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Maximizing the Sum of Radii of Balls Inscribed in a Polyhedral Set

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Abstract. The sphere packing problem is one of the most applicable areas in mathematics which finds numerous applications in science and technology [1–4; 8; 9; 11–14]. We consider a maximization problem of a sum of radii of non-overlapping balls inscribed in a polyhedral set in Hilbert space. This problem is often formulated as the sphere packing problem. We extend the problem in Hilbert space as an optimal control problem with the terminal functional and constraints for the final moment. This problem belongs to a class of nonconvex optimal control problem and application of gradient methods does not always guarantee finding a global solution to the problem. We show that the problem in a finite dimensional case for three balls (spheres) is connected to well known Malfatti's problem [16]. Malfatti's generalized problem was examined in [6; 7] as the convex maximization problem employing the global optimality conditions of Strekalovsky [17].

Keywords: Hilbert space, maximization problem, optimality conditions, optimal control, sum of radii.

1. Statement of the problem and optimality conditions

Let X be a Hilbert space. We introduce the following sets. Denote by $B(x^0, r)$ a ball with a center $x^0 \in X$ and a radius $r \in I\!R$:

$$B(x^0, r) = \{x \in X \mid \, ||x - x^0|| \le r\}. \tag{1.1}$$

A polyhedral set $D \subset X$ is given by

$$D = \{x \in X| \ \langle a^i, x\rangle \leq b_i, \ a^i \in X, \ b_i \in I\!R, \ i = \overline{1,m}\}, \tag{1.2}$$

where $\langle \ , \ \rangle$ denotes the scalar product of two vectors in X, $||\cdot||$ is the norm on X, and $intD \neq \emptyset$. $a^i \in X, \ i = \overline{1,m}$ are linearly dependent.

Theorem 1. *$B(x^0, r) \subset D$ if and only if*

$$\langle a^i, x^0\rangle + r||a^i|| \leq b_i, \ i = \overline{1,m}. \tag{1.3}$$

The proof can be done in a similar way as in [6], [7].

Denote by $u^1, u^2, \ldots, u^K$ centers of the balls inscribed in D defined by (1.2). Let $r_1, r_2, \ldots, r_K$ be their corresponding radii.

Now we consider a problem of maximizing sum of radii of K non-overlapping balls inscribed in D.

Then this problem is formulated as follows:

$$\max f = \sum_{i=1}^{K} r_i \tag{1.4}$$

$$\langle a^i, u^j\rangle + r_j||a^i|| \leq b_i, \ i = \overline{1,m}; \ j = \overline{1,K}, \tag{1.5}$$

$$||u^l - u^j||^2 \geq (r_l + r_j)^2, \ l, j = \overline{1,K}; \ l \neq j, \tag{1.6}$$

$$r_1 \geq 0, r_2 \geq 0, \ldots, r_K \geq 0. \tag{1.7}$$

The function f in (1.4) denotes a sum of radii of K balls which has to be maximized with respect to variables $u^1, u^2, \ldots, u^K$ and $r_1, r_2, \ldots, r_K$. Condition (1.5) define that all balls are inscribed into a polyhedral set D. Conditions (1.6) describe non-overlapping balls.

In order to write optimality conditions for problem (1.4)-(1.7), we introduce the Lagrange function:

$$\mathcal{L}(u, r) = -\sum_{i=1}^{K} r_i + \sum_{i=1}^{m}\sum_{j=1}^{K} \lambda_{ij}\left[\langle a^i, u^j\rangle + r_j||a^i|| - b_i\right] + \\ + \sum_{l=1}^{K}\sum_{j\neq l}^{K} \mu_{lj}\left[(r_l + r_j)^2 - ||u^l - u^j||^2\right] - \sum_{j=1}^{K} l_j r_j$$

Then the Karush-Kuhn-Tucker(KKT)conditions applied for the problem (1.4)-(1.7) are as follows:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial r_j} = -1 + \sum\limits_{i=1}^{m} \lambda_{ij}||a^i|| + 2\sum\limits_{l=1}^{K}\sum\limits_{j\neq l}^{K} \mu_{lj}(r_l + r_j) - l_j = 0,\ j = \overline{1,K}, \\ \dfrac{\partial \mathcal{L}}{\partial u^j} = \sum\limits_{i=1}^{m} \lambda_{ij} a^i + 2\sum\limits_{l=1}^{K}\sum\limits_{j\neq l}^{K} \mu_{lj}(u^l - u^j) = 0,\ j = \overline{1,K}, \\ \lambda_{ij}(\langle a^i, u^j\rangle + r_j||a^i|| - b_i) = 0,\ i = \overline{1,m},\ j = \overline{1,K}, \\ \mu_{lj}\left[(r_l + r_j)^2 - ||u^l - u^j||^2\right] = 0,\quad l,j = \overline{1,K};\ l \neq j \\ l_j r_j = 0,\ j = \overline{1,K}. \end{cases}$$

Example 1. Let $X = L_2^p[t_0, t_1]$ be a space of functions

$$u(t) = (u_1(t), ..., u_p(t)),$$

$t_0 \leq\ t \leq\ t_1$ with the norm $||u||_{L_2} = \left(\int\limits_{t_0}^{t_1} |u(t)|^2 dt\right)^{1/2}$ and scalar product

$$\langle u, v\rangle_{L_2} = \int\limits_{t_0}^{t_1} \langle u(t), v(t)\rangle dt, \quad u, v \in X$$

Problem (1.4)-(1.7) has the form:

$$\begin{cases} \max f = \sum\limits_{i=1}^{K} r_i, \\ \int\limits_{t_0}^{t_1} \langle a^i(t), u^j(t)\rangle dt + r_j \left(\int\limits_{t_0}^{t_1} |a^i(t)|^2 dt\right)^{1/2} \leq b_i,\ i = \overline{1,m},\ j = \overline{1,K}, \\ \int\limits_{t_0}^{t_1} |u^l(t) - u^j(t)|^2 dt \geq (r_l + r_j)^2,\ l,j = \overline{1,K},\ l \neq j, \\ r_1 \geq 0,\ r_2 \geq 0, \ldots, r_K \geq 0,\ u \in L_2^p[t_0, t_1]. \end{cases} \tag{1.8}$$

Introduce the new variables $x_{ij}(t),\ y_{ij}(t),\ v_i(t),\ z_i(t)$ and $\gamma_i(t)$ as follows:

$$
\begin{aligned}
&x_{ij}(t) = \int\limits_{t_0}^{t} \langle a^i(t), u^j(t)\rangle dt,\ i = \overline{1,m},\ j = \overline{1,K},\\
&x_{ij}(t_0) = 0,\\
&\gamma_l(t) = \left(\int\limits_{t_0}^{t_1} |a^l(t)|^2 dt\right)^{1/2},\ l, j = \overline{1,K},\ l \neq j,\\
&y_{lj}(t) = \int\limits_{t_0}^{t} |u^l(t) - u^j(t))|^2 dt,\ l, j = \overline{1,K},\ l \neq j,\\
&y_{lj}(t_0) = 0,\\
&v_l = r_l,\ l = \overline{1,K},\\
&v_l'(t) = 0,\\
&y_{lj}(t_1) \geq (v_l(t_1) + v_j(t_1))^2,\ l, j = \overline{1,K},\ l \neq j,\ t \in [t_0, t_1].
\end{aligned}
$$

The problem (1.8) is formulated as the following optimal control problem with the terminal functional and constraints.

$$
\begin{aligned}
&\max \mathcal{J}(u, v, y) = \sum_{i=1}^{K} v_i(t_1),\\
&x_{ij}'(t) = \langle a^i(t), u^j(t)\rangle, i = \overline{1,m},\ j = \overline{1,K},\\
&y_{lj}'(t) = ||u^l(t) - u^j(t))||^2,\ l, j = \overline{1,K},\ l \neq j,\\
&v_i'(t) = 0,\ i = \overline{1,K},\\
&x_{ij}(t_0) = 0,\ i = \overline{1,m},\ j = \overline{1,K},\\
&x_{ij}(t_1) + v_j(t_1)\gamma_i \leq b_i,\ i = \overline{1,m},\ j = \overline{1,K},\\
&y_{ij}(t_0) = 0,\ i = \overline{1,m},\ j = \overline{1,K},\\
&y_{lj}(t_1) \geq (v_l(t_1) + v_j(t_1))^2,\ l, j = \overline{1,K},\ l \neq j.
\end{aligned}
$$

Existence of a solution of the above problem and numerical solutions will be discussed in a next paper.

Example 2. Let $X = \mathbb{R}^2$ and D be a triangle set. Assume that $K = 3$. In this case, we can reformulate problem (1.4)-(1.7) as the perimeter maximization problem:

$$
\begin{cases}
\max f = 2\pi \sum_{i=1}^{3} r_i,\\
\langle a^i, u^j\rangle + r_j ||a^i|| \leq b_i,\ i, j = \overline{1,3},\\
||u^i - u^j||^2 \geq (r_i + r_j)^2,\ i, j = \overline{1,3},\ i \neq j,\\
r_1 \geq 0,\ r_2 \geq 0, r_3 \geq 0.
\end{cases}
\tag{1.9}
$$

This problem is maximization of a linear function over the nonconvex set and belongs to a class of nonconvex optimization problem.

2. Connection of the perimeter maximization problem to Malfatti's problem

We also consider the following problem of maximizing total area of 3 balls inscribed in a triangle set. This problem in the literature [16], [18], [10] is called Malfatti's problem. In [5] it was shown that the global optimality conditions by Strekalovsky [17] can be applied to Malfatti's problem. Also, numerical methods and algorithms for solving Malfatti's problem have been developed in [6] and [7]. The Malfatti's problem first formulated in [5] as the convex maximization problem as:

$$\begin{cases} \max S = \pi \sum\limits_{i=1}^{3} r_i^2, \\ \langle a^i, u^j \rangle + r_j ||a^i|| \le b_i,\ i = \overline{1,3},\ j = \overline{1,3}, \\ (r_i + r_j)^2 - ||u^i - u^j||^2 \le 0,\ i,j = \overline{1,3},\ i \ne j, \\ r_1 \ge 0,\ r_2 \ge 0, r_3 \ge 0. \end{cases} \tag{2.1}$$

Theorem 2. *A solution of the perimeter maximization problem* (1.9) *is a stationary point of Malfatti's problem* (2.1)*.*

Proof. We write down the lagrange functions for the problems (1.9) and (2.1), respectively

$$\mathcal{L}(u,r) = -2\pi \sum_{i=1}^{3} r_i + \sum_{i=1}^{3}\sum_{j=1}^{3} \lambda_{ij} \left[\langle a^i, u^j \rangle + r_j ||a^i|| - b_i\right] + \\ + \sum_{i=1}^{3}\sum_{j\ne i}^{3} \mu_{ij} \left[(r_i + r_j)^2 - ||u^i - u^j||^2\right] - \sum_{j=1}^{3} l_j r_j;$$

$$\overline{\mathcal{L}}(u,r) = -\pi \sum_{i=1}^{3} r_i^2 + \sum_{i=1}^{3}\sum_{j=1}^{3} \lambda_{ij} \left[\langle a^i, u^j \rangle + r_j ||a^i|| - b_i\right] + \\ + \sum_{i=1}^{3}\sum_{j\ne i}^{3} \mu_{ij} \left[(r_i + r_j)^2 - ||u^i - u^j||^2\right] - \sum_{j=1}^{3} l_j r_j;$$

Let a point $z = (\bar{u}^1, \bar{u}^2, \bar{u}^3, \bar{r}_1, \bar{r}_2, \bar{r}_3)$ be a solution to problem (1.9) with $\bar{r}_i > 0,\ i = 1, 2, 3.$ Then there exist Lagrange multipliers $(\bar{\lambda}_{ij}, \bar{\mu}_{ij}, \bar{l}_j)$ such that the optimality conditions (KKT) are satisfied as:

$$\begin{cases} \dfrac{\partial \mathcal{L}(z)}{\partial r_i} = -2\pi + \displaystyle\sum_{j=1}^{3} \bar{\lambda}_{ij}||a^i|| + \sum_{j\neq i}^{3} 2\bar{\mu}_{ij}(\bar{r}_i + \bar{r}_j) - \bar{l}_i = 0,\ i = \overline{1,3}, \\ \dfrac{\partial \mathcal{L}(z)}{\partial u^j} = \displaystyle\sum_{i=1}^{3} \bar{\lambda}_{ij} a^i + \sum_{j\neq i}^{3} 2\bar{\mu}_{ij}(\bar{u}^i - \bar{u}^j) = 0,\ j = \overline{1,K}, \\ \bar{\lambda}_{ij}(\langle a^i, \bar{u}^j\rangle + \bar{r}_j||a^i|| - b_i) = 0,\ i,j = \overline{1,3}, \\ \bar{\mu}_{ij}\left[(\bar{r}_i + \bar{r}_j)^2 - ||\bar{u}^i - \bar{u}^j||^2\right] = 0,\ \ i,j = \overline{1,3};\ i \neq j, \\ \bar{l}_j \bar{r}_j = 0,\ j = \overline{1,3}. \end{cases}$$

On the other hand, it can be checked that the point z satisfies the KKT conditions for problem (2.1) with the Lagrange multipliers $\tilde{\lambda}_{ij} = \dfrac{\bar{\lambda}_{ij}}{\bar{r}_i}$, $\tilde{\mu}_{ij} = \dfrac{\bar{\mu}_{ij}}{\bar{r}_i}$, $\tilde{l}_i = \dfrac{\bar{l}_i}{\bar{r}_i}$, $i,j = \overline{1,3}$.

Indeed, we have

$$\begin{cases} \dfrac{\partial \bar{\mathcal{L}}(z)}{\partial r_i} = -2\pi + \displaystyle\sum_{i=1}^{3} \tilde{\lambda}_{ij}||a^i|| + \sum_{j\neq i}^{3} 2\tilde{\mu}_{ij}(\bar{r}_i + \bar{r}_j) - \tilde{l}_i = 0,\ i = \overline{1,3}, \\ \dfrac{\partial \bar{\mathcal{L}}(z)}{\partial u^j} = \displaystyle\sum_{i=1}^{3} \tilde{\lambda}_{ij} a^i + \sum_{j\neq i}^{3} 2\tilde{\mu}_{ij}(\bar{u}^i - \bar{u}^j) = 0,\ j = \overline{1,3}, \\ \tilde{\lambda}_{ij}(\langle a^i, \bar{u}^j\rangle + \bar{r}_j||a^i|| - b_i) = 0,\ i,j = \overline{1,3}, \\ \tilde{\mu}_{ij}\left[(\bar{r}_i + \bar{r}_j)^2 - ||u^i - u^j||^2\right] = 0,\ \ i,j = \overline{1,3};\ i \neq j. \end{cases}$$

which means that the point z is a stationary point of Malfatti's problem.

Inversely, we can show that if $\bar{z}$ is a solution to Malfatti's problem then $\bar{z}$ is a stationary point of problem (1.9). □

3. Conclusion

In this paper, we formulate the problem of maximizing a sum of radii of non-overlapping balls inscribed in a polyhedral set in Hilbert space as an optimal control problem with the terminal functional and constraints for the final moment. The problem belongs to a class of global optimization. We show that the problem in a finite dimensional case is connected to Malfatti's problem via its optimality conditions.

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Максимизация суммы радиусов шаров вписанных в многогранник

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Аннотация. Задача упаковки шаров имеет множество приложений в различных областях науки и техники. Мы рассматриваем задачу максимизации суммы радиусов непересекающихся шаров, вписанных в многогранное множество в гильбертовом пространстве. Такая задача часто формулируется как задача упаковки. Рассматривая задачу в гильбертовом пространстве, мы формулируем ее как задачу оптимального управления с терминальным функционалом и терминальными ограничениями на конечный момент времени. Эта задача принадлежит к классу невыпуклых задач оптимального управления, поэтому применение градиентного метода не всегда гарантирует нахождения глобального решения для данной задачи. В работе показано, что задача для трех кругов в конечномерном пространстве является хорошо известной задачей Мальфатти [16]. Дополнительно доказано, что максимизация суммы радиусов кругов, вписанных в треугольник, эквивалентна задаче Мальфатти. Обобщенная задача Мальфатти рассматривалась как задача выпуклой максимизации в работах [6; 7] с применением условия глобальной оптимальности А. С. Стрекаловского [17].

Ключевые слова: гильбертово пространство, выпуклая максимизация, условия оптимальности, оптимальное управление, радиус шаров.

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