B.

Definition 4. *We say that the solution tube for the integral inclusion $X(t, \varphi_0(\cdot), v(\cdot),\, V(\cdot))$ is asymptotically stable, if it is stable, and also equality is validity*

$$\lim_{t \longrightarrow \infty} \rho(X(t, x_0, v(\cdot),\, V(\cdot)), X(t, \overline{x}, v(\cdot),\, V(\cdot))) = 0.$$

2. Stability of discontinuous solutions tubes

The results obtained below are a generalization of [9] for systems with delay.

Theorem 1. *Let the fundamental matrix $Y(t, s)$ of the system $\dot{x} = A(t)x$ satisfy the estimate*

$$\|Y(t, s)\| \le c e^{-(\alpha(t-s))}, \tag{2.1}$$

where α and c are some constants such that $\alpha > 0$, $c \ge 1$, $t \ge s \ge t_0$. In addition, suppose that the estimates

$$\|D_j(t)\| \le K, \ \|G(t, s)\| \le K \ \forall t \in [t_0, \infty), s \in [t_0, t], j \in \overline{1, m-1} \tag{2.2}$$

Here K - is a positive constant. Then if $x(t)$ and $\overline{x}(t)$ are integral inclusion solutions generated by the initial conditions $\varphi_0(t)$ and $\varphi_1(t)$, as well as the same system of functions $\eta^{(\overline{t})}(\xi)$ which generates jumps of the trajectorys $x(t)$ and $\overline{x}(t)$, then the following estimate holds:

$$\|\overline{x}(t) - x(t)\| \le \sup_{\xi \in [t_0 - \tau, t_0]} \|\varphi_1(\xi) - \varphi_0(\xi)\| \times e^{-(\alpha(t-t_0) - cKe(t-t_0 + V(t)))} \tag{2.3}$$

Proof. According to [6], $x(t)$ and $\overline{x}(t)$ will satisfy the integral equation

$$x(t) = \varphi(t_0) + \int_{t_0}^{t} A(\xi)x(\xi)\, d\xi + \sum_{j=1}^{m-1} \int_{t_0}^{t} D_j(\xi)x(\xi)\, dv_j^c(\xi) + \int_{t_0}^{t} g(\xi)\, dv_m^c(\xi) +$$

$$+ \int_{t_0}^{t} A_\tau(\xi)x(\xi - \tau)\, d\xi + \int_{t_0}^{t} f(\xi)\, d\xi + \sum_{t_i \le t,\, t_i \in \Omega_-} \widetilde{S}(t_i, x(t_i - 0), \eta^{(t_i - 0)}(\cdot), V(t_i - 0),$$

$$\Delta V(t_i - 0)) + \sum_{t_i < t,\, t_i \in \Omega_+} \widetilde{S}(t_i, x(t_i), \eta^{(t_i)}(\cdot), V(t_i), \Delta V(t_i)), \tag{2.4}$$

where $\varphi(t_0)$ will be equal to $\varphi_0(t_0)$ and $\varphi_1(t_0)$ for $x(t)$ and $\overline{x}(t)$ respectively,

$$\widetilde{S}(\overline{t}, x(\overline{t}), \eta^{(\overline{t})}(\cdot), V(\overline{t}), \Delta V(\overline{t})) = z(\Delta V(\overline{t})) - x(\overline{t}),$$

where $z(\xi)$ is a solution to the equation

$$\dot{z}(\xi) = \sum_{j=1}^{m-1} D_j(\overline{t})z(\xi)\eta_j^{(\overline{t})}(\xi) + g\overline{t})\eta_m^{(\overline{t})}(\xi), \ \ z(0) = x(\overline{t}). \qquad (2.5)$$

Using the Cauchy formula [10] we get that the solutions $x(t)$ and $\overline{x}(t)$ will satisfy the integral equation

$$x(t) = Y(t,t_0)\varphi(t_0) + \sum_{j=1}^{m-1}\int_{t_0}^{t} Y(t,\xi)D_j(\xi)x(\xi)\,dv_j^c(\xi) + \int_{t_0}^{t} Y(t,\xi)g(\xi)\,dv_m^c(\xi) +$$

$$+ \int_{t_0}^{t} Y(t,\xi)A_\tau(\xi)x(\xi-\tau)\,d\xi + \int_{t_0}^{t}\int_{-\tau}^{0} Y(t,\xi)G(\xi,s)x(\xi-s)\,d\xi\,ds +$$

$$+ \sum_{t_i \le t,\, t_i \in \Omega_-} Y(t,t_i)\widetilde{S}(t_i, x(t_i-0), \eta^{(t_i-0)}(\cdot), V(t_i-0), \Delta V(t_i-0)) +$$

$$+ \sum_{t_i < t,\, t_i \in \Omega_+} Y(t,t_i)\widetilde{S}(t_i, x(t_i), \eta^{(t_i)}(\cdot), V(t_i), \Delta V(t_i)) + \int_{t_0}^{t} Y(t,\xi)f(\xi)\,d\xi.$$

$$(2.6)$$

According to (2.6), $\overline{x}(t) - x(t)$ will satisfy the integral equation

$$\overline{x}(t) - x(t) =$$

$$= Y(t,t_0)(\varphi_1(t_0) - \varphi_1(t_0)) + \sum_{i=1}^{m-1}\int_{t_0}^{t} Y(t,\xi)D_i(\xi)(\overline{x}(\xi) - x(\xi))\,dv_i^c(\xi) +$$

$$+ \int_{t_0}^{t} Y(t,\xi)A_\tau(\xi)(\overline{x}(\xi-\tau) - x(\xi-\tau))\,d\xi + \int_{t_0}^{t} Y(t,\xi)\int_{-\tau}^{0} G(\xi,s)(\overline{x}(\xi-s) -$$

$$- x(\xi-s))\,ds\,d\xi + \sum_{t_i \le t,\, t_i \in \Omega_-} Y(t,t_i)\big(\tilde{S}(t_i, \overline{x}(t_{i-0}), \eta^{(t_{i-0})}(\cdot), V(t_{i-0}), \Delta V(t_{i-0})) -$$

$$- \tilde{S}(t_i, x(t_i-0), \eta^{(t_i-0)}(\cdot), V(t_i-0), \Delta V(t_i-0))\big) +$$

$$+ \sum_{t_i < t,\, t_i \in \Omega_+} Y(t,t_i)\big(\widetilde{S}(t_i, \overline{x}(t_i), \eta^{(t_i)}(\cdot), V(t_i), \Delta V(t_i-0)) -$$

$$- \tilde{S}(t_i, x(t_i), \eta^{(t_i)}(\cdot), V(t_i), \Delta V(t_i+0))\big). \quad (2.7)$$

As shown in [9] (the equation describing the jump of the trajectory does not depend on the delay) fair inequality

$$\|\overline{z}(\Delta V(t) - \overline{x}(t) - (z(\Delta V(t)) - x(t)))\| \leq K\|\overline{x}(t) - x(t)\| \cdot \int_0^{\Delta V(t)} \|\eta^{(t)}(\xi)\| \, d\xi +$$

$$+ K \int_0^{\Delta V(t)} \|\overline{z}(\xi) - \overline{x}(t) - (z(\xi) - x(t))\| \|\eta^{(t)}(\xi\| \, d\xi).$$

According to the Gronwall-Bellman lemma [1] rom the last inequality we get

$$\|\overline{z}(\Delta V(t)) - \overline{x}(t) - (z(\Delta V(t)) - x(t))\| \leq K \int_0^{\Delta V(t)} \|\eta^t(\xi)\| d\xi \|\overline{x}(t) - x(t)\| \times$$

$$\times (e^{K \int_0^{\Delta V(t)} \|\eta^{(t)}(\xi)\| d\xi}). \tag{2.8}$$

Using the obvious estimate $ae^a \leq e^{\beta a} - 1$ for all $a \geq 0$ and $\beta \geq e$ the inequality (2.8) with $\|\eta^{(t)}(\xi)\| \leq 1$ leads to the inequality

$$\|\overline{z}(\Delta V(t)) - \overline{x}(\Delta V(t)) - (z(1) - x(t))\| \leq \|\overline{x}(t) - x(t)\| (e^{eK\Delta V(t)} - 1). \tag{2.9}$$

We introduce the notation

$$y(t) = \overline{x}(t) - x(t). \tag{2.10}$$

We calculate the norms of the left and right sides in (2.7) taking into account (2.1), (2.2), (2.8), (2.9) and obvious inequality $(c \geq 1)$ $c(e^a - 1) < e^{ca} - 1$, from (2.8) we get

$$\|y(t)\| \leq c[e^{-\alpha(t-t_0)}\|y(t_0)\| + K \int_{t_0}^t e^{-\alpha(t-\xi)}\|y(\xi)\| d \operatorname*{var}_{[t_0, \xi]} v^c(\cdot) +$$

$$+ K \int_{t_0}^t e^{-\alpha(t-\xi)}\|y(\xi - \tau)\| \, d\xi + K \int_{t_0}^t e^{-\alpha(t-\xi)} \int_{-\tau}^0 \|y(\xi - s)\| \, d\xi \, ds] +$$

$$+ \sum_{t_i < t, \, t_i \in \Omega_-} e^{-\alpha(t-t_i)} \left(e^{cKe\|\Delta V(t_i - 0)\|} - 1 \right) \|y(t_i - 0)\| +$$

$$+ \sum_{t_i < t, \, t_i \in \Omega_+} e^{-\alpha(t-t_i)} \left(e^{cKe\|\Delta V(t_i)\|} - 1 \right) \|y(t_i)\|.$$

We multiply the last inequality by $e^{\alpha(t-t_0)}$ and introduce the notation

$$q(t) = e^{\alpha(t-t_0)}\|y(t)\|, \tag{2.11}$$

we get:

$$q(t) \le c\|y(t_0)\| + cK \int_{t_0}^{t} q(\xi) \, d \operatorname*{var}_{[t_0,\xi]} v^c(\cdot) +$$

$$+ cK \int_{t_0}^{t} e^{\alpha\tau} q(\xi - \tau) \, d\xi + \sum_{t_i < t, \, t_i \in \Omega_-} (e^{cKe\|\Delta V(t_i-0)\|} - 1) q(t_i - 0) +$$

$$+ \sum_{t_i < t, \, t_i \in \Omega_+} (e^{cKe\|\Delta V(t_i)\|} - 1) q(t_i).$$

We introduce another notation $h(t) = \sup_{[t-\tau, t]} q(\cdot)$. Then the last inequality can be rewritten as

$$h(t) \le ch(t_0) + cK \int_{t_0}^{t} h(\xi) \, d((e^{\alpha\tau} + \tau)\xi + \operatorname*{var}_{[t_0,\xi]} v^c(\cdot)) +$$

$$+ \sum_{t_i < t, \, t_i \in \Omega_-} (e^{cKe\|\Delta V(t_i-0)\|} - 1) h(t_i - 0) + \sum_{t_i < t, \, t_i \in \Omega_+} (e^{cKe\|\Delta V(t_i)\|} - 1) h(t_i).$$

Multiply the integral on the right-hand side of the last inequality by e and replace $\operatorname*{var}_{[t_0,\xi]} v^c(\cdot)$ to $V^c(t)$. As a result, we obtain the inequality

$$h(t) \le ch(t_0) + cKe \int_{t_0}^{t} h(\xi) \, d((e^{\alpha\tau} + \tau)\xi + V^c(\xi)) +$$

$$+ \sum_{t_i < t, \, t_i \in \Omega_-} (e^{cKe\|\Delta V(t_i-0)\|} - 1) h(t_i - 0) + \sum_{t_i < t, \, t_i \in \Omega_+} (e^{cKe\|\Delta V(t_i)\|} - 1) h(t_i).$$

$$\tag{2.12}$$

According to Lemma 5.4.3 from [10] every solution to the inequality (2.12) will satisfy the estimate

$$h(t) \le e^{cKe(t-t_0+V(t))} c \sup_{\xi \in [t_0-\tau, t_0]} \|\varphi_1(\xi) - \varphi_0(\xi)\|.$$

Multiplying this inequality by $e^{-\alpha(t-t_0)}$ and taking into account the designation (2.11) we obtain the estimate (2.3). $\qquad \square$

Theorem 2. *Under the assumptions of the theorem 1 inequality is fair*

$$\rho(X(t, \varphi(\cdot), x(\cdot), v(\cdot), V(\cdot)), X(t, \bar{\varphi}(\cdot)\bar{x}(\cdot), v(\cdot), V(\cdot))) \leq$$

$$\leq c \sup_{\xi \in [t_0 - \tau, t_0]} \|\varphi(\xi) - \bar{\varphi}_0(\xi)\| \times e^{-(\alpha(t-t_0) - cKe(t-t_0+V(t)))}. \qquad (2.13)$$

Proof. Between the sets of V — solutions $X(t, \varphi(\cdot), x(\cdot), v(\cdot), V(\cdot))$ and $X(t, \bar{\varphi}(\cdot), \bar{x}(\cdot), v(\cdot), V(\cdot))$ one-to-one correspondence is established: every trajectory from $X(t, \varphi(\cdot), x(\cdot), v(\cdot), V(\cdot))$ is associated with a trajectory from $X(t, \bar{\varphi}(\cdot), \bar{x}(\cdot), v(\cdot), V(\cdot))$ by the rule - the initial conditions are different ($\varphi(\cdot)$ and $\bar{\varphi}(\cdot)$), and the system of functions η_k, that defines jumps is the same.

First, note that according to [5] the sets $X(t, \varphi(\cdot), x(\cdot), v(\cdot), V(\cdot))$ and $X(t, \bar{\varphi}(\cdot)\bar{x}(\cdot), v(\cdot), V(\cdot))$ are closed. Their boundedness follows from the previous theorem. Then

$$\rho(X(t, \varphi(\cdot), x(\cdot), v(\cdot), V(\cdot)), X(t, \bar{\varphi}(\cdot)\bar{x}(\cdot), v(\cdot), V(\cdot))) =$$

$$= \max\{ \max_{x \in X(t, \varphi(\cdot), x(\cdot), v(\cdot), V(\cdot))} \min_{y \in X(t, \bar{\varphi}(\cdot)\bar{x}(\cdot), v(\cdot), V(\cdot))} \|x - y\|;$$

$$\max_{y \in X(t, \bar{\varphi}(\cdot)\bar{x}(\cdot), v(\cdot), V(\cdot))} \min_{x \in X(t, \varphi(\cdot), x(\cdot), v(\cdot), V(\cdot))} \|x - y\|\}. \qquad (2.14)$$

Let the extremum in (2.14) achieved when $\bar{x} \in X(t, \varphi(\cdot), x(\cdot), v(\cdot), V(\cdot))$ and $\bar{y} \in X(t, \bar{\varphi}(\cdot)\bar{x}(\cdot), v(\cdot), V(\cdot))$ i.e.

$$\rho(X(t, \varphi(\cdot), x(\cdot), v(\cdot), V(\cdot)), X(t, \bar{\varphi}(\cdot)\bar{x}(\cdot), v(\cdot), V(\cdot))) =$$

$$= \max_{x \in X(t, \varphi(\cdot), x(\cdot), v(\cdot), V(\cdot))} \min_{y \in X(t, \bar{\varphi}(\cdot)\bar{x}(\cdot), v(\cdot), V(\cdot))} \|x - y\| =$$

$$= \min_{y \in X(t, \bar{\varphi}(\cdot)\bar{x}(\cdot), v(\cdot), V(\cdot))} \|\bar{x} - y\| = \|\bar{x} - \bar{y}\|.$$

The element $\bar{x}$ can be matched $\tilde{x} \in X(t, \bar{\varphi}(\cdot)\bar{x}(\cdot), v(\cdot), V(\cdot))$. It's obvious that

$$\min_{y \in X(t, \bar{\varphi}(\cdot)\bar{x}(\cdot), v(\cdot), V(\cdot))} \|\bar{x} - y\| = \|\bar{x} - \bar{y}\| \leq \|\bar{x} - \tilde{x}\|.$$

Then from $(1.2), (1.3), (2.1)$ and the theorem 1 implies the validity of the theorem 2.

$$\square$$

Corollary 1. *Let the assumptions of theorem 1 holds. Then if*

$$(\alpha - cKe)(t - t_0 + V(t)) \geq q,$$

for all $t \in [t_0, \infty)$, where q — some constant, then the tube of solutions of the unperturbed motion $X(t, \varphi_0(\cdot), v(\cdot), V(\cdot))$ will be stable, and if

$$\lim_{t \longrightarrow \infty} (\alpha - cKe)(t - t_0 + V(t)) = \infty,$$

then the tube of solutions of the unperturbed motion $X(t, \varphi_0(\cdot), v(\cdot), V(\cdot))$ will be asymptotically stable.

Next, we consider another variant of the sufficient stability conditions for the solution tubes. First we give the Cauchy formula for a linear system with delay [1]

$$\dot{x}(t) = A(t)x(t) + A_\tau(t)x(t - \tau) + \int\limits_{-\tau}^{0} G(t, s)x(t + s)ds + p(t), \quad (2.15)$$

with the initial condition $\varphi(t)$, defined on the interval $[t_0 - \tau, t_0]$, $p(t)$ — is an integrable function. The matrix functions $A(t)$, $A_\tau(t)$, $G(t, s)$ satisfy the same conditions as in the equation (1.1)

According to [1] in the case when $p(t)$ — is an integrable function, , the solution of the equation (2.15) with the initial condition $\varphi(t) = x(t)$, $t_0 - \tau \leq t \leq t_0$ exists and is unique.

Denote Q as the square in the plane s and t , where $t_0 \leq s \leq \vartheta$ $t_0 \leq t \leq \vartheta$. According to [1] under the assumption that $p(t)$ — is an integrable function the solution of the equation (2.15) can be represented as

$$x(t) = F(t, t_0)\varphi(t_0) + \int\limits_{t_0-\tau}^{t_0} F(t, s + \tau)A_\tau(s + \tau)\varphi(s)\, ds + \int\limits_{t_0}^{t} F(t, s)p(s)\, ds.$$
$$(2.16)$$

The function $F(t, s)$ is a solution to the equation

$$\frac{\partial F(t, s)}{\partial s} = -F(t, s)A(s) - F(t, s + \tau)A_\tau(s + \tau) \quad (2.17)$$

with initial condition

$$F(t, t - 0) = E; \quad F(t, s) \equiv 0 \ s > t. \quad (2.18)$$

Now apply the formula (2.16) to the equation (1.1) under the assumption that $v(t)$ is an absolutely continuous function. As a result, we get:

$$x(t) = F(t, t_0)\varphi(t_0) + \int_{t_0-\tau}^{t_0} F(t, s + \tau)A_\tau(s + \tau)\varphi(s)\, ds +$$

$$+ \sum_{j=1}^{m-1} \int\limits_{t_0}^{t} F(t, \xi)D_j(\xi)x(\xi)\dot{v}_j(\xi)\, ds + \int\limits_{t_0}^{t} F(t, \xi)g(\xi)\dot{v}_m(\xi)\, ds +$$

$$+ \int\limits_{t_0}^{t} F(t, \xi)f(\xi)\, d\xi. \quad (2.19)$$

We substitute in (1.2) $v(t) = v^{(k)}(t)$, where $v^{(k)}(t)$ is a sequence of absolutely continuous functions that converge pointwise to the function of bounded variation $v(t)$. By $x^{(k)}(t)$ we denote the sequence of absolutely continuous solutions of the equation (1.1), generated by the sequence $v^{(k)}(t)$. It is not difficult to show that the sequence $x^{(k)}(t)$ is bounded and the sequence of variations of these functions will also be uniformly bounded. Then according to Helly's theorem, from this sequence we can distinguish a subsequence $x^{(k_i)}(t)$ which converges pointwise to some function of bounded variation $x(t)$. In (1.2) we pass to the limit for $k_i \longrightarrow \infty$. The main difficulty in performing this limit transition takes place in the expression $\sum_{j=1}^{m-1} \int_{t_0}^{t} F(t,\xi) \ D_j(\xi) x(\xi) \dot{v}_j(\xi) \, ds$. The passage to the limit in this expression can be done by replacing the time $\xi = t + \operatorname*{var}_{[t_0,t]} v^{(k_i)}$ in the same way as in [6; 7; 10]. As a result, we get that $x(t)$ will satisfy the integral equation

$$x(t) =$$

$$= F(t,t_0)\varphi(t_0) + \sum_{j=1}^{m-1} \int_{t_0}^{t} F(t,\xi)D_j(\xi)x(\xi)\,dv_j^c(\xi) + \int_{t_0}^{t} F(t,\xi)g(\xi)\,dv_m^c(\xi) +$$

$$+ \sum_{t_i \leq t,\, t_i \in \Omega_-} F(t,t_i)\widetilde{S}(t_i, x(t_i - 0), \eta^{(t_i-0)}(\cdot), V(t_i - 0), \Delta V(t_i - 0)) +$$

$$+ \sum_{t_i < t,\, t_i \in \Omega_+} F(t,t_i)\widetilde{S}(t_i, x(t_i), \eta^{(t_i)}(\cdot), V(t_i), \Delta V(t_i)) + \int_{t_0}^{t} F(t,\xi)f(\xi)\,d\xi.$$

$$(2.20)$$

Suppose that, as in the theorem 1 $x(t)$ and $\overline{x}(t)$ are integral inclusion solutions generated by the initial conditions $\varphi_0(t)$ and $\varphi_1(t)$ as well as the same system of functions $\eta_K(\xi)$ which generates jumps in the trajectories $x(t)$ and $\overline{x}(t)$. Then for the difference $\bar{x}(t) - x(t)$ the expression is true

$$\bar{x}(t) - x(t) = F(t,t_0)(\varphi_1(t_0) - \varphi_1(t_0)) + \sum_{i=1}^{m-1} \int_{t_0}^{t} F(t,\xi)D_i(\xi)(\overline{x}(\xi) -$$

$$-x(\xi))\,dv_i^c(\xi) + \sum_{t_i \leq t,\, t_i \in \Omega_-} F(t,t_i)\Big(\tilde{S}(t_i, \bar{x}(t_{i-0}), \eta^{(t_i-0)}(\cdot), V(t_{i-0}),$$

$$\Delta V(t_{i-0})) - \tilde{S}(t_i, x(t_i - 0), \eta^{(t_i-0)}(\cdot), V(t_i - 0), \Delta V(t_i - 0))\Big) +$$

$$+ \sum_{t_i < t,\, t_i \in \Omega_+} F(t,t_i)\Big(\widetilde{S}(t_i, \overline{x}(t_i), \eta^{(t_i)}(\cdot), V(t_i), \Delta V(t_i - 0)) -$$

$$-\tilde{S}(t_i, x(t_i), \eta^{(t_i)}(\cdot), V(t_i), \Delta V(t_i))\Big). \tag{2.21}$$

Further we will assume that the Cauchy matrix $F(t,s)$ satisfies the inequality

$$||F(t,s)|| \le ce^{-\alpha(t-s)}, \tag{2.22}$$

and also evaluations are performed

$$||D_j(t)|| \le K, j \in \overline{1,m}, t \in [t_0, \infty). \tag{2.23}$$

As in the theorem 1, we calculate the norms of the left and right sides in (2.21), using the estimates (2.22), (2.23) and using notation (2.10). As a result, we obtain

$$||y(t)|| \le ce^{-\alpha(t-t_0)}||y(t_0)|| + cK \int_{t_0}^{t} e^{-\alpha(t-\xi)}||y(\xi)||d \operatorname*{var}_{[t_0,\xi]} v^c(\cdot) +$$

$$+ \sum_{t_i < t, \, t_i \in \Omega_-} e^{-\alpha(t-t_i)}(e^{cKe||\Delta v(t_i-0)||} - 1)||y(t_i - 0)|| +$$

$$+ \sum_{t_i < t, \, t_i \in \Omega_+} e^{-\alpha(t-t_i)}(e^{cKe||\Delta v(t_i+0)||} - 1)||y(t_i)|| \tag{2.24}$$

Multiply (2.24) by $e^{\alpha(t-t_0)}$ and introduce the notation

$$p(t) = e^{\alpha(t-t_0)}||y(t)||. \tag{2.25}$$

As a result, we obtain

$$p(t) \le c||y(t_0)|| + cK \int_{t_0}^{t} p(\xi)d \operatorname*{var}_{[t_0,\xi]} v^c(\cdot) + \sum_{t_i < t, \, t_i \in \Omega_-} (e^{cKe||\Delta v(t_i-0)||} - 1)p(t_i-0) +$$

$$+ \sum_{t_i < t, \, t_i \in \Omega_+} (e^{cKe||\Delta v(t_i+0)||} - 1)p(t_i)$$

Now, as in the theorem 1, we introduce the notation

$$h(t) = \sup_{[t-\tau,t]} p(\cdot). \tag{2.26}$$

Given the notation (2.26) the last inequality can be written as

$$h(t) \le ch(t_0) + cK \int_{t_0}^{t} h(\xi)d \operatorname*{var}_{[t_0,\xi]} v^c(\cdot) + \sum_{t_i < t, \, t_i \in \Omega_-} (e^{cKe||\Delta v(t_i-0)||} - 1)p(t_i-0) +$$

$$+ \sum_{t_i < t, \, t_i \in \Omega_+} (e^{cKe\|\Delta v(t_i+0)\|} - 1)p(t_i) \qquad (2.27)$$

Applying Lemma 5.4.3 from [10] to (2.27) we obtain the estimate

$$h(t) \leq e^{cKV(t)}ch(t_0).$$

Multiply the last inequality by $e^{-\alpha(t-t_0)}$ and then take into account (2.25), (2.26) and (2.10). As a result we will receive

$$\|\bar{x}(t) - x(t)\| \leq c \sup_{\xi \in [t_0-\tau, t_0]} \|\varphi_1(\xi) - \varphi_0(\xi)\| e^{-(\alpha(t-t_0)-cKeV(t))}. \qquad (2.28)$$

As a result, the following theorem is proved.

Theorem 3. *Suppose that the fundamental matrix $F(t,s)$ of the system (2.15) satisfies the estimate (2.22) and the matrices $D_i(t)$ satisfy the estimates (2.23). Then the solutions of the integral inclusion (1.3), generated by the initial conditions $\varphi_0(t)$ and $\varphi_1(t)$, as well as the same system of functions $\eta_k(\xi)$, which determines the jumps of the solutions $x(t)$ and $\overline{x}(t)$, satisfy the inequality (2.28).*

Similar to theorem 2 and corollary 1 we can state the following theorem

Theorem 4. *Under the conditions of theorem 3, the inequality is fair*

$$\rho(X(t, \varphi(\cdot), x(\cdot), v(\cdot), V(\cdot)), X(t, \bar{\varphi}(\cdot)\bar{x}(\cdot), v(\cdot), V(\cdot)))$$

$$\leq c \sup_{\xi \in [t_0-\tau, t_0]} \|\varphi_1(\xi) - \varphi_0(\xi)\| e^{-(\alpha(t-t_0)-cKeV(t))}. \qquad (2.29)$$

Therefore, if

$$\alpha(t - t_0) - cKeV(t)) \geq q$$

for all $t \in [t_0, \infty)$, where q — is some constant, then the tube of solutions $X(t, \varphi(\cdot), x(\cdot), v(\cdot), V(\cdot))$ is stable, and if

$$\lim_{t \to \infty} (\alpha(t - t_0) - cKeV(t)) = \infty,$$

then the solution tube $X(t, \varphi(\cdot), x(\cdot), v(\cdot), V(\cdot))$ is asymptotically stable.

3. Conclusion

We investigated the stability property of solutions of a bilinear system with a generalized actions and delay. A distinctive feature of the system under consideration is that a non-unique reaction is possible on the generalized actions. In this regard, the paper gives a formalization of

the concept of stability of discontinuous solution tubes and two sufficient conditions are obtained that ensure the stability of discontinuous solution tubes. The results of the paper generalize the corresponding theorems for systems without delay obtained in [9].

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Об устойчивости трубок разрывных решений билинейных систем с запаздыванием

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Аннотация.   Исследуется свойство устойчивости трубок разрывных решений билинейной системы с обобщенным воздействием в правой части и запаздыванием. Особенностью рассматриваемой системы является то, что на обобщенное (импульсное) воздействие возможна неединственная реакция системы. В результате единственное обобщенное воздействие в качестве реакции системы порождает некоторую совокупность разрывных решений, которую в работе будем называть трубкой разрывных решений. Формализовано понятие устойчивости трубок разрывных решений. Получены два варианта достаточных условий асимптотической устойчивости. В первом случае устойчивость системы обеспечивается свойством устойчивости однородной системы без запаздывания, во втором случае свойство устойчивости обеспечивается свойством устойчивости однородной системы с запаздыванием. Эти результаты обобщают аналогичные результаты для систем без запаздывания.

Ключевые слова: стабилизация, обратная связь, децентрализованное управление.

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