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Ultraparabolic Equations with Operator Coefficients at the Time Derivatives *

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Abstract. The article is devoted to the study of the solvability of boundary value problems for third-order Sobolev-type differential equations of the third order with two time variables (such equations are also called composite-type equations or equations not solved for the derivative). The peculiarities of the equations under study are, firstly, that the differential operators acting at the time derivatives are not assumed inverse, and, secondly, that the statements of boundary value problems for them are determined by the coefficients of these differential operators. For the problems proposed in the article, we prove existence and uniqueness theorems for regular solutions (solutions having all weak derivatives in the sense of Sobolev involved in the equation). The technique of proving the existence theorems is based on a special regularization of the equations under study, a priori estimates, and passage to the limit.

Keywords: ultraparabolic equations, irreversible operator coefficients, boundary problems, regular solutions, existence, uniqueness.

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1. Introduction

The equations studied in the article can be called Sobolev-type equations, or composite-type equations. Among the numerous works on the theory of Sobolev-type equations (see [1–7; 10; 11; 13]), distinguish the works devoted to equations with degenerate (noninvertible) operator at the time derivative — see [2; 10; 13]. In the present paper, we also consider equations with noninvertible operators at the higher part but, firstly, the nature of the noninvertibility of the operator coefficients is different than in the works of the predecessors, and, secondly, in contrast to the numerous earlier works, we study equations with two time variables.

2. Statement of the Problem

Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ (of class C^2), T and A are given positive numbers, Q is the cylinder $\Omega \times (0,T)\times(0,A)$ of variables x, t, a, $S = \Gamma\times(0,T)\times(0,A)$ is the lateral boundary of Q, $\alpha_0(t,a)$, $\alpha_1(t,a)$, $\beta_0(t,a)$, $\beta_1(t,a)$, $m^{ij}(x,t,a)$, $m^i(x,t,a)$, $i,j = 1,\dots,n$, $m_0(x,t,a)$, $f(x,t,a)$ are given functions, defined for $x \in \overline{\Omega}$, $t \in [0,T]$, $a \in [0,A]$. Furthermore, let L_α, L_β, and M be differential operators whose action at a given function $v(x,t,a)$ is defined by the equalities

$$L_\alpha v = \alpha_0(t,a)v + \alpha_1(t,a)\Delta v, \quad L_\beta v = \beta_0(t,a)v + \beta_1(t,a)\Delta v,$$

$$Mv = m^{ij}(x,t,a)v_{x_i x_j} + m^i(x,t,a)v_{x_i} + m_0(x,t,a)v$$

(Δ is the Laplace operator with respect to the variables $x_1, \dots, x_n$, Here and below, repeating indices imply summation from 1 to n).

The aim of the article is the study of the solvability of boundary value problems for the equations

$$L_\alpha u_t + L_\beta u_a - Mu = f(x,t,a). \tag{1}$$

Introduce the notations

$$\gamma_1 = \{(x,t,a):\ x\in\Omega,\ t=0,\ a\in(0,A)\},$$

$$\gamma_2 = \{(x,t,a):\ x\in\Omega,\ t=T,\ a\in(0,A)\},$$

$$\gamma_3 = \{(x,t,a):\ x\in\Omega,\ t\in(0,T),\ a=0\},$$

$$\gamma_4 = \{(x,t,a):\ x\in\Omega,\ t\in(0,T),\ a=A\}.$$

As we will show below, in well-posed boundary value problems for equations (1), on each of the sets γ_i, $i=\overline{1,4}$, boundary conditions can be given or not given.

Let l_i, $i = \overline{1,4}$, be numbers equal to 0 or 1. Refer as the $P_{l_1 l_2 l_3 l_4}$*-condition for equation (1) to the condition that the value of the solution* $u(x,t,a)$ *is given on* γ_i *if* $l_i = 1$*, and, respectively, the value* $u(x,t,a)$ *is not given if* $l_i = 0$.

The Boundary Value Problem $P_{l_1 l_2 l_3 l_4}$: *Find a function* $u(x,t,a)$ *that is a solution to equation (1) in the cylinder* Q *and satisfies the* $P_{l_1 l_2 l_3 l_4}$*-condition and also the condition*

$$u(x,t,a)|_S = 0. \tag{2}$$

Obviously, there are 16 different problems of the given form. But it is also obvious that, among these problems, there are those similar to each other, or those reducible to one another by the change $t' = T - t$ or $a' = A - a$ (for example, the problems P_{1100} and P_{0011} are in essence identical, the problems P_{1000} and P_{0100} are reduced to each other by the change $t' = T - t$ etc.). An easy analysis makes it possible to distinguish six basic problems among all the problems $P_{l_1 l_2 l_3 l_4}$ — the problems P_{1111}, P_{1110}, P_{1100}, P_{1010}, P_{1000}, and P_{0000}. It is for these problems that we will prove existence theorems for regular solutions below.

Denote by V_0 the linear space of functions $v(x,t,a)$ belonging to $L_2(Q)$ and such that their weak derivatives $v_t(x,t,a)$, $v_a(x,t,a)$, $v_{x_i}(x,t,a)$, $v_{x_i t}(x,t,a)$, $v_{x_i a}(x,t,a)$, $v_{x_i x_j}(x,t,a)$, $v_{x_i x_j t}(x,t,a)$, $v_{x_i x_j a}(x,t,a)$, $i,j = 1,\dots,n$, exist and also belong to $L_2(Q)$. Normalize this space:

$$\|v\|_{V_0} = \left(\int\limits_Q \left[v^2 + v_t^2 + v_a^2 + \sum_{i=1}^n \left(v_{x_i}^2 + v_{x_i t}^2 + v_{x_i a}^2 \right) + \right.\right.$$

$$\left.\left. + \sum_{i,j=1}^n \left(v_{x_i x_j}^2 + v_{x_i x_j t}^2 + v_{x_i x_j a}^2 \right) \right] dx\,dt\,da \right)^{1/2}.$$

Obviously, endowed with this norm, V_0 becomes a Banach space.

It is in the Banach space V_0 that we will establish the solvability of the boundary value problems under study.

3. Solvability of the Boundary Value Problems P_{1111} and P_{0000}

The boundary value problems P_{1111} and P_{0000} can be called dual to each other — in the first of them, boundary conditions are given on all the sets γ_1, γ_2, γ_3, and γ_4 (and hence, with account taken of condition (2), the boundary value problem P_{1111} becomes the Dirichlet problem with defining the boundary data on the whole boundary of the cylinder Q), and in the second, — on the contrary, all the sets γ_1, γ_2, γ_3, and γ_4 are free from boundary data, and hence no boundary conditions are given with respect to the time variables t and a.

Let $w(x)$ be a function of the space $\overset{\circ}{W}{}_2^1(\Omega)$. We have the inequality

$$\int_\Omega w^2(x)\,dx \le d_0 \sum_{i=1}^n \int_\Omega w_{x_i}^2(x)\,dx, \tag{3}$$

in which the number d_0 is defined only by the domain Ω [12, Chapter I, § 9; 4, Chapter II, § 2].

Theorem 1. *Suppose the fulfillment of the following conditions:*

I) $m^{ij}(x,t,a) \in C^2(\overline{Q})$, $m^i(x,t,a) \in C^1(\overline{Q})$, $i,j = 1,\dots,n$, $m_0(x,t,a) \in C^1(\overline{Q})$, $\alpha_0(t,a) \in C^1(\overline{D})$, $\alpha_1(t,a) \in C^1(\overline{D})$, $\beta_0(t,a) \in C^1(\overline{D})$, $\beta_1(t,a) \in C^1(\overline{D})$;

II) $m^{ij}(x,t,a) = m^{ji}(x,t,a)$, $i,j = 1,\dots,n$, $(x,t,a) \in \overline{Q}$, $m^{ij}(x,t,a)\xi_i\xi_j \ge \mu_0|\xi|^2$, $\mu_0 > 0$, $(x,t,a) \in \overline{Q}$, $\xi \in \mathbb{R}^n$;

III) $\frac{1}{2}m^i_{x_i}(x,t,a) - \frac{1}{2}m^{ij}_{x_ix_j}(x,t,a) - m_0(x,t,a) \ge \mu_1 > 0$, $(x,t,a) \in \overline{Q}$;

IV) $\mu_0 - \frac{1}{2}\alpha_{1t}(t,a) - \frac{1}{2}\beta_{1a}(t,a) > 0$, $\mu_1 - \frac{1}{2}\alpha_{0t}(t,a) - \frac{1}{2}\beta_{1a}(t,a) > 0$, $(t,a) \in \overline{D}$;
$\left[\mu_1 + \frac{1}{2}\alpha_{0t}(t,a) - \frac{1}{2}\beta_{0a}(t,a)\right]\xi_0^2 + \left[\alpha_{0a}(t,a) + \beta_{0t}(t,a)\right]\xi_0\eta_0 +$
$\left[\mu_1 - \frac{1}{2}\alpha_{0t}(t,a) + \frac{1}{2}\beta_{0a}(t,a)\right]\eta_0^2 \ge 0$, $(t,a) \in \overline{D}$, $\xi_0 \in \mathbb{R}$, $\eta_0 \in$;
$\left[\mu_0 - \frac{1}{2}\alpha_{1t}(t,a) + \frac{1}{2}\beta_{1a}(t,a)\right]|\xi|^2 - \left[\alpha_{1a}(t,a) + \beta_{1t}(t,a)\right]\xi_i\eta_i +$
$\left[\mu_0 + \frac{1}{2}\alpha_{1t}(t,a) - \frac{1}{2}\beta_{1a}(t,a)\right]|\eta|^2 \ge \mu_2\left(|\xi|^2 + |\eta|^2\right)$, $\mu_2 > 0$, $(t,a) \in \overline{D}$, $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^n$;

V) $\alpha_1(0,a) \ge 0$, $\alpha_0(0,a) = \alpha_{01}(a) + \alpha_{02}(a)$, $\alpha_{01}(a) \le 0$, $\alpha_{02}(a) \ge 0$, $\alpha_1(T,a) \le 0$, $\alpha_0(T,a) = \alpha_{03}(a) + \alpha_{04}(a)$, $\alpha_{03}(a) \ge 0$, $\alpha_{04}(a) \le 0$, $\alpha_1(T,a) - d_0\alpha_{04}(a) \le 0$, $a \in [0,A]$;

VI) $\beta_1(t,0) \ge 0$, $\beta_0(t,0) = \beta_{01}(t) + \beta_{02}(t)$, $\beta_{01}(t) \le 0$, $\beta_{02}(t) \ge 0$, $\beta_1(t,0) - d_0\beta_{02}(t) \ge 0$, $\beta_1(t,A) \le 0$, $\beta_0(t,A) = \beta_{03}(t) + \beta_{04}(t)$, $\beta_{03}(t) \ge 0$, $\beta_{04}(t) \le 0$, $\beta_1(t,A) - d_0\beta_{04}(t) \le 0$, $t \in [0,T]$.

Then, for any function $f(x,t,a)$ such that $f(x,t,a) \in L_2(Q)$, $f_t(x,t,a) \in L_2(Q)$, $f_a(x,t,a) \in L_2(Q)$, the boundary value problem P_{0000} has a unique solution in V_0.

Proof. Make use of the regularization method. Let ε be a positive number and let L_ε be the differential operator whose action at a given function $v(x,t,a)$ is defined by the equality

$$L_\varepsilon v = \varepsilon(\Delta v_{tt} + \Delta v_{aa} - \Delta v) + L_\alpha v_t + L_\beta v_a - Mv.$$

Consider the boundary value problem: *Find a function $u(x,t,a)$ that is a solution in the cylinder Q to the equation*

$$L_\varepsilon u = f(x,t,a) \tag{4}$$

and satisfies (2) and also the conditions

$$u_t(x,0,a) = u_t(x,T,a) = 0, \quad x \in \Omega, \quad a \in (0,A), \tag{5}$$

$$u_a(x,t,0) = u_a(x,t,A) = 0, \quad x \in \Omega, \quad t \in (0,T). \tag{6}$$

Define the linear space $\widetilde{V}_0$ as the set of functions in V_0 whose weak derivatives $v_{tt}(x,t,a)$, $v_{aa}(x,t,a)$, $v_{x_i tt}(x,t,a)$, $v_{x_i aa}(x,t,a)$, $v_{x_i x_j tt}(x,t,a)$, $v_{x_i x_j aa}(x,t,a)$, $i,j = 1,\dots,n$, exist and belong to $L_2(Q)$. Normalize the space $\widetilde{V}_0$:

$$\|v\|_{\widetilde{V}_0} = \left(\|v\|^2_{V_0} + \sum_{i,j=1}^{n} \int_Q v^2_{x_i x_j tt}\, dx\, dt\, da + \sum_{i,j=1}^{n} \int_Q v^2_{x_i x_j aa}\, dx\, dt\, da \right)^{1/2}.$$

□

We prove that, for fixed ε, under conditions I–VI, the boundary value problem (4), (2), (5), (6) is solvable in $\widetilde{V}_0$ for any function $f(x,t,a)$ from $L_2(Q)$. Use the method of continuation in a parameter ([see 14, Chapter III, § 14]).

Let λ be a number in $[0,1]$. Consider the problem: *Find a function $u(x,t,a)$ that is a solution in the cylinder Q to the equation*

$$L_{\varepsilon,\lambda} u \equiv \varepsilon\Delta(u_{tt} + u_{aa} - \Delta u) + \lambda\left[L_\alpha u_t + L_\beta u_a - Mu\right] = f(x,t,a) \tag{4_λ}$$

and satisfying conditions (2), (5), and (6). Denote by Λ the set of those numbers λ in $[0,1]$ for which this boundary value problem is solvable in $\widetilde{V}_0$ for fixed ε and under conditions I–VI for any function $f(x,t,a)$ in $L_2(Q)$. If it turns out that this set is nonempty, open, and closed simultaneously, then it will coincide with the whole interval $[0,1]$ (see [14, Chapter III, § 14]).

It is obvious that the set Λ is not empty — it contains 0.

The openness and closedness of Λ will follow from the a priori estimate

$$\|u\|_{\widetilde{V}_0} \le N_0 \|f\|_{L_2(Q)} \tag{7}$$

for all possible solutions $u(x,t,a)$ to the boundary value problem (4_λ), (2), (5), (6) in $\widetilde{V}_0$, which is uniform over λ (again see [14, Chapter III, § 14]). Show that the desired estimate indeed holds.

Consider the equality

$$\int\limits_Q L_{\varepsilon,\lambda} uu \, dx\, dt\, da = \int\limits_Q fu \, dx\, dt\, da. \tag{8_λ}$$

Integrating by parts and using the boundary conditions, it is not hard to pass from this equality to the following:

$$\begin{gathered}
\varepsilon \sum_{i=1}^{n} \int\limits_Q \left(u_{x_i t}^2 + u_{x_i a}^2 + u_{x_i}^2\right) \, dx\, dt\, da+ \\
+\lambda \Biggl\{ \int\limits_Q \left[m^{ij} u_{x_i} u_{x_j} + \frac{1}{2}(\alpha_{1t} + \beta_{1a}) \sum_{i=1}^{n} u_{x_i}^2 \right] dx\, dt\, da+ \\
+\int\limits_Q \left[\frac{1}{2} m^i_{x_i} - \frac{1}{2} m^{ij}_{x_i x_j} - m_0 - \frac{1}{2}\alpha_{0t} - \frac{1}{2}\beta_{0a} \right] u^2 \, dx\, dt\, da- \\
+\frac{1}{2}\sum_{i=1}^{n} \int\limits_0^A \int\limits_\Omega \alpha_1(0,a) u_{x_i}^2(x,0,a)\, dx\, da - \frac{1}{2}\sum_{i=1}^{n} \int\limits_0^A \int\limits_\Omega \alpha_1(T,a) u_{x_i}^2(x,T,a)\, dx\, da+ \\
-\frac{1}{2} \int\limits_0^A \int\limits_\Omega \alpha_{01}(a) u^2(x,0,a)\, dx\, da + \frac{1}{2} \int\limits_0^A \int\limits_\Omega \alpha_{03}(a) u^2(x,T,a)\, dx\, da+ \\
+\frac{1}{2}\sum_{i=1}^{n} \int\limits_0^T \int\limits_\Omega \beta_1(t,0) u_{x_i}^2(x,t,0)\, dx\, dt - \frac{1}{2}\sum_{i=1}^{n} \int\limits_0^T \int\limits_\Omega \beta_1(t,A) u_{x_i}^2(x,t,A)\, dx\, dt- \\
-\frac{1}{2} \int\limits_0^T \int\limits_\Omega \beta_{01}(t) u^2(x,t,0)\, dx\, dt + \frac{1}{2} \int\limits_0^T \int\limits_\Omega \beta_{03}(t) u^2(x,t,A)\, dx\, dt \Biggr\} = \\
= \frac{\lambda}{2} \int\limits_0^A \int\limits_\Omega \alpha_{02}(a) u^2(x,0,a)\, dx\, da - \frac{\lambda}{2} \int\limits_0^A \int\limits_\Omega \alpha_{04}(a) u^2(x,T,a)\, dx\, da+ \\
+\frac{\lambda}{2} \int\limits_0^T \int\limits_\Omega \beta_{02}(t) u^2(x,t,0)\, dx\, dt - \frac{\lambda}{2} \int\limits_0^T \int\limits_\Omega \beta_{04}(t) u^2(x,t,A)\, dx\, dt + \int\limits_Q fu\, dx\, dt\, da.
\end{gathered} \tag{9}$$

Estimating the first four summands on the right-hand side of (9) with the use of (3) and applying Young's inequality to the last summand on the right-hand side of (9) and also inequality (3), we conclude that the conditions of the theorem imply the a priori estimate

$$\varepsilon \sum_{i=1}^{n} \int_Q (u_{x_it}^2 + u_{x_ia}^2 + u_{x_i}^2)\, dx\, dt\, da +$$

$$+\lambda \int_Q \left[u^2 + \sum_{i=1}^{n} u_{x_i}^2 \right] dx\, dt\, da \le N_1 \int_Q f^2\, dx\, dt\, da, \qquad (10_\lambda)$$

in which the number N_1 is defined by the functions $m^{ij}(x,t,a)$, $m^i(x,t,a)$, $i,j = 1,\dots,n$, $m_0(x,t,a)$, $\alpha_0(t,a)$, $\alpha_1(t,a)$, $\beta_0(t,a)$, $\beta_1(t,a)$, and also by the numbers d_0, T, A, and ε.

Consider the equality

$$-\int_Q L_{\varepsilon,\lambda} u (u_{tt} + u_{aa})\, dx\, dt\, da = -\int_Q f (u_{tt} + u_{aa})\, dx\, dt\, da. \qquad (11_\lambda)$$

Integrating by parts once again and using the boundary conditions (2), (5), and (6), applying conditions I–VI and inequality (3), we conclude that this equality implies the estimate

$$\varepsilon \sum_{i=1}^{n} \int_Q \left(u_{x_itt}^2 + u_{x_ita}^2 + u_{x_iaa}^2 + u_{x_it}^2 + u_{x_ia}^2 \right) dx\, dt\, da +$$

$$+\lambda \int_Q \left[u_t^2 + u_a^2 + \sum_{i=1}^{n} \left(u_{x_it}^2 + u_{x_ia}^2 \right) \right] dx\, dt\, da \le N_2 \int_Q f^2\, dx\, dt\, da, \quad (12_\lambda)$$

where the number N_2 is defined by the functions $m^{ij}(x,t,a)$, $m^i(x,t,a)$, $i,j = 1,\dots,n$, $m_0(x,t,a)$, $\alpha_0(t,a)$, $\alpha_1(t,a)$, $\beta_0(t,a)$, $\beta_1(t,a)$, and also by the numbers d_0, T, A, and ε.

Consider the equality

$$-\int_Q L_{\varepsilon,\lambda} u \Delta u\, dx\, dt\, da = -\int_Q f \Delta u\, dx\, dt\, da. \qquad (13_\lambda)$$

Integrating by parts, using the second main inequality for elliptic operators (see [8], [9, Chapter III, § 8], estimate (12_λ), and Young's inequality, we conclude that (13_λ) implies the estimate

$$\varepsilon \int_Q \left[(\Delta u_t)^2 + (\Delta u_a)^2 + (\Delta u)^2 \right] dx\, dt\, da +$$

$$+\lambda \sum_{i,j=1}^{n} \int\limits_{Q} u_{x_i x_j}^2 \, dx\, dt\, da \le N_3 \int\limits_{Q} f^2 \, dx\, dt\, da, \qquad (14_\lambda)$$

in which the constant N_3 is defined by the functions $m^{ij}(x,t,a)$, $m^i(x,t,a)$, $ij, = 1,\dots,n$, $m_0(x,t,a)$, $\alpha_0(t,a)$, $\alpha_1(t,a)$, $\beta_0(t,a)$, $\beta_1(t,a)$, the domain Ω, and also by the numbers T, A, and ε.

Finally, consider the equality

$$-\int\limits_{Q} L_{\varepsilon,\lambda} u(\Delta u_{tt} + \Delta u_{aa})\, dx\, dt\, da = -\int\limits_{Q} f(\Delta u_{tt} + \Delta u_{aa})\, dx\, dt\, da. \qquad (15_\lambda)$$

Integrating by parts again, using estimates (12_λ) and (14_λ), the second main inequality for elliptic operators, and Young's inequality, we conclude that solutions $u(x,t,a)$ to the boundary value problem (4_λ), (2), (5), (6) satisfy the estimate

$$\varepsilon \int\limits_{Q} \left[(\Delta u_{tt})^2 + (\Delta u_{aa})^2\right] \, dx\, dt\, da +$$

$$+\lambda \sum_{i,j=1}^{n} \int\limits_{Q} (u_{x_i x_j t}^2 + u_{x_i x_j a}^2)\, dx\, dt\, da \le N_4 \int\limits_{Q} f^2\, dx\, dt\, da, \qquad (16_\lambda)$$

with the constant N_4 defined by the functions $m^{ij}(x,t,a)$, $m^i(x,t,a)$, $i,j = 1,\dots,n$, $m_0(x,t,a)$, $\alpha_0(t,a)$, $\alpha_1(t,a)$, $\beta_0(t,a)$, $\beta_1(t,a)$, the domain Ω, and also the numbers T, A, and ε.

From (12_λ), (14_λ), and (16_λ), we have the estimate

$$\varepsilon \int\limits_{Q} \left[(\Delta u_{tt})^2 + (\Delta u_{aa})^2 + (\Delta u)^2\right] \, dx\, dt\, da \le N_5 \int\limits_{Q} f^2\, dx\, dt\, da,$$

in which the constant N_5 is defined by the functions $m^{ij}(x,t,a)$, $m^i(x,t,a)$, $i,j = 1,\dots,n$, $m_0(x,t,a)$, $\alpha_0(t,a)$, $\alpha_1(t,a)$, $\beta_0(t,a)$, $\beta_1(t,a)$, the domain Ω, and also the numbers T, A, and ε. This estimate implies the desired estimate (7).

As we already said above, estimate (7) implies that the boundary value problem (4_λ), (2), (5), (6) is solvable in $\widetilde{V}_0$ for fixed ε, under conditions I–VI, and for all $\lambda \in [0,1]$.

Thus, the boundary value problem (4_1), (2), (5), (6) has a solution $u(x,t,a)$ belonging to $\widetilde{V}_0$. Let us demonstrate that, under conditions I–VI, for $f(x,t,a)$ such that $f(x,t,a) \in L_2(Q)$, $f_t(x,t,a) \in L_2(Q)$, $f_a(x,t,a) \in L_2(Q)$, this solution satisfies a priori estimates uniform over ε.

Observe first of all that, under conditions I–VI, equality (8_1) implies the estimate

$$\varepsilon\sum_{i=1}^{n}\int\limits_{Q}\left(u_{x_it}^2+u_{x_ia}^2+u_{x_i}^2\right)\,dx\,dt\,da+\int\limits_{Q}\left[u^2+\sum_{i=1}^{n}u_{x_i}^2\right]\,dx\,dt\,da\leq$$

$$\leq\widetilde{N}_1\int\limits_{Q}f^2\,dx\,dt\,da, \tag{17}$$

with the constant $\widetilde{N}_1$ defined by the functions $m^{ij}(x,t,a)$, $m^i(x,t,a)$, $i,j=1,\dots,n$, $m_0(x,t,a)$, $\alpha_0(t,a)$, $\alpha_1(t,a)$, $\beta_0(t,a)$, and $\beta_1(t,a)$, and also the number d_0.

Further, it is not hard to transform equality (11_1) to the form

$$-\int\limits_{Q}L_{\varepsilon,1}u(u_{tt}+u_{aa})\,dx\,dt\,da=\int\limits_{Q}(f_tu_t+f_au_a)\,dx\,dt\,da.$$

Transforming the left-hand side of this equality as it was done in proving estimate (12_λ), using the conditions of the theorem and estimate (17), and applying Young's inequality, we conclude that solutions $u(x,t,a)$ to the boundary value problem (4_1), (2), (5), (6) satisfy the estimate

$$\varepsilon\sum_{i=1}^{n}\int\limits_{Q}(u_{x_itt}^2+u_{x_ita}^2+u_{x_iaa}^2)\,dx\,dt\,da+$$

$$+\int\limits_{Q}\left[u_t^2+u_a^2+\sum_{i=1}^{n}\left(u_{x_it}^2+u_{x_ia}^2\right)\right]\,dx\,dt\,da\leq\widetilde{N}_2\int\limits_{Q}(f^2+f_t^2+f_a^2)\,dx\,dt\,da, \tag{18}$$

with the constant $\widetilde{N}_2$ defined by the functions $m^{ij}(x,t,a)$, $m^i(x,t,a)$, $i,j=1,\dots,n$, $m_0(x,t,a)$, $\alpha_0(t,a)$, $\alpha_1(t,a)$, $\beta_0(t,a)$, and $\beta_1(t,a)$, and the number d_0.

Equality (13_1), estimates (17) and (18) and also the conditions of the theorem and the second main inequality for elliptic equations imply the third a priori estimate

$$\varepsilon\int\limits_{Q}\left[(\Delta u_t)^2+(\Delta u_a)^2\right]\,dx\,dt\,da+$$

$$+\sum_{i,j=1}^{n}\int\limits_{Q}u_{x_ix_j}^2\,dx\,dt\,da\leq\widetilde{N}_3\int\limits_{Q}(f^2+f_t^2+f_a^2)\,dx\,dt\,da, \tag{19}$$

in which the constant $\widetilde{N}_3$ is defined by the functions $m^{ij}(x,t,a)$, $m^i(x,t,a)$, $i,j = 1,\dots,n$, $m_0(x,t,a)$, $\alpha_0(t,a)$, $\alpha_1(t,a)$, $\beta_0(t,a)$, $\beta_1(t,a)$, and also by the domain Ω.

At the last step, consider equality (15_1). Transforming the right-hand side of this equality by integration by parts, using the conditions of the theorem, the second main inequality for elliptic operators, and estimates (17)–(19), we conclude that solutions $u(x,t,a)$ to the boundary value problem (4_1), (2), (5), (6) satisfy the fourth a priori estimate

$$\varepsilon \int\limits_Q \left[(\Delta u_{tt})^2 + (\Delta u_{aa})^2\right]\, dx\, dt\, da + \sum_{i,j=1}^{n} \int\limits_Q u^2_{x_i x_j t}\, dx\, dt\, da +$$
$$+ \sum_{i,j=1}^{n} \int\limits_Q u^2_{x_i x_j a}\, dx\, dt\, da \le \widetilde{N}_4 \int\limits_Q (f^2 + f_t^2 + f_a^2)\, dx\, dt\, da, \tag{20}$$

in which the constant $\widetilde{N}_4$ is defined by the functions $m^{ij}(x,t,a)$, $m^i(x,t,a)$, $i,j = 1,\dots,n$, $m_0(x,t,a)$, $\alpha_0(t,a)$, $\alpha_1(t,a)$, $\beta_0(t,a)$, $\beta_1(t,a)$, and also the domain Ω.

Estimates (17)–(20) are desired estimates of solutions $u(x,t,a)$ to the boundary value problem (4_1), (2), (5), (6) uniform over ε. These estimates and the reflexivity of a Hilbert space imply that there exists a sequence $\{\varepsilon_m\}_{m=1}^{\infty}$ of positive numbers $\{u_m(x,t,a)\}_{m=1}^{\infty}$ of solutions to the boundary value problem (4_1), (2), (5), (6) and a function $u(x,t,a)$ such that, as $m \to \infty$ and for $i,j = 1,\dots,m$, we have the convergences

$$\varepsilon_m \to 0,$$
$$u_m(x,t,a) \to u(x,t,a) \textit{ weakly in } W_2^1(Q),$$
$$u_{mx_ix_j}(x,t,a) \to u_{x_ix_j}(x,t,a) \textit{ weakly in } L_2(Q),$$
$$u_{mx_ix_jt}(x,t,a) \to u_{x_ix_jt}(x,t,a) \textit{ weakly in } L_2(Q),$$
$$u_{mx_ix_ja}(x,t,a) \to u_{x_ix_ja}(x,t,a) \textit{ weakly in } L_2(Q),$$
$$\varepsilon_m \Delta u_m(x,t,a) \to 0 \textit{ weakly in } L_2(Q),$$
$$\varepsilon_m \Delta u_{mtt}(x,t,a) \to 0 \textit{ weakly in } L_2(Q),$$
$$\varepsilon_m \Delta u_{maa}(x,t,a) \to 0 \textit{ weakly in } L_2(Q).$$

Obviously, the limit function $u(x,t,a)$ belongs to V_0 and is a desired solution to the boundary value problem P_{0000}.

Uniqueness in V_0 of solutions to the boundary value problem P_{0000} obviously follows from the equality

$$\int\limits_Q (L_\alpha u_t + L_\alpha u_a - Mu)u\, dx\, dt\, da = 0. \tag{21}$$

The theorem is proved.

Turn to investigating the solvability of the boundary value problem P_{1111}.

The solvability of the boundary value problem P_{1111} in V_0 will be proved again by the regularization method, we will again use equation (4) but no additional boundary conditions will be given.

Define some conditions to be used below:

VII₁. $\alpha_1(0,a) < 0$, $\alpha_0(0,a) \geq 0$, $\alpha_1(T,a) > 0$, $\alpha_0(T,a) \leq 0$, $a \in [0,A]$;
VII₂. $\alpha_1(0,a) \leq 0$, $\alpha_0(0,a) > 0$, $\alpha_1(T,a) > 0$, $\alpha_0(T,a) \leq 0$, $a \in [0,A]$;
VII₃. $\alpha_1(0,a) < 0$, $\alpha_0(0,a) \geq 0$, $\alpha_1(T,a) \geq 0$, $\alpha_0(T,a) < 0$, $a \in [0,A]$;
VI₄. $\alpha_1(0,a) \leq 0$, $\alpha_0(0,a) > 0$, $\alpha_1(T,a) \geq 0$, $\alpha_0(T,a) < 0$, $a \in [0,A]$;
VIII₁. $\beta_1(t,0) < 0$, $\beta_0(t,0) \geq 0$, $\beta_1(t,A) > 0$, $\beta_0(t,A) \leq 0$, $t \in [0,T]$;
VIII₂. $\beta_1(t,0) \leq 0$, $\beta_0(t,0) > 0$, $\beta_1(t,A) > 0$, $\beta_0(t,A) \leq 0$, $t \in [0,T]$;
VIII₃. $\beta_1(t,0) \leq 0$, $\beta_0(t,0) \geq 0$, $\beta_1(t,A) \geq 0$, $\beta_0(t,A) < 0$, $t \in [0,T]$;
VIII₄. $\beta_1(t,0) \leq 0$, $\beta_0(t,0) > 0$, $\beta_1(t,A) \geq 0$, $\beta_0(t,A) < 0$, $t \in [0,T]$.

Theorem 2. *Suppose the fulfillment of conditions I–IV, of one of conditions* VII_1–VII_4, *and of one of conditions* $VIII_1$–$VIII_4$. *Then, for any function* $f(x,t,a)$ *such that* $f(x,t,a) \in L_2(Q)$, $f_t(x,t,a) \in L_2(Q)$, $f_a(x,t,a) \in L_2(Q)$, $f(x,0,a) = f(x,T,a) = 0$ *for* $(x,a) \in \Omega \times (0,A)$, $f(x,t,0) = f(x,t,A) = 0$ *for* $(x,t) \in \Omega \times (0,T)$, *the boundary value problem* P_{1111} *has a unique solution in* V_0.

Proof. Consider the boundary value problem: *Find a function* $u(x,t,a)$ *that is a solution to equation (4) in the cylinder* Q *and satisfies the* P_{1111}*-condition and also condition (2).* The proof of the solvability of this problem in $\widetilde{V}_0$ for fixed ε and $f(x,t,a) \in L_2(Q)$ is again carried out by the method of continuation in a parameter and a priori estimates. In view of the analogy of the procedure of applying of the method of continuation in a parameter with the corresponding procedure used in the proof of Theorem 1, we just show that solutions to the above-proposed boundary value problem satisfy estimates in $\widetilde{V}_0$ valid if $f(x,t,a) \in L_2(Q)$, and then — that, under additional conditions on the function $f(x,t,a)$, there are estimates uniform over ε.

Consider equality (8_1). Using the P_{1111}–condition, condition (2), conditions I–IV, we infer that solutions $u(x,t,a)$ to the boundary value problem for equation (4) with the P_{1111}-condition and condition (2) satisfy estimate (10_1).

Now, consider equality (11_1). After integration by parts with the use of the boundary condition, this equality takes the form

$$\varepsilon \sum_{i=1}^{n} \int\limits_{Q} (u^2_{x_i tt} + 2u^2_{x_i ta} + u^2_{x_i aa} + u^2_{x_i t} + u^2_{x_i a})\, dx\, dt\, da +$$

$$+\left\{\int\limits_Q \left\{\left[\frac{1}{2}m^i_{x_i} - \frac{1}{2}m^{ij}_{x_ix_j} - m_0 + \frac{1}{2}\alpha_{0t} - \frac{1}{2}\beta_{0a}\right]u_t^2 + \right.\right.$$

$$\left.+(\beta_{0t}+\alpha_{0a})u_au_t + \left[\frac{1}{2}m^i_{x_i} - \frac{1}{2}m^{ij}_{x_ix_j} - m_0 + \frac{1}{2}\beta_{0a} - \frac{1}{2}\alpha_{0t}\right]u_a^2\right\} dx\,dt\,da +$$

$$+\int\limits_Q \left[m^{ij}u_{x_it}u_{x_jt} + \left(\frac{1}{2}\beta_{1a} - \frac{1}{2}\alpha_{1t}\right)\sum_{i=1}^n u^2_{x_it} + m^{ij}u_{x_ia}u_{x_ja} + \right.$$

$$\left.+\left(\frac{1}{2}\alpha_{1t} - \frac{1}{2}\beta_{1a}\right)\sum_{i=1}^n u^2_{x_ia} - (\beta_{1t}+\alpha_{1a})u_{x_it}u_{x_ia}\right] dx\,dt\,da +$$

$$+\int\limits_Q \left[m^{ij}_t u_{x_i}u_{x_jt} + m^{ij}_a u_{x_i}u_{x_ja} + (m^{ij}_{x_jt} - m^i_t)u_{x_i}u_t + \right.$$

$$\left.\left.+(m^{ij}_{x_ja} - m^i_a)u_{x_i}u_a - m_{0t}uu_t - m_{0a}uu_a\right] dx\,dt\,da\right\} +$$

$$+\frac{1}{2}\int\limits_0^A\int\limits_\Omega \alpha_0(0,a)u_t^2(x,0,a)\,dx\,da - \frac{1}{2}\sum_{i=1}^n\int\limits_0^A\int\limits_\Omega \alpha_1(0,a)u^2_{x_it}(x,0,a)\,dx\,da +$$

$$-\frac{1}{2}\int\limits_0^A\int\limits_\Omega \alpha_0(T,a)u_t^2(x,T,a)\,dx\,da + \frac{1}{2}\sum_{i=1}^n\int\limits_0^A\int\limits_\Omega \alpha_1(T,a)u^2_{x_it}(x,T,a)\,dx\,da +$$

$$+\frac{1}{2}\int\limits_0^T\int\limits_\Omega \beta_0(t,0)u_a^2(x,t,0)\,dx\,dt - \frac{1}{2}\sum_{i=1}^n\int\limits_0^T\int\limits_\Omega \beta_1(t,0)u^2_{x_ia}(x,t,0)\,dx\,dt -$$

$$-\frac{1}{2}\int\limits_0^T\int\limits_\Omega \beta_0(t,A)u_t^2(x,t,A)\,dx\,dt - \frac{1}{2}\sum_{i=1}^n\int\limits_0^T\int\limits_\Omega \beta_1(t,A)u^2_{x_ia}(x,t,A)\,dx\,dt =$$

$$= -\int\limits_Q f(u_{tt} + u_{aa})\,dx\,dt\,da.$$

Under the fulfillment of one of conditions *VII*$_1$, *VII*$_2$, *VII*$_3$, or *VII*$_4$, one of conditions *VIII*$_1$, *VIII*$_2$, *VIII*$_3$, or *VIII*$_4$, the sum of the last four summands on the left-hand side of this equality is nonnegative. Using this fact and conditions I–IV, we infer that a solution $u(x,t,a)$ to the boundary value problem for equation (4) with the P_{1111}-condition and condition (2) satisfies estimate (12_1).

Further analysis of equalities (13_1) and (15_1) gives the fulfillment of estimates (14_1) and (16_1) for a solution $u(x,t,a)$ to the boundary value problem for equation (4) with the P_{1111}-condition and condition (2). The sum of estimates (10_1), (12_1), (14_1), and (16_1) gives estimate (7); this estimate implies the existence in $\widetilde{V}_0$ of a function $u(x,t,a)$ that is a solution to equation (4) in the cylinder Q and satisfies the P_{1111}-condition and also condition (2).

The presence of a priori estimates uniform over ε for solutions $u(x,t,a)$ to the boundary value problem for equation (4) with the P_{1111}-condition and condition (2) is proved by an additional integration by parts in (11_1) and (15_1). The possibility of choosing a sequence converging to a solution to the boundary value problem P_{1111} stems from the obtained estimates uniform over ε and the reflexivity of a Hilbert space.

The uniqueness of solutions to the boundary value problem P_{1111} in V_0 is easy to show with the use of equality (21) and conditions I–IV.

The theorem is completely proved. □

4. Solvability of the Boundary Value Problems P_{1110}, P_{1010}, P_{1100}, and P_{1000}.

The boundary value problems P_{1110}, P_{1010}, P_{1100}, and P_{1000} are problems intermediate between the problems P_{0000} and P_{1111} — in them, part of the sets γ_1, γ_2, γ_3, and γ_4, are free of boundary value conditions, whereas the value of the solution is given on the remaining part. The solvability of these problems in the space V_0 is studied by a combination of the methods of investigating the problems P_{0000} and P_{1111}. More exactly, we once again use the regularization method, where equation (4) again serves as the regularizing equation. Only the boundary conditions change — for each of the problems P_{1110}, P_{1010}, P_{1100}, and P_{1000}, on the sets on which the values of the solution are not given, in the regularizing problem, the values of the derivative u_t or u_a are given (for example, in the problem regularizing the problem P_{1110}, the additional condition

$$u_a(x,t,a) = 0 \quad \textit{for} \quad x \in \Omega, \quad t \in 0, T$$

is given). The technique of the necessary a priori estimates as for fixed ε as of estimates uniform over ε completely corresponds to the technique used in the proof of Theorems 1 and 2. The convergent sequence is chosen in a standard manner: with the use of the estimates obtained and the reflexivity property of a Hilbert space.

Let us give the exact results for each of the problems.

Theorem 3. *Suppose the fulfillment of conditions I–IV, of one of conditions VII_1, VII_2, VII_3, or VII_4, of the condition*

$$\beta_1(t, A) \le 0, \quad \beta_0(t, A) = \beta_{03}(t) + \beta_{04}(t), \quad \beta_{03}(t) \ge 0, \quad \beta_{04}(t) \le 0,$$

$$\beta_1(t,A) - d_0\beta_{04}(t) \le 0, \quad t \in [0,T];$$

and also of one of the conditions

$$\beta_1(t,0) < 0, \quad \beta_0(t,0) \ge 0, \quad \in [0,T];$$

or

$$\beta_1(t,0) \le 0, \quad \beta_0(t,0) > 0, \quad \in [0,T].$$

Then, for any function $f(x,t,a)$ *such that* $f(x,t,a) \in L_2(Q)$, $f_t(x,t,a) \in L_2(Q)$, $f_a(x,t,a) \in L_2(Q)$, $f(x,0,a) = f(x,T,a) = 0$ *for* $(x,a) \in \Omega\times(0,A)$, $f(x,t,0) = 0$ *for* $(x,t) \in \Omega \times (0,T)$, *the boundary value problem* P_{1110} *has a unique solution in* V_0.

Theorem 4. *Suppose the fulfillment of conditions I–IV, of one of conditions VII*$_1$, *VII*$_2$, *VII*$_3$, *or VII*$_4$, *and also of the conditions*

$$\beta_1(t,0) \ge 0, \quad \beta_0(t,0) = \beta_{01}(t) + \beta_{02}(t), \quad \beta_{01}(t) \le 0, \quad \beta_{02}(t) \ge 0,$$

$$\beta_1(t,0) - d_0\beta_{02}(t) \ge 0, \quad t \in [0,T];$$

$$\beta_1(t,A) \le 0, \quad \beta_0(t,A) = \beta_{03}(t) + \beta_{04}(t), \quad \beta_{03}(t) \ge 0, \quad \beta_{04}(t) \le 0,$$

$$\beta_1(t,A) - d_0\beta_{04}(t) \le 0, \quad t \in [0,T].$$

Then, for any function $f(x,t,a)$ *such that* $f(x,t,a) \in L_2(Q)$, $f_t(x,t,a) \in L_2(Q)$, $f_a(x,t,a) \in L_2(Q)$, $f(x,0,a) = f(x,T,a) = 0$ *for* $(x,a) \in \Omega \times (0,A)$, $f(x,t,0) = f(x,T,a) = 0$ *for* $(x,t) \in \Omega \times (0,T)$, *the boundary value problem* P_{1100} *has a unique solution in* V_0.

Theorem 5. *Suppose the fulfillment of conditions I–IV, of one of the conditions*

$$\alpha_1(0,a) < 0, \quad \alpha_0(0,a) \ge 0, \quad a \in [0,A];$$

or

$$\alpha_1(0,a) \le 0, \quad \alpha_0(0,a) > 0, \quad a \in [0,A];$$

of one of the conditions

$$\beta_1(t,0) < 0, \quad \beta_0(t,0) \ge 0, \quad \in [0,T];$$

or

$$\beta_1(t,0) \le 0, \quad \beta_0(t,0) > 0, \quad \in [0,T],$$

and also of the conditions

$$\alpha_1(T,a) \le 0, \quad \alpha_0(T,a) = \alpha_{03}(a) + \alpha_{04}(a), \quad \alpha_{03}(a) \ge 0, \quad \alpha_{04}(a) \le 0,$$

$$\alpha_1(T,a) - d_0\alpha_{04}(a) \le 0, \quad a \in [0,A];$$

$$\beta_1(t,A) \le 0, \quad \beta_0(t,A) = \beta_{03}(t) + \beta_{04}(t), \quad \beta_{03}(t) \ge 0, \quad \beta_{04}(t) \le 0,$$

$$\beta_1(t,A) - d_0\beta_{04}(t) \le 0, \quad t \in [0,T].$$

Then, for any function $f(x,t,a)$ *such that* $f(x,t,a) \in L_2(Q)$*,* $f_t(x,t,a) \in L_2(Q)$*,* $f_a(x,t,a) \in L_2(Q)$*,* $f(x,0,a) = 0$ *for* $(x,a) \in \Omega \times (0,A)$*,* $f(x,t,0) = 0$ *for* $(x,t) \in \Omega \times (0,T)$*, the boundary value problem* P_{1010} *has a unique solution in* V_0*.*

Theorem 6. *Suppose the fulfillment of conditions I–IV, of one of the conditions*

$$\alpha_1(0,a) < 0, \quad \alpha_0(0,a) \ge 0, \quad a \in [0,A];$$

or

$$\alpha_1(0,a) \le 0, \quad \alpha_0(0,a) > 0, \quad a \in [0,A];$$

and also of the condition

$$\alpha_1(T,a) \le 0, \quad \alpha_0(T,a) = \alpha_{03}(a) + \alpha_{04}(a), \quad \alpha_{03}(a) \ge 0, \quad \alpha_{04}(a) \le 0,$$

$$\alpha_1(T,a) - d_0\alpha_{04}(a) \le 0, \quad a \in [0,A];$$

$$\beta_1(t,0) \ge 0, \quad \beta_0(t,0) = \beta_{01}(t) + \beta_{02}(t), \quad \beta_{01}(t) \le 0, \quad \beta_{02}(t) \ge 0,$$

$$\beta_1(t,0) - d_0\beta_{02}(t) \ge 0, \quad t \in [0,T];$$

$$\beta_1(t,A) \le 0, \quad \beta_0(t,A) = \beta_{03}(t) + \beta_{04}(t), \quad \beta_{03}(t) \ge 0, \quad \beta_{04}(t) \le 0,$$

$$\beta_1(t,A) - d_0\beta_{04}(t) \le 0, \quad t \in [0,T].$$

Then, for any function $f(x,t,a)$ *such that* $f(x,t,a) \in L_2(Q)$*,* $f_t(x,t,a) \in L_2(Q)$*,* $f_a(x,t,a) \in L_2(Q)$*,* $f(x,0,a) = 0$ *for* $(x,a) \in \Omega \times (0,A)$*, the boundary value problem* P_{1000} *is uniquely solvable in* V_0*.*

5. Conclusion

We have studied the solvability of boundary value problems for third-order Sobolev-type equations with two time variables not solved for the derivatives. Existence and uniqueness theorems are prove for regular solutions.

The equations under study have model form. It is not hard to obtain analogous theorems on the solvability of the corresponding boundary value problems also for general equations. For example, the Laplace operator can be replaced by an arbitrary second-order elliptic operator whose coefficients α_0, α_1, β_0, and β_1 can depend also on the variables $x_1, \dots, x_n$, and the number of time variables can be arbitrary. The calculations and conditions for such general equations will ne substantially more cumbersome but the essence of the results on solvability will not change.

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Ультрапараболические уравнения с операторными коэффициентами при временных производных

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Аннотация. Работа посвящена исследованию разрешимости краевых задач для дифференциальных уравнений соболевского типа третьего порядка с двумя временными переменными (подобные уравнения называются также уравнениями составного типа, или уравнениями, неразрешенными относительно производной). Отличительными особенностями изучаемых уравнений являются, во-первых, то, что дифференциальные операторы, действующие на временные производные, не предполагаются обратными, во-вторых, то, что постановки краевых задач для них определяются коэффициентами этих дифференциальных операторов. Для предложенных задач в работе доказываются теоремы существования и единственности регулярных решений (решений, имеющих все обобщенные по С. Л. Соболеву производные, входящие в уравнение). Техника доказательств теорем существования основана на специальной регуляризации изучаемых уравнений, априорных оценках и предельном переходе.

Ключевые слова: ультрапараболические уравнения, необратимые операторные коэффициенты, краевые задачи, регулярные решения, существование, единственность.

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