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A Short Calculation of the Multiple Sum of Krivokolesko-Leinartas with Linear Constraints on Summation Indices

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Abstract. The method of integral representation and calculation of the combinatorial sums of various type (the method of coefficients) using the formal Laurent power series over $\mathbb{C}$, the theory of analitical functions and the theory of multiple residues in $\mathbb{C}^n$ were proposed by the author in the late seventies. This method was applied in various fields of mathematics. The method of coefficients is important for a difficult problem of calculation of the multiple sums with linear constraints on summation indices. Various combinatorial problems can be formulated in terms of such constraints. The calculation of the multiple sum with q-binomial coefficients and linear recurrent constraints on summation indices was published by the author in ≪The Bulletin of Irkutsk State University. Series Mathematics.≫in 2016. This problem appears at the enumeration of all own t-dimensional subspaces of the space V_m over field $GF(q)$. V.P. Krivokolesko and E.K. Leinartas in ≪The Bulletin of Irkutsk State University. Series Mathematics.≫ in 2012, using the Hadamard composition have proved the multiple identity with polynomial coefficients and various constraints on the limits of summation, containing the family of free parameters. This identity is generalisation of the identities studied earlier by several authors, since constructions of the Deubechies filters in the wavelets theory. Using the author's method of coefficients the short and simple calculation of Krivokolesko–Leinartas sum is carried out. These calculations also automatically provides an equivalent way of calculation of the specified sum by means of a traditional method of generation functions, using only the well-known operations over corresponding multiple formal Laurent power series.

Keywords: combinatorial sums, the method of coefficients, integral representations, generating functions

1. Introduction

Recently, V.P. Krivokolesko and E.K. Leinartas [8] with the Hadamard composition properties proved the following statement: *Let μ be the given natural number, $1 \leqslant \mu \leqslant n$, and complex numbers $z = (z_1, \ldots, z_n)$ such that*

$$|z_1| + \ldots + |z_{\mu-1}| < 1, \ z_1 + z_2 + \ldots + z_n = 1. \tag{1.1}$$

Then the following multiple summation formula with polynomial coefficients and linear constraints on the summation indices holds:

$$S_n = \sum_{j=\mu}^{n} z_j \sum_{\beta \in B^s_{\mu,j}} \binom{\beta_1 + \ldots + \beta_n}{\beta_1, \ldots, \beta_n} \prod_{i=1}^{n} z_i^{\beta_i} \equiv 1. \tag{1.2}$$

Here $\beta = (\beta_1, \ldots, \beta_n)$ and for $j = \mu, \mu+1, \ldots, n$ set

$$B^s_{\mu,j} = \{\beta \in \mathbb{Z}^n_+ : \beta_\mu \leqslant s_\mu, \ldots, \beta_{j-1} \leqslant s_{j-1}, \beta_j = s_j, \beta_{j+1} \leqslant s_{j+1}, \ldots, \beta_n \leqslant s_n\}$$

and polynomial coefficient

$$\binom{\beta_1 + \ldots + \beta_n}{\beta_1, \ldots, \beta_n} = \frac{(\beta_1 + \ldots + \beta_n)!}{\beta_1! \ldots \beta_n!}.$$

Generalization of combinatorial identities studied in [2], [11], [10], [4] gives us formula (1.2).

Let us conduct multiple calculation (1.2) using the general scheme of the coefficient method [3], [6]. Many applications of this method in various fields of mathematics were found. Here readers may refer to [12], [9], [7], [1], [5]. First of all, let us give a brief description of the coefficient method.

2. Coefficients method

Let L be the set of Laurent power set with real or complex coefficients, containing only a finite number of terms with negative powers; $L_k = \{A(z)\}$ is set of formal Laurent power series of order k,

$$A(z) = \sum_{i=0}^{k} a_{-i} z^{-i} + \sum_{i=1}^{\infty} a_i z^i, \ a_k \neq 0.$$

For $A(z) \in L$ we define *formal residue operator*

$$res_z\{A(z)\} := a_{-1}. \tag{2.1}$$

Two series are similar $A(z) = \sum_k a_k z^k$ and $B(z) = \sum_k b_k z^k$ from L if and only if $a_k = b_k$ for all k. Operator $res_z\{A(z)\}$ is defined on the set

L Laurent formal power series with the well-known operations of addition, multiplication, superposition and inversion, as well as the operations of differentiation and integration of series (see, for example, [3]). If

$$A(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then from (2.1) it follows

$$a_n = res_z\{A(z)z^{-n-1}\},\ n = 0, 1, \ldots .$$

For example, one of the possible representations for a binomial coefficient

$$\binom{n}{k} = res_z\{(1+z)^n z^{-k-1}\},\ k = 0, 1, \ldots, n,$$

and for polynomial coefficient

$$\binom{\beta_1 + \ldots + \beta_n}{\beta_1, \ldots, \beta_n} = res_{z_1,\ldots,z_n}\left\{\frac{1}{(1 - z_1 - \ldots - z_n)}\prod_{i=1}^{n} z_i^{-\beta_i - 1}\right\}. \quad (2.2)$$

From the operator definition $res_z\{A(z)\}$ and series properties L the rules of action (output rules) for the operator $res_z\{A(z)\}$ follow directly. We will omit the replacement, inverse, differentiation and integration rules and present only those that are used by us in the calculations.

Operator output rules $res_z\{A(z)\}$

Let us consider the following two series $A(z) = \sum_k a_k z^k$ and $B(z) = \sum_k b_k z^k$ from L.

Deduction rule. *Equality*

$$res_z\{A(z)z^{-k-1}\} = res_z\{B(z)z^{-k-1}\}$$

is fulfilled for all $k = 0, 1, \ldots,$ *if and only if* $A(z) = B(z)$.

Linearity rule. *For all* $\alpha, \beta \in \mathbb{C}$

$$\alpha res_z\{A(z)\} + \beta res_z\{B(z)\} = res_z\{\alpha A(z) + \beta B(z)\}.$$

By induction, it follows that the operators $res_z\{A(z)\}$ *and summation are commutative for any finite number of terms.*

Substitution rule.

$$\sum_k w^k res_z\{A(z)\} = [A(z)]_{z=w} = A(w).$$

If series $A(z)$ *converges in a punctured neighborhood of the origin, then we assume*

$$res_z\{A(z)\} = res_{z=0}\{A(z)\},$$

where $res_{z=0}\{A(z)\}$ *is classic residue at* $z = 0$ *in the theory of functions of complex variables, if necessary, we can use theory of residues for one or more complex variables* [3].

3. Main result

Let

$$S(w) = \sum_{s_\mu \ldots s_n = 0}^{\infty} S_n \, w_\mu^{s_\mu} \ldots w_n^{s_n} \tag{3.1}$$

be the generating function of power type from complex variables $w = (w_\mu, \ldots, w_n)$ for sequence $\{S_n\}$ from (1.2) non-negative integer parameters $s_\mu, \ldots, s_n$.

Theorem 1. *(a) The following formula is valid:*

$$S(w) = \prod_{i=\mu}^{n} \frac{1}{1 - w_i}. \tag{3.2}$$

(b) If condition (1.1) holds then identity (1.2) is fulfield.

Proof. Using the integral formula (2.2) for a polynomial coefficient, by the rule of linearity we get

$$S_n = \sum_{j=\mu}^{n} z_j \sum_{\beta \in B_{\mu,j}^{s}} \binom{\beta_1 + \ldots + \beta_n}{\beta_1, \ldots, \beta_n} \prod_{i=1}^{n} z_i^{\beta_i} =$$

$$\sum_{j=\mu}^{n} z_j^{s_j+1} \sum_{\beta_1 \ldots \beta_{\mu-1}=0}^{\infty} \sum_{\beta_\mu \ldots [j] \ldots \beta_n = 0}^{s_\mu \ldots [j] \ldots s_n} \prod_{i=1, i \neq j}^{n} z_i^{\beta_i} \times$$

$$res_{x_1 \ldots x_n} \left\{ \frac{1}{(1 - x_1 - \ldots - x_n)} x^{-s_j - 1} \prod_{i=1, i \neq j}^{n} x_i^{-\beta_i - 1} \right\} =$$

$$res_{x_\mu \ldots x_n} \left\{ \sum_{j=\mu}^{n} \frac{z_j}{x_j} \left(\frac{z_j}{x_j} \right)^{s_j} \prod_{i=1,\, i \neq j}^{n} \frac{1}{x_i} \sum_{\beta_j = 0}^{s_j} \left(\frac{z_i}{x_i} \right)^{\beta_i} \times \right.$$

$$\left. \left[\sum_{\beta_1 \ldots \beta_{\mu-1}=0}^{\infty} res_{x_1 \ldots x_{\mu-1}} \left\{ \frac{1}{(1 - x_1 - \ldots - x_n)} x^{-s_j - 1} \prod_{i=1}^{\mu-1} x_i^{-\beta_i - 1} \right\} \right] \right\}.$$

Summing up the indices $i=\mu,\ldots,n,\ i\neq j,\ (n-\mu)$-times according to the formula of finite geometric progression

$$\frac{1}{x_i}\sum_{\beta_j=0}^{s_j}\left(\frac{z_i}{x_i}\right)^{\beta_i}=\frac{1-(z_i/x_i)^{s_i+1}}{x_i-z_i},$$

and summation over indices $\beta_1,\ldots,\beta_{\mu-1}$ in square brackets according to the substitution rule $(\mu-1)$-times: replacements $x_i=z_i$, $i=1,\ldots,\mu-1$. We have

$$S_n=res_{x_\mu\ldots x_n}\left\{\sum_{j=\mu}^{n}\frac{z_j}{x_j}\left(\frac{z_j}{x_j}\right)^{s_j}\prod_{i=1,\,i\neq j}^{n}\frac{1-(z_i/x_i)^{s_i+1}}{x_i-z_i}\times\right.$$
$$\left.\left[\frac{1}{(1-x_1-\ldots-x_n)}\right]_{x_i=z_i,\,i=1,\ldots,\mu-1}\right\}=$$
$$res_{x_\mu\ldots x_n}\left\{\sum_{j=\mu}^{n}\frac{z_j}{x_j}\left(\frac{z_j}{x_j}\right)^{s_j}\prod_{i=1,\,i\neq j}^{n}\frac{1-(z_i/x_i)^{s_i+1}}{x_i-z_i}\times\right.$$
$$\left.\frac{1}{(1-z_1-\ldots-z_{\mu-1}-x_\mu-\ldots-x_n)}\right\}. \quad (3.3)$$

Using the integral formula (3.3) for the sum S_n, by the linearity rule we get

$$S(w)=\sum_{s_\mu\ldots s_n=0}^{\infty}w_\mu^{s_\mu}\ldots w_n^{s_n}\times$$
$$\times\, res_{x_\mu\ldots x_n}\left\{\sum_{j=\mu}^{n}\frac{z_j}{x_j}\left(\frac{z_j}{x_j}\right)^{s_j}\prod_{i=1,\,i\neq j}^{n}\frac{1-(z_i/x_i)^{s_i+1}}{x_i-z_i}\times\right.$$
$$\left.\frac{1}{(1-z_1-\ldots-z_{\mu-1}-x_\mu-\ldots-x_n)}\right\}=$$
$$res_{x_\mu\ldots x_n}\left\{\sum_{j=\mu}^{n}\frac{z_j}{x_j}\sum_{s_j=0}^{\infty}\left(\frac{z_jw_j}{x_j}\right)^{s_j}\prod_{i=1,\,i\neq j}^{n}\frac{1}{x_i-z_i}\sum_{s_i=0}^{\infty}\left(w_i^{s_i}-\frac{z_i}{x_i}\left(\frac{z_iw_i}{x_i}\right)^{s_i}\right)\times\right.$$
$$\left.\frac{1}{(1-z_1-\ldots-z_{\mu-1}-x_\mu-\ldots-x_n)}\right\}. \quad (3.4)$$

Let us find the sums by indices $s_\mu,\ldots,s_n$ using the formula of infinite geometric progression in assumption $|w_iz_i/x_i|<1$, $i=\mu,\ldots,n$ and $|w_i|<1$, $i=\mu,\ldots,n$, $i\neq j$. We have

$$\frac{1}{x_i-z_i}\sum_{s_i=0}^{\infty}\left(w_i^{s_i}-\frac{z_i}{x_i}\left(\frac{z_iw_i}{x_i}\right)^{s_i}\right)=\frac{1}{x_i-z_i}\left(\frac{1}{1-w_i}-\frac{z_i}{x_i-z_iw_i}\right)=$$

$$\frac{1}{(1-w_i)(x_i-z_iw_i)},\ i=\mu,\ldots,n,\ i\neq j;\ \ \frac{z_j}{x_j}\sum_{s_j=0}^{\infty}\left(\frac{z_jw_j}{x_j}\right)^{s_j}=\frac{z_j}{x_j-z_jw_j}.$$

Hence, in accordance with (3.4), we have

$$S(w)=res_{x_\mu\ldots x_n}\left\{\sum_{j=\mu}^{n}\frac{z_j}{x_j-z_jw_j}\prod_{i=1,\,i\neq j}^{n}\frac{1}{(1-w_i)(x_i-z_iw_i)}\times\right.$$

$$\left.\frac{1}{(1-z_1-\ldots-z_{\mu-1}-x_\mu-\ldots-x_n)}\right\}=$$

$$\prod_{i=\mu}^{n}\frac{1}{(1-w_i)}\times\sum_{j=\mu}^{n}z_j(1-w_j)\times$$

$$res_{x_\mu\ldots x_n}\left\{\frac{1}{(1-z_1-\ldots-z_{\mu-1}-x_\mu-\ldots-x_n)}\prod_{i=\mu}^{n}\frac{1}{(x_i-z_iw_i)}\right\}.$$

In the last expression for $S(w)$ let us use the first-order residues for each variable x_i for $x_i=z_iw_i$, and taking into account conditions (1.1) $1-z_1-\ldots-z_{\mu-1}=z_\mu+\ldots+z_n$, we have

$$S(w)=\prod_{i=\mu}^{n}\frac{1}{(1-w_i)}\times\sum_{j=\mu}^{n}\frac{z_j(1-w_j)}{w_\mu-\ldots-w_n-w_\mu z_\mu-\ldots-w_nz_n}=$$

$$\prod_{i=\mu}^{n}\frac{1}{(1-w_i)}.$$

Thus, item (a) is proved, and according to (3.1) and (3.2)

$$S(n)=\sum_{s_\mu\ldots s_n=0}^{\infty}res_{x_\mu\ldots x_n}\left\{S(w)\prod_{i=\mu}^{n}w_i^{-s_i-1}\}\right\}=$$

$$\sum_{s_\mu\ldots s_n=0}^{\infty}res_{x_\mu\ldots x_n}\left\{\prod_{i=\mu}^{n}\frac{1}{1-w_i}\prod_{i=\mu}^{n}w_i^{-s_i-1}\right\}=$$

$$res_{x_\mu\ldots x_n}\left\{\sum_{l_\mu\ldots l_n=0}^{\infty}w_\mu^{l_\mu}\ldots w_n^{l_n}\prod_{i=\mu}^{n}w_i^{-s_i-1}\right\}\equiv 1.$$

□

4. Conclusion

In this article the standard scheme of the coefficients method is employed for a new, direct and simple calculation of the sum S_n from (1.2) which does not require knowledge of the answer. This proves the effectiveness of the method of coefficients when calculating multiple sums with polynomial coefficients and linear constraints on the summation indices, containing a family of free parameters. In this case, linear constraints are determined by specifying finite and infinite multiple sums with independent summation indices. Finally, the sum of S_n in Theorem 1 means that they can easily be rewritten using the conventional method of generating functions, using only well-known operations over the corresponding Laurent multiple power series.

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Короткое вычисление кратной суммы Кривоколеско – Лейнартаса с линейными ограничениями на индексы суммирования

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Аннотация. В конце 1970-х гг. автором был разработан метод интегрального представления и вычисления комбинаторных сумм различного типа (метод коэффициентов) с использованием формальных степенных рядов Лорана над $\mathbb{C}$, теории аналитических функций и теории кратных вычетов в $\mathbb{C}^n$. С тех пор этот метод нашел многочисленные применения в различных областях математики в нашей стране и за рубежом. На наш взгляд, особенно интересно и актуально использование метода коэффициентов при решении трудной проблемы вычисления кратных сумм с линейными ограничениями на индексы суммирования. Проблемы такого типа нередко возникают на практике при решении различных комбинаторных задач. Например, в 2016 г. автором в статье, опубликованной в журнале «Известия Иркутского государственного университета. Серия Математика», была вычислена кратная сумма с q-биномиальными коэффициентами и линейными рекуррентными соотношениями на индексы суммирования, возникшая при перечислении всех собственных t-мерных подпространств V_m над полем $GF(q)$.

В 2012 году В. П. Кривоколеско и Е. К. Лейнартас в журнале «Известия Иркутского государственного университета. Серия Математика» доказали с использованием композиции Адамара кратное тождество с полиномиальными коэффициентами и ограничениями различного типа на пределы суммирования, содержащее семейство свободных параметров. Это тождество является обобщением тождеств, изученных ранее несколькими авторами, начиная с построения фильтров Добеши в вейвлет-теории. Здесь по стандартной схеме метода коэффициентов проведено, не зная ответа, короткое и простое вычисление кратной суммы Кривоколеско – Лейнартаса. Это вычисление также автоматически дает эквивалентный способ вычисления указанной суммы с помощью традиционного метода производящих функций, используя лишь хорошо известные операции над соответствующими кратными степенными рядами Лорана.

Ключевые слова: комбинаторные суммы, метод коэффициентов, интегральные представления, производящие функции.

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