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# Ranks for Families of Permutation Theories *

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**Abstract.** The notion of rank for families of theories, similar to Morley rank for fixed theories, serves as a measure of complexity for given families. There arises a natural problem of describing a rank hierarchy for a series of families of theories.

In this article, we answer the question posed and describe the ranks and degrees for families of theories of permutations with different numbers of cycles of a certain length. A number examples of families of permutation theories that have a finite rank are given, and it is constructed a family of permutation theories having a specified countable rank and degree $n$. It is proved that in the family of permutation theories any theory equals a theory of a finite structure or it is approximated by finite structures, i.e. any permutation theory on an infinite set is pseudofinite. Topological properties of the families under consideration were studied.

**Keywords:** family of theories, pseudofinite theory, permutation, rank, degree.

A rank for the families of theories, similar to Morley rank and defined in [9], can be considered as a measure for complexity or richness of these families. Thus increasing the rank by extensions of families we produce more rich families obtaining families with the infinite rank that can be considered “rich enough”.

Permutation theories and theories in the language of one unary function have been studied in a number of papers, including [1; 2; 5; 7; 8]. In the present paper, we describe ranks and degrees for families of permutation theories, partially answering a question in [9].

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## 1. Preliminaries

Throughout hereinafter we consider families $\mathcal{T}$ of complete first-order theories of a language $\Sigma = \Sigma(\mathcal{T})$ and use the following terminology from [3; 6; 9; 10].

**Definition 1.** [10] Let $\mathcal{T}$ be a family of theories and $T$ be a theory, $T \notin \mathcal{T}$. The theory $T$ is called $\mathcal{T}$-approximated, or approximated by $\mathcal{T}$, or $\mathcal{T}$ - approximable, or a pseudo-$\mathcal{T}$ -theory, if for any formula $\varphi \in T$ there is $T' \in \mathcal{T}$ such that $\varphi \in T'$.

If $T$ is $\mathcal{T}$-*approximated* then $\mathcal{T}$ is called an *approximating family* for $T$, theories $T' \in \mathcal{T}$ are *approximations* for $T$, and $T$ is an *accumulation point* for $\mathcal{T}$.

We put $\mathcal{T}_\varphi = \{T \in \mathcal{T} \mid \varphi \in T\}$. Any set $\mathcal{T}_\varphi$ is called the $\varphi$-*neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$.

An approximating family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite.

It was shown in [10] that any *e-minimal* family $\mathcal{T}$ has unique accumulation point $T$ with respect to neighbourhoods $\mathcal{T}_\varphi$, and $\mathcal{T} \cup \{T\}$ is also called *e-minimal*.

**Proposition 1.** [10] *A theory $T \notin \mathcal{T}$ is $\mathcal{T}$-approximated if and only if $T \in \mathrm{Cl}_E(\mathcal{T})$.*

**Definition 2.** [6] An infinite structure $\mathcal{M}$ is pseudofinite if every sentence true in $\mathcal{M}$ has a finite model.

If $T = Th(\mathcal{M})$ for pseudofinite $\mathcal{M}$ then $T$ is called pseudofinite as well.

We denote by $\overline{\mathcal{T}}$ the class of all complete elementary theories, by $\overline{\mathcal{T}}_{fin}$ the subclass of $\overline{\mathcal{T}}$ consisting of all theories with finite models.

**Proposition 2.** [10] *For any theory $T$ the following conditions are equivalent:*

(1) *$T$ is pseudofinite;*
(2) *$T$ is $\overline{\mathcal{T}}_{fin}$-approximated;*
(3) *$T \in Cl_E(\overline{\mathcal{T}}_{fin}) \setminus \overline{\mathcal{T}}_{fin}$.*

Following [9] we define the *rank* $\mathrm{RS}(\cdot)$ for the families of theories, similar to Morley rank [4], and a hierarchy with respect to these ranks in the following way.

$\mathrm{RS}(\mathcal{T}) = -1$, if the family $\mathcal{T}$ is empty;
$\mathrm{RS}(\mathcal{T}) = 0$, if the family $\mathcal{T}$ is finite and non-empty;
$\mathrm{RS}(\mathcal{T}) \geq 1$, if the family $\mathcal{T}$ is infinite.

For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent $\Sigma(\mathcal{T})$-sentences $\varphi_n$, $n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.

If $\alpha$ is a limit ordinal then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.

We set $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.

If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any $\alpha$, we put $\mathrm{RS}(\mathcal{T}) = \infty$.

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.

**Proposition 3.** [9] *If an infinite family $\mathcal{T}$ does not have e-minimal subfamilies $\mathcal{T}_\varphi$ then $\mathcal{T}$ is not totally transcendental.*

If $\mathcal{T}$ is $e$-totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences $\varphi_i$ such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

It is described in [3] the ranks and degrees for the families $\mathcal{T}_\Sigma$ of all theories of an arbitrarily given language $\Sigma$ and (non-) totally transcendental families were characterized.

**Theorem 1.** [3] *If $\Sigma$ is a language containing an $m$-ary predicate symbol, for $m \geq 2$, or an $n$-ary functional symbol, for $n \geq 1$, then $RS(\mathcal{T}_\Sigma) = \infty$.*

Furthermore, it is shown an applications of these characteristics for the families $\mathcal{T}_{\Sigma,n}$ of all theories of languages $\Sigma$ and having $n$-element models, where $n \in \omega$, as well as for the $\mathcal{T}_{\Sigma,\infty}$ families of all theories of languages $\Sigma$ and having infinite models.

Clearly, for any language $\Sigma$, $\mathcal{T}_\Sigma = \bigcup\limits_{n\in\omega} \mathcal{T}_{\Sigma,n} \cup \mathcal{T}_{\Sigma,\infty}$. Therefore, by monotony of RS, we have for any $n \in \omega$ the following relations are true:

$$\mathrm{RS}(\mathcal{T}_{\Sigma,n}) \leq \mathrm{RS}(\mathcal{T}_\Sigma),$$

$$\mathrm{RS}(\mathcal{T}_{\Sigma,\infty}) \leq \mathrm{RS}(\mathcal{T}_\Sigma).$$

**Theorem 2.** [3] *For any language $\Sigma$ either* $\mathrm{RS}(\mathcal{T}_{\Sigma,n}) = 0$, *if $\Sigma$ is finite or $n = 1$ and $\Sigma$ has finitely many predicate symbols, or* $\mathrm{RS}(\mathcal{T}_{\Sigma,n}) = \infty$, *otherwise.*

**Theorem 3.** [3] *For any language $\Sigma$ either* $\mathrm{RS}(\mathcal{T}_{\Sigma,\infty})$ *is finite, if $\Sigma$ is finite and without predicate symbols of arities $m \geq 2$ as well as without functional symbols of arities $n \geq 1$, or* $\mathrm{RS}(\mathcal{T}_{\Sigma,\infty}) = \infty$, *otherwise.*

**Proposition 4.** [10] *Any family $\mathcal{T}$ of theories can be expanded till a family $\mathcal{T}'$ with the least generating set.*

**Definition 3.** [9] A family $\mathcal{T}$, with infinitely many accumulation points, is called *a-minimal* if for any sentence $\varphi \in \Sigma(T)$, $\mathcal{T}_\varphi$ or $\mathcal{T}_{\neg\varphi}$ has finitely many accumulation points.

Let $\alpha$ be an ordinal. A family $\mathcal{T}$ of rank $\alpha$ is called *$\alpha$-minimal* if for any sentence $\varphi \in \Sigma(T)$, $\mathrm{RS}(\mathcal{T}_\varphi) < \alpha$ or $\mathrm{RS}(\mathcal{T}_{\neg\varphi}) < \alpha$.

**Proposition 5.** [9] (1) *A family $\mathcal{T}$ is* 0*-minimal if and only if $\mathcal{T}$ is a singleton.*

(2) *A family $\mathcal{T}$ is* 1*-minimal if and only if $\mathcal{T}$ is e-minimal.*

(3) *A family $\mathcal{T}$ is* 2*-minimal if and only if $\mathcal{T}$ is a-minimal.*

(4) *For any ordinal $\alpha$ a family $\mathcal{T}$ is $\alpha$-minimal if and only if* $\mathrm{RS}(\mathcal{T}) = \alpha$ *and* $\mathrm{ds}(\mathcal{T}) = 1$.

**Theorem 4.** [9] *For any family $\mathcal{T}$, $\mathrm{RS}(\mathcal{T}) = 2$ with $\mathrm{ds}(\mathcal{T}) = n$, if and only if $\mathcal{T}$ is represented as a disjoint union of subfamilies $\mathcal{T}_{\varphi_1}, \ldots, \mathcal{T}_{\varphi_n}$, for some pairwise inconsistent sentences $\varphi_1, \ldots, \varphi_n$, such that each $\mathcal{T}_{\varphi_i}$ is a-minimal.*

**Proposition 6.** [9] *For any family $\mathcal{T}$, $\mathrm{RS}(\mathcal{T}) = \alpha$ with $\mathrm{ds}(\mathcal{T}) = n$, if and only if $\mathcal{T}$ is represented as a disjoint union of subfamilies $\mathcal{T}_{\varphi_1}, \ldots, \mathcal{T}_{\varphi_n}$, for some pairwise inconsistent sentences $\varphi_1, \ldots, \varphi_n$, such that each $\mathcal{T}_{\varphi_i}$ is $\alpha$-minimal.*

## 2. Ranks for families of permutation theories

In this section, we describe the ranks for families of permutation theories.

Let a language $\Sigma$ consist of the permutation $f$. Denote by $\mathcal{T}_\Sigma$ the family of all permutation theories of language $\Sigma$.

Each permutation $f$ has the axiom $\forall y \exists^{=1} x (f(x) = y)$. The *length* of a permutation cycle is the number of its elements. The *type* of permutation $f$ is the vector $\lambda(f) = (\lambda_1(f), \ldots, \lambda_n(f), \ldots)$, where $\lambda_i(f)$ is the number of cycles of length $i$ in the permutation $f$. Note that for any permutation $f$ the value $\sum_{i=1}^{n} i \cdot \lambda_i(f)$ is equal to the power of the set of elements that make up the cycles.

Let $T$ be a permutation theory, $\mathcal{M} \models T$. An element $a \in M$ is called *acyclic* if $a$ does not belong to any cycle.

Note that by the compactness theorem the theories with cycles of unbounded length generate acyclic elements in some their models. If the cycle lengths are limited in aggregate, then there are theories with both acyclic elements and without acyclic elements.

For a given theory of $T$ permutations, we can consider the set of pairs $(n, \lambda_n)$, where $n \in \omega$ and $\lambda_n \in \omega \cup \{\infty\}$ is the number of cycles of length $n$. As $\varepsilon \in \{0, 1\}$, indicating the absence / presence of successor function, we can take the value 0 if there is no successor function and 1 if there is such a function. Moreover, $\varepsilon = 1$ if $\{\lambda_n > 0 \mid n \in \omega\}$ is infinite. In particular, the closure contains the theory with $\varepsilon = 1$, if there are theories in this family with $\lambda_n > 0$ for an infinite number of values of $n$.

Note that the characteristics $(n, \lambda_n)$, $n \in \omega$, and $\varepsilon$ uniquely define this permutation theory. The ranks of families of permutation theories are given by sets of these characteristics. Below we consider all possible cases.

**Families with a bounded number of positive $\lambda_n$ with $\varepsilon = 0$**

A family $\mathcal{T} \subseteq \mathcal{T}_\Sigma$ of permutations can be infinite only if there are unbounded values of $\lambda_n$, and if the family has a finite number of variants for $\lambda_n$, then it is finite, $RS(\mathcal{T}_\Sigma) = 0$ and $ds(\mathcal{T}_\Sigma)$ is equal to the number of these variants; if $\lambda_n$ has an infinite number of variants in the theories from this family, then we need to look at the number of accumulation points, with $\lambda_n = \infty$, depending on whether the remaining $\lambda_n$ are fixed or not. If the number of accumulation points is finite, then $RS(\mathcal{T}_\Sigma) = 1$ and $ds(\mathcal{T}_\Sigma)$ correspond to the number of these accumulation points. If the number of accumulation points is infinite, then $RS(\mathcal{T}_\Sigma) \geq 2$ and we need to look at how many accumulation points the accumulation points themselves generate.

**Example 1.** Let $\mathcal{T}'_\Sigma$ be the family for all identical permutations. Then $RS(\mathcal{T}'_\Sigma) = 1$ and $ds(\mathcal{T}'_\Sigma) = 1$, therefore, $\mathcal{T}'_\Sigma$ is $e$-minimal. The only accumulation point is the theory of identical permutations on an infinite set.

**Example 2.** If we consider the cycles of length $n_0$ and $n_1$ and the number of these cycles satisfies $\lambda_{n_0} < k, \lambda_{n_1} < l$, then for the family $\mathcal{T}_{\lambda_{n_0},\lambda_{n_1}}$ of theories with these relations, we have $k \cdot l$ variants, $RS(\mathcal{T}_{\lambda_{n_0},\lambda_{n_1}}) = 0$, and $ds(\mathcal{T}_{\lambda_{n_0},\lambda_{n_1}}) = k \cdot l$.

We denote by $\mathcal{T}_n$ the set of all theories from $\mathcal{T}_\Sigma$ with one arbitrary value $\lambda_n$, where $\lambda_m = 0$ for $m \neq n$ and the models of these theories do not have acyclic elements.

**Proposition 7.** *Each family $\mathcal{T}_n$ is e-minimal.*

*Proof.* The family $\mathcal{T}_n$ consists of theories $T_m$ with $m \in \omega\setminus\{0\}$ cycles of length $n$ and the theory $T_\infty$ with an infinite number of cycles of length $n$. The theory $T_\infty$ is the unique accumulation point for $\mathcal{T}_n$. Thus, $RS(\mathcal{T}_n) = 1$, $ds(\mathcal{T}_n) = 1$ and therefore the family $\mathcal{T}_n$ is $e$-minimal. □

**Example 3.** If we allow cycles of different lengths $n_0$ and $n_1$, then we get a countable number of variants $(\lambda_{n_0}, \lambda_{n_1})$, where $\lambda_{n_0}$ is the number of cycles of length $n_0$, and $\lambda_{n_1}$ is the number of cycles of length $n_1$. Thus, there are countably many theories with cycles of length $n_0$ and $n_1$, forming the family $\mathcal{T}_{n_0,n_1}$. Here, each theory with one infinite $\lambda_{n_0}$ or $\lambda_{n_1}$ has $RS(\mathcal{T}_{n_0,n_1}) = 1$, and the only limit point c $\lambda_{n_0} = \lambda_{n_1} = \infty$, has infinitely many cycles of length $n_0$, infinitely many cycles of length $n_1$ and $RS(\mathcal{T}_{n_0,n_1}) = 2$. Thus, for a given family $\mathcal{T}_{n_0,n_1}$, we obtain $RS(\mathcal{T}_{n_0,n_1}) = 2$ and $ds(\mathcal{T}_{n_0,n_1}) = 1$. Therefore, the family is $a$-minimal.

**Example 4.** If we consider cycles of different lengths $n_0, n_1$ and $n_2$, then you also obtain countably many possibilities $(\lambda_{n_0}, \lambda_{n_1}, \lambda_{n_2})$, where $\lambda_{n_0}$ is the number of cycles of length $n_0$, $\lambda_{n_1}$ is the number of cycles of length $n_1$, $\lambda_{n_2}$ is the number of cycles of length $n_2$. Each $e$-minimal subfamily with one infinite $\lambda_{n_0}, \lambda_{n_1}$ or $\lambda_{n_2}$ containing theories with only one positive $\lambda_{n_i}$ has the rank $RS = 1$. Theories with nonzero $\lambda_{n_i}, \lambda_{n_j}$ and $\lambda_{n_k} = 0$, $\{i, j, k\} = \{0, 1, 2\}$ have $RS = 2$ and $ds = 1$. A family of all theories with $\lambda_m = 0$ for $m \notin \{n_0, n_1, n_2\}$ has $RS = 3$ and $ds = 1$.

Thus, adding new $\lambda_n$ for cycles of a certain length, one can unboundedly increase the rank to any pre-given natural number.

**Families with a bounded number of positive $\lambda_n$ with $\varepsilon = 1$**

As in the previous case, families are $e$-totally transcendental and may contain $e$-minimal, $a$-minimal, $\alpha$-minimal subfamilies. As well as their model may contain copies of successor function $(\mathbb{Z}, s)$ on integers. In this case, the rank of a family is ordinal. Repeating the argument for the previous case, we obtain the ranks $RS = \alpha$ and $ds = m$.

By the definition of $\alpha$-minimality and Proposition 7, the family $\mathcal{T}$ of theories of permutations with $RS(\mathcal{T}) = \alpha$ and $ds(\mathcal{T}) = m$ can be represented as a disjoint union of subfamilies $\mathcal{T}_{\lambda_{n_0}}, \ldots, \mathcal{T}_{\lambda_{n_{m-1}}}$, for some different $\lambda_{n_0}, \ldots, \lambda_{n_{m-1}}$, such that each $\mathcal{T}_{\lambda_i}$ is $\alpha$-minimal.

The following theorem shows that there is a family of permutation theories, having a countable rank.

**Theorem 5.** *For any countable ordinal $\alpha$ and natural $k \geq 1$ there exists a family $\mathcal{T} \subseteq \mathcal{T}_\Sigma$, such that $RS(\mathcal{T}) = \alpha$ and $ds(\mathcal{T}) = k$.*

*Proof.* Realizations of finite ranks by families of permutation theories shows that, in order to prove the theorem, it suffices to construct a family of theories that has a specified countable rank and degree $n$.

We choose the number $s \in \omega \setminus \{0\}$ and let the considered theories of permutations $T$ have an arbitrary number of cycles of length $s$ and at most one cycle of length $m$ for each $m \neq s$.

This countable rank $\alpha$ for a family of permutation theories $T$ can be constructed using countable sets $X$ consisting of natural numbers $m \neq s$, each of which symbolizes the presence of a single cycle of length $m$ in models of theory $T$ and the absence of cycles of length $m' \neq s$ with $m' \notin X$. Since the equality $\mathrm{RS} = \alpha$ implies that the corresponding Boolean algebra is superatomic, the families $X$ must form a hierarchy in which every transition from the set $X$, specifying a family of rank $\beta \leq \alpha$, to the sets $X_i$, $i \in \omega$, specifying disjoint subfamilies of theories of lower rank, must satisfy the following conditions:

$$X \subset X_i, |X_i \setminus X| = \omega,$$

$$(X_i \cap X_j) \setminus X = \emptyset,$$

where $i, j \in \omega, i \neq j$.

Herewith the chains with respect to the inclusion consisting of the sets $X$ should be well ordered. The sets $X$ are indexed by the pairs $\langle \beta, k \rangle$, where $\beta \leq \alpha$, $k < m$ for $\beta = \alpha$, and $k \in \omega$ for $\beta < \alpha$. Thus, the sets $X_\beta$ for the ordinal $\beta = \gamma + 1$ are expanded by a countable family of sets $X_{\langle\gamma,k\rangle}$ pairwise disjoint over $X_\beta$.

Each set $X_{\langle 1,k\rangle}$ defines a suitable $e$-minimal family of $\mathcal{T}_{\langle 1,k\rangle}$ theories having one cycle of each length $m \in X_{\langle 1,k\rangle}$ and an arbitrary number of cycles of length $s$. Denote by $\mathcal{T}$ the union of all families $\mathcal{T}_{\langle 1,k\rangle}$. Using induction, it is easy to show that the formulas $\varphi_\beta$, describing the presence of cycles of length $m' \in \langle\beta, k'\rangle$, they define neighborhoods of $\mathcal{T}_{\varphi_\beta}$ with rank $\beta$. Thus, $\mathrm{RS}(\mathcal{T}) = \alpha$ and $\mathrm{ds}(\mathcal{T}) = n$ is established. □

**Families with infinitely many positive $\lambda_n$**

In the family $\mathcal{T}_\Sigma$ of permutation theories there are theories with infinite cycles and cycles with unbounded lengths. As well as in the models of these theories we observe automatically the presence of a copy of successor function $(\mathbb{Z}, s)$ on integers. In this case, there is an infinite 2-tree formed thus, the family has $RS(\mathcal{T}_\Sigma) = \infty$.

**Theorem 6.** *Any theory $T$ of a permutation on an infinite set is pseudofinite.*

*Proof. Case 1.* Let $T$ have a finite number of cycles. Then $T$ has a model $\mathcal{M} = \mathcal{M}_0 \sqcup \mathcal{M}_1$, where $\mathcal{M}_0$ is a subsystem consisting of cycles, and $\mathcal{M}_1$ is a subsystem without cycles. For this model, the following is true:

$$\mathcal{M} = \lim_{i\to\infty} \mathcal{M}'_i,$$

where $\mathcal{M}'_i = \mathcal{M}_0 \sqcup \mathcal{N}_i$ is finite and $\mathcal{N}_i$ is the structure consisting of one cycle of length $i$. Thus $\{Th(\mathcal{N}_i) \mid i \in \omega\}$ approximates the theory $Th(\mathcal{M}_1)$, and $\{Th(\mathcal{M}_i) \mid i \in \omega\}$ approximates the theory of $T$.

*Case 2.* Let the theory $T$ have infinitely many cycles and the lengths of the cycles are bounded in the aggregate. Let $n_0, \ldots, n_k$ be the cycle lengths, $\lambda_0, \ldots, \lambda_k$ are the number of cycles of length $n_0, \ldots, n_k$, $\lambda_0, \ldots, \lambda_r$ finite, $\lambda_{r+1}, \ldots, \lambda_k$ are infinite, $r < k$ and $\mathcal{M}_0 \sqcup \mathcal{N}_i$, where $\mathcal{N}_i$ consists of $i$ cycles of length $n_{r+1}, \ldots, n_k$. Then the set $\{Th(\mathcal{N}_i) \mid r \in \omega\}$ approximates the theory $Th(\mathcal{M}_1)$, where $\mathcal{M}_1$ consists of an infinite number of cycles of length $n_{r+1}, \ldots, n_k$. Thus, $\{Th(\mathcal{M}_0 \sqcup \mathcal{N}_i) | i \in \omega\}$ approximates the theory $T = Th(\mathcal{M}_0 \sqcup \mathcal{N}_\omega)$.

*Case 3.* Let $T$ have infinitely many cycles and the lengths of the cycles are not limited in aggregate. Let $n_0, \ldots, n_k, \ldots$ be the cycle lengths. In this case, the theory is $T = Th(\mathcal{M}_0 \sqcup \mathcal{M}_1)$, where $\mathcal{M}_0$ is a subsystem consisting of cycles, and $\mathcal{M}_1$ is a subsystem without cycles. The $\mathcal{N}_i$ subsystem consists of $\leq i$ cycles of length $j \leq i$ and does not contain cycles of length $> i$. Let $n_s$ be the cycle length, $\lambda_s$ be the number of cycles of length $n_s$ in the model of the theory $T$. Then $\mathcal{N}_i$ contains $min\{i, \lambda_s\}$ cycles of length $n_s$. The subsystem $\mathcal{M}_1$ is obtained by the compactness theorem. So the $T$ theory is approximated by a family of theories $Th(\mathcal{N}_i)$. □

Since $E$-closures of families of theories preserve the rank [9, Theorem 2.10], Theorem 6 shows that to calculate the ranks of families of permutation theories, it suffices to consider suitable families of permutation theories on finite sets.

## 3. Conclusion

In the paper the ranks and degrees for families of permutation theories with different numbers of cycles of a certain length are described. Several examples of families of finite rank permutation theories are given. A family of permutation theories is constructed that has a specified countable rank and degree $n$. It is proved that any permutation theory on an infinite set is pseudofinite. Topological properties of families of permutation theories are studied.

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## Ранги для семейств теорий подстановок

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**Аннотация**. Понятие ранга для семейств теорий, аналогичное рангу Морли для фиксированных теорий, служит мерой сложности для данных семейств. Возникает естественная проблема описания иерархии ранга для ряда семейств теорий.

В данной статье мы, отвечая на поставленный вопрос, описываем ранги и степени для семейств теорий подстановок с разным числом циклов определенной длины. Приведено несколько примеров семейств теорий подстановок, которые имеют конечный ранг, а также построено семейство теорий подстановок, имеющее данный счетный ранг и данную степень $n$. Доказано, что в семействе теорий подстановок любая теория является теорией конечной структуры или аппроксимируется теориями конечных структур, т. е. любая теория подстановки на бесконечном множестве является псевдоконечной. Изучены топологические свойства рассматриваемых семейств.

**Ключевые слова:** семейство теорий, псевдоконечная теория, подстановки, ранг, степень.

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