On a Method for Finding Extremal Controls in Systems with Constraints *

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Abstract. In the class of controlled systems with constraints, the conditions for improving and optimality of control are constructed and analyzed in the form of fixed point problems. This form allows one to obtain enhanced necessary optimality conditions in comparison with the known conditions and makes it possible to apply and modify the theory and methods of fixed points to search for extreme controls in optimization problems of the class under consideration. Fixed-point problems are constructed using the transition to auxiliary optimal control problems without restrictions with Lagrange functionals. An iterative algorithm is proposed for constructing a relaxation sequence of admissible controls based on the solution of constructed fixed point problems. The considered algorithm is characterized by the properties of nonlocal improvement of admissible control and the fundamental possibility of rigorous improvement of non-optimal controls satisfying the known necessary optimality conditions, in contrast to gradient and other local methods. The conditions of convergence of the control sequence for the residual of fulfilling the necessary optimality conditions are substantiated. A comparative analysis of the computational and qualitative efficiency of the proposed iterative algorithm for finding extreme controls in a model problem with phase constraints is carried out.

Keywords: the controlled system with constraints, extreme controls, conditions for improving control, fixed point problem, iterative algorithm.

1. Introduction

Extreme search, i.e. satisfying the necessary optimality conditions, controls in the optimization problems of systems with constraints are usually carried out for two alternative possible cases: not regular (or degenerate)

and regular (or non-degenerate). In the regular case, a widespread approach for the numerical implementation of the necessary optimality conditions consists of the construction of relaxation sequences of controls using local methods for improving control of the gradient type [8;10]. Moreover, at each iteration of control improvement, the exact fulfillment of the constraints of the task is not guaranteed.

In the work to search for extremal controls in the regular case, a new approach is proposed for constructing a relaxation sequence of controls based on the constructed systems of conditions for nonlocal improvement of control with the exact fulfillment of the constraints of the problem. These conditions can be interpreted as fixed point problems of a control operator with an additional algebraic equation. The fixed point approach under consideration for improving control is the development and extension of the nonlocal control improvement approach, which was initially developed in linear and linear-quadratic state control problems without restrictions [8]. This approach is based on the development of non-standard formulas for incrementing problem functionals that do not contain residual expansion terms. Fixed point methods were constructed and substantiated in the classes of nonlinear optimal control problems [2–5]. This paper describes a new method for searching for extremal controls in the considered class of problems with constraints based on the fixed-point approach.

2. Problem with constraints

We consider the class of optimal control problems with constraints, which can be reduced to the following canonical form:

$$\dot{x}(t) = f(x(t), u(t), t), x(t_0) = x^0, u(t) \in U, t \in T = [t_0, t_1], \qquad (2.1)$$

$$\Phi_0(u) = \varphi_0(x(t_1)) + \int_T F_0(x(t), u(t), t)dt \to \inf_{u \in V}, \qquad (2.2)$$

$$\Phi_1(u) = \varphi_1(x(t_1)) = 0, \qquad (2.3)$$

where $x(t) = (x_1(t), ..., x_n(t))$ is state vector, $u(t) = (u_1(t), ..., u_m(t))$ is a vector of control functions, $U \subseteq R^m$ is closed convex set. Interval T is fixed. As available control functions, we consider the set of V piecewise continuous on T functions with values in the set U: $V = \{v \in PC(T) : v(t) \in U, t \in T\}$. The functions $\varphi_0(x)$ and $\varphi_1(x)$ are continuously differentiable on R^n, the functions $F_0(x, u, t)$, $f(x, u, t)$ and their partial derivatives with respect to x, u are continuous in the set of arguments on the set $R^n \times U \times T$. The function $f(x, u, t)$ satisfies the Lipschitz condition with respect to x in $R^n \times U \times T$ with constant $L > 0$: $\|f(x, u, t) - f(y, u, t)\| \leq L \|x - y\|$.

To type (2.1) – (2.3) by standard penalties for violating restrictions, many optimal control problems with phase and terminal restrictions can be reduced.

Available control $u \in V$ is called admissible if functional constraint (2.3) is satisfied. The set of admissible controls is denoted $D = \{v \in V : \Phi_1(v) = \varphi_1(x(t_1)) = 0\}$.

Consider an auxiliary problem without restrictions based on the Lagrange functional:

$$\dot{x}(t) = f(x(t), u(t), t), x(t_0) = x^0, u(t) \in U, t \in T = [t_0, t_1], \qquad (2.4)$$

$$L^\lambda(u) = \lambda_0 \Phi_0(u) + \lambda_1 \Phi_1(u) \to \inf_{u \in V}, \lambda = (\lambda_0, \lambda_1) \in R^2, \lambda \neq 0. \qquad (2.5)$$

The Pontryagin function with the conjugate variable $\psi \in R^n$ and the standard conjugate system in the Lagrange problem (2.4), (2.5) have the form

$$H^\lambda(\psi, x, u, t) = \langle \psi, f(x, u, t) \rangle - \lambda_0 F_0(x, u, t),$$

$$\begin{aligned}
\dot{\psi}(t) &= -H_x^\lambda(\psi(t), x(t), u(t), t), \quad t \in T, \\
\psi(t_1) &= -\varphi_x^\lambda(x(t_1)), \quad \varphi^\lambda(x) = \lambda_0 \varphi_0(x) + \lambda_1 \varphi_1(x).
\end{aligned} \qquad (2.6)$$

For an available control $v \in V$, let $x(t, v)$, $t \in T$ denote the solution of system (2.1) for $u(t) = v(t)$. We denote by $\psi^\lambda(t, v)$, $t \in T$ the solution of the standard conjugate system (2.6) for $x(t) = x(t, v)$ and $u(t) = v(t)$.

The well-known necessary optimality condition (maximum principle) for an admissible control $v \in V$ in problem (2.1) – (2.3) for some $\lambda \neq 0$ in the notation introduced is written in the form:

$$v(t) = \arg \max_{w \in U} H^\lambda(\psi^\lambda(t, v), x(t, v), w, t), t \in T. \qquad (2.7)$$

As a consequence, this implies the well-known weakened necessary condition (differential maximum principle), which is presented in the projection form:

$$v(t) = P_U(v(t) + \alpha H_u^\lambda(\psi^\lambda(t, v), x(t, v), v(t), t)), t \in T, \alpha > 0. \qquad (2.8)$$

Here we introduce the notation P_U for the operator of projection onto the set $U \subset R^m$ in the Euclidean norm.

Note that condition (2.8) is sufficient to verify for at least one $\alpha > 0$. In the control linear problem (2.1) – (2.3) (the functions $f(x, u, t)$, $F_0(x, u, t)$ are linear in the argument u), conditions (2.7) and (2.8) are equivalent.

The degenerate case ($\lambda_0 = 0$) of necessary optimality conditions in specific problems of optimal control, as a rule, is studied analytically taking into account constraints (2.3). In the regular case ($\lambda_0 = 1$), in order to search for extreme controls, constraint condition (2.3) is added to the necessary optimality conditions, and the resulting systems of equations (2.7), (2.3) and (2.8), (2.3) for a pair of unknown $(v, \lambda_1) \in V \times R$ are solved by numerical methods.

In this paper, for a regular case, we propose a method for constructing a relaxation sequence of admissible controls for which the residual value

of the differential maximum principle in problem (2.1) – (2.3) tends to zero. The differential maximum principle is formulated in terms of a fixed point problem that characterizes the conditions under consideration for the nonlocal improvement of an admissible control.

3. Conditions for improving control

We use the following notation for the partial increment of an arbitrary vector function $g(y_1, ..., y_l)$ with respect to the variables y_{s_1}, y_{s_2}:

$$\Delta_{z_{s_1}, z_{s_2}} g(y_1, ..., y_l) = g(y_1, ..., z_{s_1}, ..., z_{s_2}, ..., y_l) - g(y_1, ..., y_{s_1}, ..., y_{s_2}, ..., y_l).$$

Consider the problem of improving the available control in the regular Lagrange problem (2.4), (2.5): for a given available control $v^I \in V$, it is necessary to find an available control $v \in V$ with the condition $\Delta_v L^\lambda(v^I) = L^\lambda(v) - L^\lambda(v^I) \leq 0$.

In accordance with [2], we introduce a modified differential-algebraic conjugate system including an additional phase variable $y(t) = (y_1(t), ..., y_n(t))$,

$$\dot{p}(t) = -H_x^\lambda(p(t), x(t), u(t), t) - r(t), \tag{3.1}$$

$$\langle H_x^\lambda(p(t), x(t), u(t), t) + r(t), y(t) - x(t)\rangle = \\ = \Delta_{y(t)} H^\lambda(p(t), x(t), u(t), t), \tag{3.2}$$

$$p(t_1) = -\varphi_x^\lambda(x(t_1)) - q, \tag{3.3}$$

$$\left\langle \varphi_x^\lambda(x(t_1)) + q, y(t_1) - x(t_1) \right\rangle = \Delta_{y(t_1)} \varphi^\lambda(x(t_1)), \tag{3.4}$$

in which, by definition, we set $r(t) = 0$, $q = 0$ in the case of linearity of the functions f, F_0, φ_0, φ_1 with respect to x (state-linear problem (2.1) – (2.3)), as well as in the case $y(t) = x(t)$ for the corresponding $t \in T$.

In the state-linear problem (2.1) – (2.3), the modified conjugate system (3.1) – (3.4), by definition, coincides with the standard conjugate system (2.6).

In problem nonlinear in state (2.1) – (2.3), algebraic equations (3.2) and (3.4) can always be analytically solved with respect to the quantities $r(t)$ and q in the form of explicit or conditional formulas (possibly not in a unique way).

Thus, the differential-algebraic conjugate system (3.1) – (3.4) can always be reduced (possibly not uniquely) to a differential conjugate system with uniquely defined quantities $r(t)$ and q.

For the available controls $v \in V$, $v^I \in V$, let $p^\lambda(t, v^I, v)$, $t \in T$ be the solution of the modified adjoint system (3.1) – (3.4) for $x(t) = x(t, v^I)$, $y(t) = x(t, v)$, $u(t) = v^I(t)$. The definition implies the obvious equality $p^\lambda(t, v, v) = \psi^\lambda(t, v)$, $t \in T$.

According to [2], the projection conditions for improving the available control $v^I \in V$ in the Lagrange problem with a given projection parameter $\alpha > 0$ take the form:

$$v(t) = P_U(v^I(t) + \alpha(H_u^\lambda(p^\lambda(t, v^I, v), x(t, v), v^I(t), t) + s(t))), t \in T, \quad (3.5)$$

$$\Delta_{v(t)} H^\lambda(p^\lambda(t, v^I, v), x(t, v), v^I(t), t) =$$
$$= \left\langle H_u^\lambda(p^\lambda(t, v^I, v), x(t, v), v^I(t), t) + s(t), v(t) - v^I(t) \right\rangle, \quad (3.6)$$

in which in equation (3.6), by definition, $s(t) = 0$ is assumed in the case of a linear control problem (2.1) – (2.3), or in the case $v(t) = v^I(t)$ at $t \in T$.

In the nonlinear control problem (2.1) – (2.3), equation (3.6) can always be uniquely analytically solved with respect to $s(t)$ (possibly not in a unique way).

Thus, system (3.5), (3.6) can always be reduced to an equation in the form (3.5) with respect to the control v with the uniquely determined right-hand side. The obtained equation can be interpreted as the fixed point problem with respect to control v for the control operator uniquely determined by the right-hand side of the equation.

According to [2], the solution v of system (3.5), (3.6) provides an improvement in the control $v^I \in V$ for any parameter $\alpha > 0$ with an estimate of the functional improvement:

$$\Delta_v L^\lambda(v^I) \leq -\frac{1}{\alpha} \int_T \left\| v(t) - v^I(t) \right\|^2 dt.$$

Moreover, the improvement of control is guaranteed not only in a sufficiently small neighborhood of the original control $v^I \in V$, i.e. the considered improvement procedure has the property of nonlocality, in contrast to the known gradient methods and other local methods for improving control.

Consider the problem of improving the admissible control in problem (2.1) – (2.3) in the following statement: for a given admissible control $v^I \in D$, we need to find an admissible control $v \in D$ with the condition

$$\Delta_v \Phi_0(v^I) = \Phi_0(v) - \Phi_0(v^I) \leq 0.$$

To implement the problem in the regular case, it is enough to solve the system (3.5), (3.6) with the additional condition that constraint (2.3) is fulfilled. In this case, the improvement will be carried out with the assessment:

$$\Delta_v \Phi_0(v^I) \leq -\frac{1}{\alpha} \int_T \left\| v(t) - v^I(t) \right\|^2 dt. \quad (3.7)$$

Conditions for improving control (3.5), (3.6), (2.3) with a given method for the unambiguous resolution of equations (3.2), (3.4) and (3.6) with

respect to the corresponding quantities $r(t)$, q and $s(t)$ can be considered as a fixed point problem relative to control v with an additional algebraic equation (2.3). Such an interpretation makes it possible to apply and modify the known [7] fixed-point search algorithms for solving the improvement problem on the set of admissible controls.

4. Conditions for optimal control

Between the necessary optimality conditions for an admissible control in the regular case and the conditions for nonlocal improvement of an admissible control (3.5), (3.6), (2.3), a connection can be established that is determined by the following statements.

Let $D^\alpha(v^I) \subseteq D$ be the set of fixed points of problem (3.5), (3.6) satisfying condition (2.3).

Theorem 1. *In problem (2.1) – (2.3), the control $v^I \in D$ satisfies the necessary optimality condition (2.8) if and only if there is $\alpha > 0$ for which the condition*

$$v^I \in D^\alpha(v^I). \tag{4.1}$$

Proof. Let $v^I \in D^\alpha(v^I)$ for some $\alpha > 0$, then obviously v^I satisfies condition (2.8) with the factor $\lambda_1 \in R$ for which the pair (v^I, λ_1) satisfies conditions (3.5), (3.6), (2.3). Conversely, let $v^I \in D$ satisfy condition (2.8) with some $\lambda_1 \in R$. Then control v^I is a solution to system (3.5), (3.6), (2.3) at $v = v^I$ for all $\alpha > 0$. □

Corollary 1. *Let the control $v^I \in D$ be optimal in the regular problem (2.1) – (2.3). Then there exists $\alpha > 0$ for which condition $v^I \in D^\alpha(v^I)$ is satisfied.*

From the obtained statements, other simple statements in the regular problem (2.1) – (2.3) follow.

1) The fixed point problem (3.5), (3.6) with the additional equation (2.3) is always solvable for a control satisfying the differential maximum principle.

2) In the case of non-uniqueness of the solution of the fixed point problem (3.5), (3.6), (2.3) for a control satisfying the differential maximum principle, this control can be strictly improved by virtue of estimate (3.7).

3) The absence of fixed points in problem (3.5), (3.6), (2.3) indicates the non-optimal control.

Evaluation (3.7) of the functional improvement makes it possible to obtain a strengthened necessary condition for optimality of control in problem (2.1) – (2.3) in comparison with the differential maximum principle.

Theorem 2. *Let the control $v^I \in D$ be optimal in the regular problem (2.1) – (2.3). Then for all $\alpha > 0$ the condition is satisfied:*

$$D^\alpha(v^I) = \{v^I\}. \tag{4.2}$$

Proof. If $v \in D^\alpha(v^I)$, $v \neq v^I$ exists for some $\alpha > 0$, then according to estimate (3.7) the control v strictly improves v^I, which contradicts the optimality of the control. $\qquad\square$

Note that in the control linear problem (2.1) – (2.3), condition (4.2) strengthens the maximum principle (2.7).

5. Iterative algorithm

To solve the fixed point problem (3.5), (3.6), (2.3), it is proposed to use the following modification of the simple iteration algorithm for $k \geq 0$:

$$v^{k+1}(t) = P_U(v^I(t) + \alpha(H_u^\lambda(p^\lambda(t, v^I, v^k), x(t, v^k), v^I(t), t) + s(t))), \quad (5.1)$$

$$\begin{aligned}\Delta_{v^k(t)} H^\lambda(p^\lambda(t, v^I, v^k), x(t, v^k), v^I(t), t) = \\ = \langle H_u^\lambda(p^\lambda(t, v^I, v^k), x(t, v^k), v^I(t), t) + s(t), v^k(t) - v^I(t) \rangle,\end{aligned} \tag{5.2}$$

$$\varphi_1(x(t_1, v^{k+1})) = 0. \tag{5.3}$$

For $k = 0$, the initial approximation $v^0 \in V$ is specified, for which, in the practice of computing, the control $v^I \in V$ is usually chosen.

In this modification, at each iteration of the algorithm, the exact fulfillment of constraint (2.3) is required in contrast to other algorithms [4] for solving similar problems on a fixed point with an additional condition for fulfilling the constraint. The solution of equation (5.3) at each iteration of algorithm (5.1) – (5.3) reduces to solving the implicit equation with respect to the scalar Lagrange multiplier $\lambda_1 \in R$.

The convergence conditions for the iterative process (5.1) – (5.3) can be obtained similarly to [3; 7] based on the requirements that provide the well-known "compression" property for the operator of the right-hand side of the fixed point problem.

Iterations over the $k \geq 0$ index are carried out until the first strict improvement of the $v^I \in V$ control over the target functional: $\Phi_0(v^{k+1}) < \Phi_0(v^I)$. Next, a new fixed-point problem is constructed to improve the obtained design control and the iterative process is repeated.

If a strict control improvement does not occur, then the iterative process is carried out until the condition:

$$\left\| v^{k+1} - v^k \right\|_{C(T)} \leq \varepsilon,$$

where $\varepsilon > 0$ is given the accuracy of calculating the fixed point problem. At this iteration of the calculation of successive problems, the improvements in the control of the proposed algorithm end.

The control sequence u^s, $s \geq 0$ formed as a result of the calculation by the objective functional can start from any available starting control $v^I \in V$ for the initial problem of improving control. Beginning with the second improvement problem, improved v^I control becomes valid: $v^I \in D$. Thus, only the initial term developed by the method of constructing the relaxation sequence u^s can be an unacceptable control. The possible inadmissibility of the start control for index $s = 0$ greatly simplifies the implementation of the proposed method for finding extreme controls.

Let us analyze the convergence of the relaxation sequence u^s constructed in the class of admissible controls.

For each index $s \geq 1$, we consider the quantity

$$\delta(u^s) = \Phi_0(u^s) - \Phi_0(u^{s+1}) \geq 0.$$

If $\delta(u^s) = 0$, then, by virtue of estimate (3.7), we obtain that $u^s(t) = u^{s+1}(t)$, $t \in T$, i.e. the control u^s satisfies the condition of the differential maximum principle (2.8). Thus, the quantity $\delta(u^s)$ in the regular problem $(2.1) - (2.3)$ can be interpreted as the residual (measure) of the differential principle of maximum for the control u^s.

Theorem 3. *Suppose that in the regular problem (2.1) – (2.3) the family of phase trajectories of system (2.1) in the aggregate is bounded:*

$$x(t, u) \in X, t \in T, u \in V,$$

where the set $X \in R^n$ is convex and compact. Then the relaxation sequence of admissible controls u^s for $s \geq 1$ converges in the residual of the differential maximum principle:

$$\delta(u^s) \to 0, s \to \infty.$$

Proof. Due to the boundedness of the family of phase trajectories, the sequence $\Phi_0(u^s)$ for the index $s \geq 1$ is bounded below. Therefore, taking into account relaxation, this sequence is convergent, i.e.

$$\delta(u^s) = \Phi_0(u^s) - \Phi_0(u^{s+1}) \to 0, s \to \infty.$$

$\square$

6. Example

The comparative effectiveness of the proposed method for finding extreme controls is illustrated by the well-known example of a linear problem with two bilateral phase constraints [6; 9]:

$$\begin{cases} \dot{x}_1(t) = u_1(t), \\ \dot{x}_2(t) = x_1(t) + u_2(t), \end{cases} \quad \begin{cases} x_1(0) = -1, \\ x_2(0) = 2, \end{cases}$$

$$-8 \le x_1(t) \le 0, -4 \le x_2(t) \le 2, t \in T = [0, 10],$$

$$\Phi(u) = 3x_1(10) - x_2(10) \to \inf_{u \in V},$$

$$V = \{u = (u_1, u_2) \in PC(T) : u(t) \in U, t \in T\},$$

$$U = \{u = (u_1, u_2) : |u_1| \le 1, \quad 1 \le u_2 \le 2\}.$$

By introducing additional phase variables using cubic penalty functions, the problem reduces to an equivalent problem with one terminal restriction-equality:

$$\Phi_0(u) = 3x_1(10) - x_2(10) \to \inf_{u \in V},$$

$$\Phi_1(u) = x_3(10) + x_4(10) = 0.$$

$$\begin{cases} \dot{x}_1(t) = u_1(t), \\ \dot{x}_2(t) = x_1(t) + u_2(t), \\ \dot{x}_3(t) = Q_1(x_1(t)), \\ \dot{x}_4(t) = Q_2(x_2(t)), \end{cases} \quad \begin{cases} x_1(0) = -1, \\ x_2(0) = 2, \\ x_3(0) = 0, \\ x_4(0) = 0, \end{cases} \quad t \in T,$$

$$Q_1(x_1) = \begin{cases} x_1^3, & x_1 > 0, \\ 0, & x_1 \in [-8, 0], \\ (-x_1 - 8)^3, & x_1 < -8, \end{cases} \quad Q_2(x_2) = \begin{cases} (x_2 - 2)^3, & x_2 > 2, \\ 0, & x_2 \in [-4, 2], \\ (-x_2 - 4)^3, & x_2 < -4, \end{cases}$$

Consider the regular case of the Lagrange problem with the multiplier $\lambda_1 \in R$:

$$L^\lambda(u) = (3x_1(10) - x_2(10)) + \lambda_1(x_3(10) + x_4(10)) \to \inf.$$

The Pontryagin function and the differential-algebraic conjugate system for the Lagrange problem take the following form:

$$H(p, x, u, t) = p_1 u_1 + p_2(x_1 + u_2) + p_3 Q_1(x_1) + p_4 Q_2(x_2),$$

$$\begin{cases} \dot{p}_1(t) = -p_2(t) - p_3(t) G_1(x_1(t)) - r_1(t), \\ \dot{p}_2(t) = -p_4(t) G_2(x_2(t)) - r_2(t), \\ \dot{p}_3(t) = -r_3(t), \\ \dot{p}_4(t) = -r_4(t), G_i(x_i) = \frac{\partial Q_i}{\partial x_i}, i = 1, 2, \end{cases} \quad \begin{cases} p_1(10) = -3, \\ p_2(10) = 1, \\ p_3(10) = -\lambda_1, \\ p_4(10) = -\lambda_1, \end{cases}$$

Moreover, the quantity $r(t) = (r_1(t), \ r_2(t), \ r_3(t), r_4(t))$ is determined from the algebraic equation with an additional phase variable $z(t) = (z_1(t), z_2(t), z_3(t), z_4(t))$:

$$p_2(t)(z_1(t) - x_1(t)) + p_3(t)(Q_1(z_1(t)) - Q_1(x_1(t))) +$$

$$+p_4(t)(Q_2(z_2(t)) - Q_2(x_2(t))) =$$

$$= (p_2(t) + p_3(t)G_1(x_1(t)) + r_1(t))(z_1(t) - x_1(t)) +$$

$$+ (p_4(t)G_2(x_2(t)) + r_2(t))(z_2(t) - x_2(t)) + r_3(t)(z_3(t) - x_3(t)) +$$

$$+r_4(t)(z_4(t) - x_4(t)).$$

We fix the following method of unambiguous resolution of the quantity $r(t) = (r_1(t), \ r_2(t), \ r_3(t), r_4(t))$:

1) if $z_1(t) \neq x_1(t)$, then $r_2(t) = 0, r_3(t) = 0, r_4(t) = 0$, and $r_1(t)$ analytically determined in the form of a formula from the above equation.

2) if $z_1(t) = x_1(t)$ and $z_2(t) \neq x_2(t)$, then $r_1(t) = 0, r_3(t) = 0, r_4(t) = 0$, and $r_2(t)$ determined as a formula from the equation.

3) if $z_1(t) = x_1(t)$ and $z_2(t) = x_2(t)$, then $r_1(t) = 0, r_2(t) = 0, r_3(t) = 0, r_4(t) = 0$.

From here we get $p_3(t) = -\lambda_1$, $p_4(t) = -\lambda_1$, $t \in [0, 10]$.

Due to the linearity of the Lagrange problem for control, the fixed-point problem for improving control $v^I = (v_1^I, v_2^I) \in V$ with an additional condition for fulfilling the terminal constraint takes the form:

$$(v_1(t), v_2(t)) = P_U(v_1^I(t) + \alpha p_1(t, v_1^I, v), v_2^I(t) + \alpha p_2(t, v_2^I, v)),$$

$$x_3(10, v) + x_4(10, v) = 0.$$

Where $\alpha > 0$ – preset projection parameter.

To solve the fixed-point problem with an additional algebraic equation, we considered the process of simple iteration at $k \geq 0$ with the initial approximation at $v^0 \in V$ for $k = 0$:

$$(v_1^{k+1}(t), v_2^{k+1}(t)) = P_U(v_1^I(t) + \alpha p_1(t, v_1^I, v^k), v_2^I(t) + \alpha p_2(t, v_2^I, v^k)),$$

$$x_3(10, v^{k+1}) + x_4(10, v^{k+1}) = 0.$$

At each iteration of the process, the implementation of the constraint reduces to solving an implicit equation with respect to the factor $\lambda_1 \in R$.

The computational implementation of the extremal control search algorithm is characterized by the following features.

The phase and conjugate Cauchy problems were numerically solved using the Runge–Kutta–Werner method of variable (5–6) order of accuracy using

the DIVPRK program of the IMSL Fortran PowerStation 4.0 library [1]. The values of controlled, phase and conjugate variables were stored in nodes of a fixed uniform grid T_h with a sampling step $h > 0$ on the interval T. In the intervals between adjacent nodes of the grid T_h, the value of the control function was assumed to be constant and equal to the value in the left node.

A numerical solution of the algebraic equation with respect to the parameter $\lambda_1 \in R$ was carried out using the DUMPOL program [1], which implements the method of a deformable polyhedron. The accuracy of the solution of the equation was controlled by the criterion:

$$\Gamma = \max\{\Gamma_1(x_1(t, v^{k+1})), \ \Gamma_2(x_2(t, v^{k+1})), t \in T_h\} \le \varepsilon_1,$$

where $\varepsilon_1 > 0$ – specified the accuracy of the implementation of the corresponding phase constraints, $\Gamma_i(x_i) = \sqrt[3]{Q_i(x)}$, $i = 1, 2$.

The iterations of calculating the problem of a fixed point in $k \ge 0$ continued until the first implementation of a strict improvement of the control of $v^I \in V$:

$$L^\lambda(v^{k+1}) < L^\lambda(v^I).$$

In this case, a new fixed point problem was constructed to improve the obtained calculated control, and the iterative process was repeated. In this case, as the initial approximation of the control $v^0 \in V$ for $k = 0$ in the iterative process, the obtained calculated control was chosen.

If a strict improvement in control was not achieved, then the numerical calculation of the fixed point problem was carried out before the condition

$$\max\{|v_1^{k+1}(t) - v_1^k(t)|, |v_2^{k+1}(t) - v_2^k(t)|, t \in T_h\} \le \varepsilon_2,$$

where $\varepsilon_2 > 0$ is given the accuracy of calculating the fixed point problem. On this, the process of constructing and calculating successive problems to improve control ended.

The calculation was carried out with various available starting controls $v^I \in V$ for the initial problem of improving control, various discretization steps $h > 0$ and project parameters $\alpha > 0$. In particular, the control $\bar{v}^I \in V$, which is an approximate copy of the calculation control shown in the figure in the work [9], was considered as one of the starting controls. As another, the initial approximation $(v_1^I \equiv -1, v_2^I \equiv 1)$ used in the calculations was used [6].

The specificity of the linearity of the control problem determines the increased requirements for the accuracy of the calculation of controlled variables. The control values according to the calculation formula of the iterative process in the problem under consideration depend only on the values of the conjugate variables. Therefore, to increase the accuracy of calculating the conjugate system, the step $h > 0$ of the discretization grid
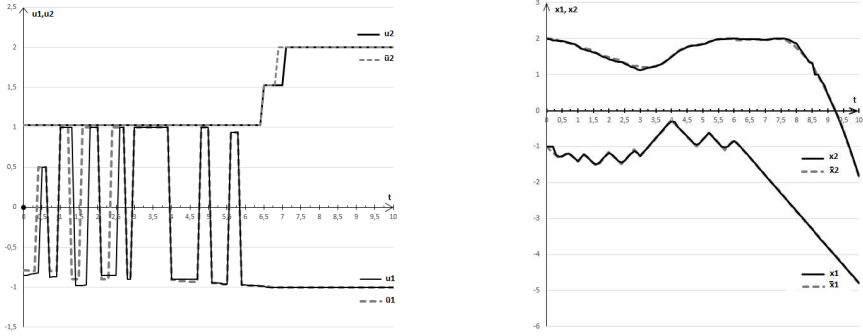
Figure 1. $u1$, $u2$ and $x1$, $x2$ are trajectories of calculated controlled and phase variables for start control ($v_1^I \equiv 0$, $v_2^I \equiv 1$). $\bar{u}1$, $\bar{u}2$ and $\bar{x}1$, $\bar{x}2$ are trajectories of calculated controlled and phase variables for start control $\bar{v}^I$.

T_h, at which the phase variables necessary for calculating the conjugate system are stored, was chosen to be sufficiently small.

Table 1 shows the calculation results for the three start controls $v^I \in V$ for $h = 10^{-4}$, $\alpha = 10^{-2}$, $\varepsilon_1 = 10^{-2}$, $\varepsilon_2 = 10^{-16}$. Here Φ^I and Γ^I are the values of the target functional and the Γ index of the phase deviation at the start control, Φ^* and Γ^* are the calculated values of the target functional and Γ index of the phase deviation. N is the total number of the calculated phase and conjugate Cauchy problems.

With the decreasing step $h > 0$ of the discretization grid T_h, the accuracy of the approximation of the controlled variables increases. In this case, the calculated values of the target functional improve and come closer to the calculated optimal value $\Phi^* \approx -12.5$ indicated in [6; 9]. As the step $h > 0$ increases, the calculated Φ^* values deteriorate. The qualitative structure of the calculated controlled and phase variables do not change.

Figure 1 show the calculated controlled variables $u_1(t)$, $u_2(t)$ and the corresponding phase trajectories $x_1(t)$, $x_2(t)$ obtained by the proposed method for the two starting controls indicated in Table 1.

Table 1

Calculation results

v^I	Φ^I	Γ^I	Φ^*	Γ^*	N
$\bar{v}^I$	-9.2	3.1	-12.4232	1.0×10^{-2}	134
($v_1^I \equiv -1$, $v_2^I \equiv 1$)	-7.3	3.4	-12.0988	1.0×10^{-2}	98218
($v_1^I \equiv 0$, $v_2^I \equiv 1$)	-6.1	4.9	-12.1225	1.0×10^{-2}	102236

On the whole, the convergence of the construct relaxation sequence substantially depends on the tuning parameter $\alpha > 0$, which is selected experimentally for a specific optimal control problem. As this parameter

decreases, the total number N of Cauchy computational problems increases and the convergence rate of the iterative process slows down. With an increase in the parameter $\alpha > 0$, the quality of the calculated control deteriorates until the convergence is lost. This parameter also regulates a fairly wide area of convergence in the initial starting control. The calculation of the model problem demonstrates the computational and qualitative efficiency of the proposed method that is acceptable for practice in comparison with the known methods [6;9] that implement combined multi-method computing technologies.

7. Conclusion

The developed method for searching for extremal controls in the considered class of problems with constraints is characterized by the following features: nonlocality of improvement of controls; the absence of a time-consuming procedure of needle or convex variation of control in a small neighborhood of the improved control characteristic of gradient methods; exact implementation of restrictions; the fundamental possibility of a rigorous improvement of non-optimal controls that satisfy the differential maximum principle; the presence of one main tuning parameter $\alpha > 0$, which regulates the speed, quality, and region of convergence of the iterative process. The relaxation sequence of admissible controls constructed by the proposed method, under broad assumptions, converges in magnitude characterizing the measure of fulfilling the necessary optimality conditions. The indicated properties are essential factors for increasing the computational and qualitative efficiency of solving nonlinear optimal control problems with restrictions.

References

1. Bartenev O.V. *Fortran dlya professionalov. Matematicheskaya biblioteka IMSL. Chast' 2* [Fortran for Professionals. Mathematical Library IMSL. Part 2]. Moscow, Dialog-MIFI Publ., 2001, 320 p. (in Russian)
2. Buldaev A.S. Metody nepodvizhnykh tochek na osnove operacij proektirovaniya v zadachakh optimizacii upravlyayushchikh funkcij i ametrov dinamicheskikh sistem [Fixed-Point Methods Based on Projection Operations in Optimization Problems for Control Functions and parameters of Dynamical Systems] *Vestnik Buryatskogo gosudarstvennogo universiteta. Matematika, informatika* [Bulletin of the Buryat State University. Mathematics, Computer Science], 2017, no. 1, pp. 38-54. http://journals.bsu.ru/doi/10.18101/2304-5728-2017-1-38-54. (in Russian)
3. Buldaev A.S. *Metody vozmushchenij v zadachakh uluchsheniya i optimizatsii upravlyaemykh sistem* [Perturbation Methods in Improvement and Optimization Problems for Controllable Systems]. Ulan-Ude, Publishing House of the Buryat State University, 2008, 260 p. (in Russian)
4. Buldaev A.S., Burlakov I.D. About One Approach to Numerical Solution of Nonlinear Optimal Speed Problems. *Bulletin of the South Ural State University. Series*

Mathematical Modeling, Programming & Computer Software, 2018, vol. 11, no. 4, pp. 55-66. https://doi.org/10.14529/mmp180404

5. Buldaev A.S., Khishektueva I.-Kh. The Fixed Point Method in ametric Optimization Problems for Systems. *Automation and Remote Control*, 2013, vol. 74, no. 12, pp. 1927-1934. https://doi.org/10.1134/S0005117913120011

6. Grachev N.I., *Filkov A.N. Reshenie zadach optimal'nogo upravleniya v sisteme DISO* [The Solution of Optimal Control Problems in the DISO System]. Moscow, Publishing House of Computer Center of the Academy of Sciences USSR, 1986, 66 p. (in Russian)

7. Samarsky A.A., Gulin A.V. *Chislennye metody* [Numerical Nethods]. Moscow, Nauka Publ., 1989, 432 p. (in Russian)

8. Srochko V.A. *Iteracionnye metody resheniya zadach optimal'nogo upravleniya* [Iterative Methods for Solving Optimal Control Problems]. Moscow, Fizmatlit Publ., 2000, 160 p. (in Russian)

9. Tyatyushkin A.I. *Mnogometodnaya tekhnologiya optimizacii upravlyaemyh sistem* [Multi-Method Technology for Optimizing Controlled Systems]. Novosibirsk, Nauka Publ., 2006, 343 p. (in Russian)

10. Vasiliev O.V. *Lekcii po metodam optimizacii* [Lectures on Optimization Methods]. Irkutsk, Publishing House of Irkutsk State University, 1994, 340 p. (in Russian)

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Об одном методе поиска экстремальных управлений в системах с ограничениями

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Аннотация.   В классе управляемых систем с ограничениями конструируются и анализируются условия улучшения и оптимальности управления в форме задач о неподвижной точке. Такая форма позволяет получить усиленные необходимые условия оптимальности по сравнению с известными условиями, дает возможность применить и модифицировать теорию и методы неподвижных точек для поиска экстремальных управлений в задачах оптимизации рассматриваемого класса. Задачи о неподвижной точке строятся с помощью перехода к вспомогательным задачам оптимального управления без ограничений с функционалами Лагранжа. Предлагается итерационный алгоритм построения релаксационной последовательности допустимых управлений на основе решения конструируемых задач о неподвижной точке. Рассматриваемый алгоритм характеризуется свойствами нелокального улучшения допустимого управления и принципиальной возможностью строгого улучшения неоптимальных управлений, удовлетворяющих известным необходимым условиям

оптимальности, в отличие от градиентных и других локальных методов. Обосновываются условия сходимости последовательности управлений по невязке выполнения необходимых условий оптимальности. Проводится сравнительный анализ вычислительной и качественной эффективности предлагаемого итерационного алгоритма поиска экстремальных управлений в модельной задаче с фазовыми ограничениями.

Ключевые слова: управляемая система с ограничениями, экстремальные управления, условия улучшения управления, задача о неподвижной точке, итерационный алгоритм.

Список литературы

1. Бартеньев О. В. Фортран для профессионалов. Математическая библиотека IMSL. Ч. 2. М. : Диалог-МИФИ, 2001. 320 с.

2. Булдаев А. С. Методы неподвижных точек на основе операций проектирования в задачах оптимизации управляющих функций и параметров динамических систем // Вестник Бурятского государственного университета. Математика, информатика. 2017. № 1. С. 38–54. http://journals.bsu.ru/doi/10.18101/2304-5728-2017-1-38-54.

3. Булдаев А. С. Методы возмущений в задачах улучшения и оптимизации управляемых систем. Улан-Удэ : Изд-во Бурятского госуниверситета, 2008. 260 с.

4. Buldaev A. S., Burlakov I. D. About One Approach to Numerical Solution of Nonlinear Optimal Speed Problems // Bulletin of the South Ural State University. Series Mathematical Modeling, Programming & Computer Software. 2018. Vol. 11, N 4. P. 55–66. https://doi.org/10.14529/mmp180404

5. Buldaev A. S., Khishektueva I.-Kh. The Fixed Point Method in ametric Optimization Problems for Systems // Automation and Remote Control. 2013. Vol. 74, N 12. P. 1927–1934. https://doi.org/10.1134/S0005117913120011

6. Грачев Н. И., Фильков А. Н. Решение задач оптимального управления в системе ДИСО. М. : ВЦ АН СССР, 1986. 66 с.

7. Самарский А. А., Гулин А. В. Численные методы. М. : Наука, 1989. 432 с.

8. Срочко В. А. Итерационные методы решения задач оптимального управления. М. : Физматлит, 2000. 160 с.

9. Тятюшкин А. И. Многометодная технология оптимизации управляемых систем. Новосибирск : Наука, 2006. 343 с.

10. Васильев О.В. Лекции по методам оптимизации. Иркутск : Изд-во Иркутского университета, 1994. 340 с.

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