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Existence of periodic solution to one dimensional free boundary problems for adsorption phenomena

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Abstract. In this paper we consider a drying and wetting process in porous medium to create a mathematical model for concrete carbonation. The process is assumed to be characterized by the growth of the air zone and a diffusion of moisture in the air zone. Under the assumption we proposed a one-dimensional free boundary problem describing adsorption phenomena in a porous medium. The free boundary problem it to find a curve representing the air zone and the relative humidity of the air zone. For the problem we also established existence, uniqueness and a large time behavior of solutions. Here, by improving the method for uniform estimates we can show the existence of a periodic solution of the problem. Also, the extension method is applied in the proof. This idea is quite important and new since the value of the humidity on the free boundary is unknown.

Keywords: free boundary problem, periodic solution, fixed point argument.

1. Introduction

Recently, we have proposed and studied the following free boundary problem (1.1) – (1.5) to investigate concrete carbonation process. In this model we consider a drying and wetting processes in one hole of a porous medium and regard the hole as a one-dimensional interval $[0, L]$, where L is the length of the hole. Also, we suppose that the interval consists of the water-drop (liquid) zone $[0, s(t))$ and the air zone $(s(t), L]$, and denote by u the relative humidity in the air zone, where $t \in [0, T]$ is the time variable and $x = s$ is a curve with $0 \leq s < 1$ on $[0, T]$ for $T > 0$ (see Figure 1).

The problem is to find a pair of the curve $x = s(t)$ and the function u on

$$Q_s(T) := \{(t, x); 0 < t < T, s(t) < x < L\}$$

(see Figure 2) satisfying

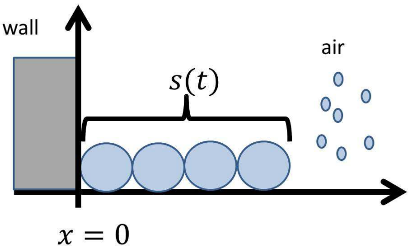

Figure 1.: Image of one hole

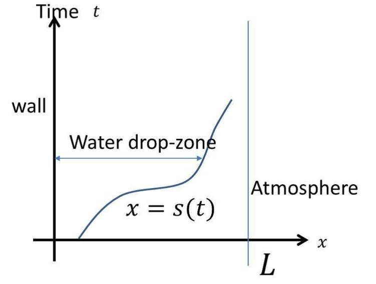

Figure 2.: Domain of the problem

$$\rho_g u_t - \kappa u_{xx} = 0 \quad \text{in } Q_s(T), \tag{1.1}$$

$$u(t, L) = b(t) \quad \text{for } 0 < t < T, \tag{1.2}$$

$$\dot{s}(t)(:= \frac{d}{dt} s(t)) = a(u(t, s(t)) - \varphi(s(t))) \quad \text{for } 0 < t < T, \tag{1.3}$$

$$\kappa u_x(t, s(t)) = (\rho_a - \rho_g u(t, s(t)))\dot{s}(t) \quad \text{for } 0 < t < T, \tag{1.4}$$

$$s(0) = s_0, u(x, 0) = u_0(x) \quad \text{for } s_0 < x < L, \tag{1.5}$$

where ρ_a and ρ_g are constants of the density of the aqueous-H_2O and the gaseous-H_2O, respectively, κ is a diffusion constant of the gaseous-H_2O, a positive constant a and a continuous function φ on $\mathbb{R}$ indicate the growth rate of the liquid zone, b is a given boundary function on $[0, T]$, and s_0 and u_0 are initial data.

This problem was first proposed in 2013 [5]. The main idea of the modeling is to regard that the degree of saturation is ratio of the length of the liquid region to the whole length. By this assumption we can describe the relationship between the degree of saturation and the relative humidity by the free boundary problem. Before this research we adopted a play operator as a mathematical description for the relationship, for example, [2;3;10]. However, the system with the play operator model has a difficulty on regularities of solutions. Then, we arrived at the free boundary model to overcome the difficulty.

In [5; 12] under suitable conditions for a, φ, b, ρ_a, ρ_g, s_0 and u_0 we have proved that the above initial-boundary value problem P(b, s_0, u_0) := {(1.1) – (1.5)} admits a unique solution $\{s, u\}$ on $[0, T_0]$ for some $0 < T_0 \leq T$. Also, we [6] showed the global existence of a solution in time and convergence of the solution to the stationary state as $t \to \infty$ by getting uniform estimates of the solution with respect to t. Moreover, this free boundary problem is a part of a two-scale model for concrete carbonation process in three dimensional domain $\Omega \subset \mathbb{R}^3$ (see [4]). On this subject Kumazaki [8;9] obtained continuity and measurability of solutions of $\mathrm{P}(b(\xi), s_0(\xi), u_0(\xi))$ with respect to $\xi \in \Omega$, when the function $b(\xi) = b(t, \xi)$, $s_0(\xi)$ and $u_0(\xi) = u_0(x, \xi)$ are defined for $t \in (0, T), x \in (s_0(\xi), L), \xi \in \Omega$. Furthermore, in our recent work [11] the local existence of a solution of the two-scale model was proved.

In this paper for $0 < T_* < \infty$ we consider a periodic problem $\mathrm{PP}(b) := \{(1.1) - (1.4), (1.6)\}$,

$$s(0) = s(T_*), u(0) = u(T_*) \text{ on } (s(0), L), \tag{1.6}$$

and establish the existence of a periodic solution by improving a way to get uniform estimates. There are a lot of results dealing with periodic solutions of a classical one-dimensional Stefan problem which is one kind of free boundary problems. For example, in [1] existence and uniqueness of a periodic solution were proved. Particularly, in the proof of the uniqueness the weak formulation called enthalpy formulation plays a very important role (also see [7]).

However, in the present model a weak formulation is not found, yet. Thus, the uniqueness of the periodic solution of $\mathrm{PP}(b)$ is open problem, now.

We define a solution of our problem and give a statement of our main theorem in the next section. In section 3 we shall obtain some uniform estimates for solutions with respect to t. Finally, we prove the theorem in section 4.

2. Main result

We begin with assumption for given data φ, a, ρ_g, ρ_a, L and b.

(A1) $\varphi \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, $\varphi = 0$ on $(-\infty, 0]$, $\varphi \leq 1$ on $\mathbb{R}$ and $\varphi' > 0$ on $(0, L]$.

(A2) a, κ, ρ_a and ρ_g are positive constants satisfying

$$\rho_a > 2\rho_g, \rho_a \geq \rho_g \left((1 + \frac{1}{L})|\varphi'|_{L^\infty(\mathbb{R})} + 2 \right) \text{ and } 9aL\rho_g^2 \leq \kappa\rho_a.$$

(A3) $b \in W^{1,2}(0, T_*)$, $b(0) = b(T_*)$ and $0 \leq b \leq b_*$ on $[0, T_*]$, where b_* is a positive constant with $b_* < \varphi(L)$. We put $d_0 = \varphi(L) - b_* > 0$.

Here, we define solutions of $\mathrm{P}(b,\, s_0,\, u_0)$ and $\mathrm{PP}(b)$ in the following way:

Definition 1. *Let s and u be functions on $[0, T]$ and $Q_s(T)$, respectively, for $T > 0$. We call that a pair $\{s, u\}$ is a solution of $P(b, s_0, u_0)$ on $[0, T]$ if the conditions (S1), (S2) and (1.1) – (1.5) hold:*

(S1) $s \in W^{1,\infty}(0,T)$, $0 \le s < L$ *on* $[0,T]$.

(S2) $u \in L^\infty(Q_s(T))$, $u_t, u_{xx} \in L^2(Q_s(T))$, $|u_x(\cdot)|_{L^2(s(\cdot),L)} \in L^\infty(0,T)$.

Also, it is called that the pair $\{s,u\}$ *is a solution of PP(b) on* $[0,T_*]$, *if (S1) and (S2) hold with* $T = T_*$, *and (1.1) – (1.4), (1.6) are valid.*

Remark 1. Let $\{s,u\}$ be a solution of P(b, s_0, u_0) on $[0,T]$, $T>0$. It is easy to see that $u \in C(\overline{Q(T)})$. Immediately, by (1.3) we have $s \in C^1([0,T])$.

The main result of this paper is as follows:

Theorem 1. *If (A1) – (A3) hold, then the periodic problem PP(b) has a solution on* $[0,T_*]$.

Before we give a sketch of a proof of Theorem 1, we recall existence and uniqueness results for $P(b,s_0,u_0)$.

Theorem 2. [6, Proposition 2.3, Theorem 2.4] *If (A1) – (A3) and (A4) hold,*

(A4) $0 \le s_0 < L, u_0 \in H^1(s_0,L), u_0(L) = b(0)$ *and* $0 \le u_0 \le 1$ *on* $[s_0,L]$,

then P(b, s_0, u_0) has one and only one solution $\{s,u\}$ *on* $[0,T_*]$ *satisfying* $0 \le u \le 1$ *on* $Q_s(T_*)$.

Next, in order to consider our problem on the cylindrical domain we introduce some notations. Let $\{s,u\}$ be a solution of P(b, s_0, u_0) on $[0,T_*]$ and put

$$\tilde{u}(t,y) = u(t,(1-y)s(t)+Ly) \text{ for } (t,y) \in Q(T_*) := (0,T_*)\times(0,1). \quad (2.1)$$

Then $\tilde{u}$ satisfies:

$$\rho_g \tilde{u}_t - \frac{\kappa}{(L-s)^2}\tilde{u}_{yy} = \frac{\rho_g(1-y)\dot{s}}{L-s}\tilde{u}_y \quad \text{in } Q(T_*), \quad (2.2)$$

$$\tilde{u}(t,1) = b(t) \quad \text{for } 0<t<T_*, \quad (2.3)$$

$$\dot{s}(t) = a(\tilde{u}(t,0) - \varphi(s(t))) \quad \text{for } 0<t<T_*, \quad (2.4)$$

$$\frac{\kappa}{L-s(t)}\tilde{u}_y(t,0) = (\rho_a - \rho_g\tilde{u}(t,0))\dot{s}(t) \quad \text{for } 0<t<T_*, \quad (2.5)$$

$$s(0) = s_0, \tilde{u}(0,y) = \tilde{u}_0(y) := u_0((1-y)s_0+Ly) \quad \text{for } 0 \le y \le 1 \quad (2.6)$$

Here, we provide the sketch of the proof in which we apply the Schauder fixed point theorem.

1) For any $M>0$ and $\delta>0$ we define a set

$$K(M,\delta) = [0,1-\delta] \times \left\{ \begin{array}{l} z \in H^1(0,L); |z_x|_{L^2(0,L)} \le M, \\ z(L) = b(0), 0 \le z \le 1 \text{ on } (0,L) \end{array} \right\}.$$

For $(s_0, u_0) \in K(M, \delta)$ we write $u_0 = u_0|_{[s_0,L]}$, again, where $u_0|_{[s_0,L]}$ is a restriction of u_0 to $[s_0, L]$. Then, Theorem 2 implies the existence of a solution $\{s, u\}$ of P(b, s_0, u_0) on $[0, T_*]$. Hence, we can define a solution operator $\Lambda : K(M, \delta) \to [0, 1) \times L^2(0, L)$ by $\Lambda(s_0, u_0) = (s(T_*), u(T_*))$, where $u(T_*)$ is an extension of $u(T_*)$ defined below:

$$u(T_*, x) = \begin{cases} u(T_*, x) & \text{for } s(T_*) \le x \le L, \\ u(T_*, s(T_*)) & \text{for } 0 \le x \le s(T_*). \end{cases} \tag{2.7}$$

2) By using uniform estimates obtained in Lemmas 2 and 4 we can take $M > 0$ and $\delta > 0$ such that $\Lambda : K(M, \delta) \to K(M, \delta)$.

3) On account of the continuity of Λ given in Lemma 5 we can apply the fixed point theorem to Λ so that we can prove Theorem 1.

3. Uniform estimates with respect to t

In this section we provide some uniform estimates for the solution of P(b, s_0, u_0).

Lemma 1. *Assume (A1) – (A4). Let $\{s, u\}$ be a solution of P(b, s_0, u_0) on $[0, T_*]$. Then it holds that*

$$\kappa |u_x(t, s(t))| \le 2a(\rho_a + \rho_g), |\dot{s}(t)| \le 2a \quad \text{for } t \in [0, T_*]. \tag{3.1}$$

Proof. Since $0 \le u \le 1$ on $Q_s(T_*)$, by (1.3) and (1.4) we see that

$$|\dot{s}(t)| = a|u(t, s(t)) - \varphi(s(t))| \le 2a,$$

and

$$\begin{aligned} \kappa |u_x(t, s(t))| &= |\rho_a - \rho_g u(t, s(t))||\dot{s}(t)| \\ &\le 2a(\rho_a + \rho_g) \quad \text{for } t \in [0, T_*]. \end{aligned}$$

Thus we have proved this lemma. □

Now, we give essential uniform estimates in the proof of Theorem 1. Although the proof is quite similar to that of [6, Lemma 3.2], it is improved and the calculations are delicate so that we show it in detail, here. To state the lemma, we put

$$\kappa_0 = \frac{\kappa}{2\rho_g L^2}, E(t) = \int_{s(t)}^{L} |u_x(t)|^2 dx \text{ for } t \in [0, T_*].$$

Lemma 2. *Under the same assumptions as in Lemma 1 let* $\{s, u\}$ *be a solution of* $P(b, s_0, u_0)$ *on* $[0, T_*]$*. Then there exists a positive constant* M_1 *depending only on* ρ_a*,* ρ_b*,* φ*,* a*,* b *and* L *(independent of* s_0 *and* u_0*) such that*

$$E(t) \leq e^{-\kappa_0 t} E(0) + M_1 \text{ for } 0 \leq t \leq T_*. \tag{3.2}$$

Proof. First, to handle the boundary condition, easily, we define a function ψ on $\mathbb{R}$ by

$$\psi(r) = \begin{cases} \frac{1}{2}\rho_a r^2 - \frac{1}{3}\rho_g & \text{for } r > 1, \\ \frac{1}{2}\rho_a r^2 - \frac{1}{3}\rho_g r^3 & \text{for } 0 \leq r \leq 1, \\ 0 & \text{for } r < 0. \end{cases} \tag{3.3}$$

Clearly, ψ is convex and continuous on $\mathbb{R}$. Moreover, we use the notation $\tilde{u}$ and $\tilde{u}_0$ given by (2.1) and (2.6), respectively, and put $v_h(t, y) = \dfrac{\tilde{u}(t, y) - \tilde{u}(t-h, y)}{h}$ for $h > 0$ and $(t, y) \in Q(T_*)$, where

$$\tilde{u}(t, y) = \begin{cases} \tilde{u}_0(y) & \text{for } t < 0, 0 \leq y \leq 1, \\ \tilde{u}_0(0) & \text{for } t < 0, y < 0, \end{cases} \tag{3.4}$$

and $s(t) = s(0) + t\dot{s}(0)$ and $b(t) = b(0)$ for $t < 0$. Here, we note that Remark 1 implies $s \in C^1([0, T_*])$ so that the extension of s is available.

We multiply (2.2) by $(L - s)v_h$ and integrate it over $(0, 1)$ with respect to y. Then, we have

$$\begin{aligned} &\rho_g (L - s(t)) \int_0^1 \tilde{u}_t(t) v_h(t) dy \\ = &-\frac{\kappa}{L - s(t)} \int_0^1 \tilde{u}_y(t) v_{hy}(t) dy + \frac{\kappa}{L - s(t)} \tilde{u}_y(t, 1) v_h(t, 1) \\ &-\frac{\kappa}{L - s(t)} \tilde{u}_y(t, 0) v_h(t, 0) + \rho_g \dot{s}(t) \int_0^1 (1 - y) \tilde{u}_y(t) v_h(t) dy \\ &(:= \sum_{i=1}^{4} I_i(t)) \quad \text{ for a.e. } t \in [0, T_*]. \end{aligned} \tag{3.5}$$

It is easy to see that

$$
\begin{aligned}
I_1(t) &\le -\frac{1}{2h}\frac{\kappa}{(L-s(t))}\int_0^1 (|\tilde{u}_y(t)|^2 - |\tilde{u}_y(t-h)|^2)dy \\
&\le -\frac{\kappa}{2h}\left(\int_{s(t)}^L |u_x(t)|^2dx - \frac{L-s(t-h)}{L-s(t)}\int_{s(t-h)}^L |u_x(t-h)|^2dx\right) \\
&\le -\frac{\kappa}{2h}\left(\int_{s(t)}^L |u_x(t)|^2dx - \int_{s(t-h)}^L |u_x(t-h)|^2dx\right) \\
&\quad +\frac{\kappa}{2h}\frac{s(t)-s(t-h)}{L-s(t)}\int_{s(t-h)}^L |u_x(t-h)|^2dx \quad \text{for a.e. } t\in[0,T_*],
\end{aligned}
$$

and

$$
\begin{aligned}
I_2(t) &= \frac{\kappa}{L-s(t)}\tilde{u}_y(t,1)\frac{b(t)-b(t-h)}{h} \\
&= \kappa u_x(t,L)\frac{b(t)-b(t-h)}{h} \quad \text{for a.e. } t\in[0,T_*].
\end{aligned}
$$

On account of (2.4) and (2.5) we infer that

$$
\begin{aligned}
I_3(t) &= -a(\rho_a-\rho_g\tilde{u}(t,0))(\tilde{u}(t,0)-\varphi(s(t)))v_h(t,0) \\
&= -a(\rho_a-\rho_g\tilde{u}(t,0))\tilde{u}(t,0)v_h(t,0) + a(\rho_a-\rho_g\tilde{u}(t,0))\varphi(s(t))v_h(t,0) \\
&:= I_{3,1}(t) + I_{3,2}(t) \quad \text{for a.e. } t\in[0,T_*].
\end{aligned}
$$

Here, since $0\le u\le 1$ on $Q(T_*)$, it holds that $(\rho_a-\rho_g\tilde{u}(t,0))\tilde{u}(t,0) = \psi'(\tilde{u}(t,0))$ for a.e. $t\in[0,T_*]$, and from the convexity of ψ it follows that

$$
I_{3,1}(t) \le -\frac{a}{h}(\psi(\tilde{u}(t,0)) - \psi(\tilde{u}(t-h,0))) \text{ for } t\in[0,T_*].
$$

Also, by using (2.4) again, we have

$$
\begin{aligned}
I_{3,2}(t) &= (\rho_a-\rho_g\tilde{u}(t,0))\varphi(s(t))\left(\frac{\dot{s}(t)-\dot{s}(t-h)}{h}\right) \\
&\quad +a(\rho_a-\rho_g\tilde{u}(t,0))\varphi(s(t))\left(\frac{\varphi(s(t))-\varphi(s(t-h))}{h}\right) \text{ for } t\in[h,T_*],
\end{aligned}
$$

and

$$
\begin{aligned}
&I_{3,2}(t) \\
&= (\rho_a-\rho_g\tilde{u}(t,0))\varphi(s(t))\left(\frac{\dot{s}(t)-\dot{s}(t-h)}{h}\right) \\
&\quad +a(\rho_a-\rho_g\tilde{u}(t,0))\varphi(s(t))\left(\frac{\varphi(s(t))-\varphi(s(t-h))}{h}\right) \\
&\quad +a(\rho_a-\rho_g\tilde{u}(t,0))\varphi(s(t))\left(\frac{\varphi(s(0)+(t-h)\dot{s}(0))-\varphi(s(0))}{h}\right) \text{ for } t\in[0,h].
\end{aligned}
$$

Here, we integrate (3.5) over $[0, t_1]$ for $0 \le t_1 \le T_*$. Accordingly, the above arguments lead to:

$$
\begin{aligned}
&\rho_g \int_0^{t_1} (L - s(t)) \int_0^1 \tilde{u}_t(t) v_h(t) dydt (:= \hat{I}_{0h}) \le \\
&\le -\frac{\kappa}{2h} \int_0^{t_1} \left(\int_{s(t)}^L |u_x(t)|^2 dx - \int_{s(t-h)}^L |u_x(t-h)|^2 dx \right) dt + \\
&+ \frac{\kappa}{2h} \int_0^{t_1} \frac{s(t) - s(t-h)}{L - s(t)} \int_{s(t-h)}^L |u_x(t-h)|^2 dydt + \\
&+ \int_0^{t_1} \kappa u_x(t, L) \frac{b(t) - b(t-h)}{h} dt - \int_0^{t_1} \frac{a}{h} (\psi(\tilde{u}(t,0)) - \psi(\tilde{u}(t-h,0))) dt + \\
&+ \rho_a \int_0^{t_1} \varphi(s(t)) \frac{(\dot{s}(t) - \dot{s}(t-h))}{h} dt + a\rho_a \int_0^{t_1} \varphi(s(t)) \frac{\varphi(s(t)) - \varphi(s(t-h))}{h} dt - \\
&- \frac{\rho_g}{a} \int_0^{t_1} \dot{s}(t) \varphi(s(t)) \frac{\dot{s}(t) - \dot{s}(t-h)}{h} dt - \rho_g \int_0^{t_1} \varphi(s(t))^2 \frac{\dot{s}(t) - \dot{s}(t-h)}{h} dt - \\
&- \rho_g \int_0^{t_1} \dot{s}(t) \varphi(s(t)) \frac{\varphi(s(t)) - \varphi(s(t-h))}{h} dt - \\
&- \rho_g \int_0^{t_1} \varphi(s(t))^2 \frac{\varphi(s(t)) - \varphi(s(t-h))}{h} dt + \\
&+ a \int_0^h (\rho_a - \rho_g \tilde{u}(t,0)) \varphi(s(t)) \left(\frac{\varphi(s(0) + (t-h)\dot{s}(0)) - \varphi(s(0))}{h} \right) + \\
&+ \rho_g \int_0^{t_1} \dot{s}(t) \int_0^1 (1-y) \tilde{u}_y(t) v_h(t) dydt \\
&\left(:= \sum_{i=1}^{12} \hat{I}_{ih} \right) \qquad \text{for any } t_1 \in [0, T_*]. \quad (3.6)
\end{aligned}
$$

First, by change of the variable we see that

$$
\begin{aligned}
\lim_{h \to 0} \hat{I}_{0h} &= \rho_g \int_0^{t_1} (L - s(t)) \int_0^1 |\tilde{u}_t(t)|^2 dydt = \\
&= \rho_g \int_0^{t_1} \int_{s(t)}^L |u_t(t)|^2 dxdt + \\
&+ 2\rho_g \int_0^{t_1} \dot{s}(t) \int_{s(t)}^L \frac{L - x}{L - s(t)} u_t(t) u_x(t) dxdt + \\
&+ \rho_g \int_0^{t_1} |\dot{s}(t)|^2 \int_0^L \left(\frac{L - x}{L - s(t)} \right)^2 |u_x(t)|^2 dxdt.
\end{aligned}
$$

Next, we have

$$\hat{I}_{1h} = -\frac{\kappa}{2h}\left(\int_{t_1-h}^{t_1}\int_{s(t)}^{L}|u_x(t)|^2dxdt - \int_{-h}^{0}\int_{s(t)}^{L}|u_x(t)|^2dxdt\right).$$

As mentioned in the proof of Lemma 3.5 in [12], the function

$$t \to \int_{s(t)}^{L}|u_x(t)|^2dx$$

is absolutely continuous on $[0, T_*]$. Then it follows that

$$\lim_{h\to 0} -\frac{\kappa}{2h}\int_{t_1-h}^{t_1}\int_{s(t)}^{L}|u_x(t)|^2dxdt = -\frac{\kappa}{2}\int_{s(t_1)}^{L}|u_x(t_1)|^2dx.$$

The extension (3.4) leads to

$$u(t,x) = \begin{cases} u_0(x) & \text{for } t < 0, s_0 \le x \le L, \\ u_0(s_0) & \text{for } t < 0, x < s_0. \end{cases}$$

Then, it is easy to see that

$$\begin{aligned} \frac{\kappa}{2h}\int_{-h}^{0}\int_{s(t)}^{L}|u_x(t)|^2dxdt &= \frac{\kappa}{2}\int_{s_0}^{L}|u_{0x}|^2dx + \frac{1}{h}\int_{-h}^{0}\int_{s_0+t\dot{s}(0)}^{s(0)}|u_{0x}|^2dxdt \\ &\to \frac{\kappa}{2}\int_{s_0}^{L}|u_{0x}|^2dx \quad \text{as } h \to 0. \end{aligned}$$

Thus, we get

$$\lim_{h\to 0}\hat{I}_{1h} = -\frac{\kappa}{2}\int_{s(t_1)}^{L}|u_x(t_1)|^2dx + \frac{\kappa}{2}\int_{s_0}^{L}|u_{0x}|^2dx.$$

Similarly,

$$\lim_{h\to 0}\hat{I}_{2h} = \frac{\kappa}{2}\int_{0}^{t_1}\frac{\dot{s}(t)}{L-s(t)}\int_{s(t)}^{L}|u_x(t)|^2dxdt.$$

By (1.1) and (1.4) it follows that

$$\lim_{h\to 0}\hat{I}_{3h} = \int_0^{t_1} \kappa u_x(t,L)\dot{b}(t)dt =$$
$$= \kappa\int_0^{t_1}\left(\int_{s(t)}^L u_{xx}(t)dx + u_x(t,s(t))\right)\dot{b}(t)dt =$$
$$= \int_0^{t_1}\left(\int_{s(t)}^L \rho_g u_t(t)dx + (\rho_a - \rho_g\varphi(s(t)))\dot{s}(t)\right)\dot{b}(t)dt \le$$
$$\le \frac{\rho_g}{2}\int_0^{t_1}\int_{s(t)}^L |u_t(t)|^2 dxdt + \frac{\rho_g L}{2}\int_0^{t_1}|\dot{b}(t)|^2 dt +$$
$$+\frac{\rho_a+\rho_g}{2}\int_0^{t_1}(|\dot{b}(t)|^2 + |\dot{s}(t)|^2)dt.$$

Since $\tilde{u}$ is continuous on $\overline{Q(T_*)}$, we can get

$$\begin{aligned}\lim_{h\to 0}\hat{I}_{4h} &= -\lim_{h\to 0}\left(\frac{a}{h}\int_{t_1-h}^{t_1}\psi(\tilde{u}(t,0))dt\right) + a\psi(\tilde{u}_0(0)) \\ &= -a(\psi(u(t_1,s(t_1))) - \psi(\tilde{u}_0(0))).\end{aligned}$$

Here, by using $\dot{s}\in C([0,T_*])$ and $\varphi(s)\in C^1([0,T_*])$ we observe that

$$\lim_{h\to 0}\hat{I}_{5h} = \rho_a\lim_{h\to 0}\int_0^{t_1-h}\frac{\varphi(s(t)) - \varphi(s(t+h))}{h}\dot{s}(t)dt +$$
$$+\rho_a\lim_{h\to 0}\left(\frac{1}{h}\int_{t_1-h}^{t_1}\varphi(s(t))\dot{s}(t)dt - \frac{1}{h}\int_0^{-h}\varphi(s(t+h))\dot{s}(0)dt\right) =$$
$$= -\rho_a\int_0^{t_1}\varphi'(s(t))|\dot{s}(t)|^2 dt + \rho_a(\varphi(s(t_1))\dot{s}(t_1) - \varphi(s(0))\dot{s}(0)),$$

$$\lim_{h\to 0}\hat{I}_{6h} = \frac{a\rho_a}{2}(\varphi(s(t_1))^2 - \varphi(s(0))^2),$$

$$\limsup_{h\to 0}\hat{I}_{7h} \le \limsup_{h\to 0}\frac{\rho_g}{2ah}\int_0^{t_1}\varphi(s(t))(\dot{s}(t-h)^2 - \dot{s}(t)^2)dt \le$$
$$\le \limsup_{h\to 0}\frac{\rho_g}{2a}\int_0^{t_1-h}\frac{\varphi(s(t+h)) - \varphi(s(t))}{h}\dot{s}(t)^2 dt +$$
$$+\frac{\rho_g}{2a}\limsup_{h\to 0}\left(\frac{1}{h}\int_{t_1-h}^{t_1}\varphi(s(t))\dot{s}(t)^2 dt - \frac{1}{h}\int_{-h}^0\varphi(s(t+h))\dot{s}(0)^2 dt\right) \le$$
$$\le \frac{\rho_g}{2a}\int_0^{t_1}\varphi'(s(t))|\dot{s}(t)|^3 dt + \frac{\rho_g}{2a}\left(\varphi(s(t_1))|\dot{s}(t_1)|^2 - \varphi(s(0))|\dot{s}(0)|^2\right),$$

$$\lim_{h\to 0} \hat{I}_{8h} = -\rho_g \lim_{h\to 0} \int_0^{t_1} \frac{\varphi(s(t))^2 - \varphi(s(t+h))^2}{h} \dot{s}(t)dt -$$
$$- \rho_g \lim_{h\to 0} \left(\frac{1}{h} \int_{t_1-h}^{t_1} \varphi(s(t))^2 \dot{s}(t)dt - \frac{1}{h} \int_{-h}^{0} \varphi(s(t+h))^2 \dot{s}(0)dt \right) =$$
$$= 2\rho_g \int_0^{t_1} \varphi'(s(t))\varphi(s(t))|\dot{s}(t)|^2 dt - \rho_g(\varphi(s(t_1))^2 \dot{s}(t_1) - \varphi(s(0))^2 \dot{s}(0)),$$

$$\lim_{h\to 0} \hat{I}_{9h} = -\frac{\rho_g}{a} \int_0^{t_1} |\dot{s}(t)|^2 \varphi(s(t))\varphi'(s(t))dt \leq 0,$$

$$\lim_{h\to 0} \hat{I}_{10h} = -\frac{\rho_g}{3}(\varphi(s(t_1))^3 - \varphi(s(0))^3),$$

and

$$|\hat{I}_{11h}| \leq a(\rho_a + \rho_g)|\varphi'|_{L^\infty(\mathbb{R})} \int_0^h \frac{|t-h||\dot{s}(0)|}{h} dt \to 0 \quad \text{as } h \to 0.$$

Moreover, we see that

$$\lim_{h\to 0} I_{12h} = \rho_g \int_0^{t_1} \dot{s}(t) \int_{s(t)}^{L} \frac{L-x}{L-s(t)} u_x(t)u_t(t)dxdt +$$
$$+ \rho_g \int_0^{t_1} |\dot{s}(t)|^2 \int_{s(t)}^{L} \left(\frac{L-x}{L-s(t)} \right)^2 |u_x(t)|^2 dxdt.$$

From the estimates for $\hat{I}_{0h}, \cdots, \hat{I}_{12h}$ it follows

$$\frac{\rho_g}{2} \int_0^{t_1} \int_{s(t)}^{L} |u_t(t)|^2 dxdt + \frac{\kappa}{2} \int_{s(t_1)}^{L} |u_x(t_1)|^2 dx + F(t_1) \leq$$
$$\leq F(0) - \rho_g \int_0^{t_1} \dot{s}(t) \int_{s(t)}^{L} \frac{L-x}{L-s(t)} u_t(t)u_x(t)dxdt +$$
$$+ \frac{\kappa}{2} \int_{s_0}^{L} |u_{0x}|^2 dx + \int_0^{t_1} \left(\frac{\kappa \dot{s}(t)}{2(L-s(t))} \right) \int_{s(t)}^{L} |u_x(t)|^2 dxdt +$$
$$+ C_1 \int_0^{t_1} (|\dot{b}(t)|^2 + 1)dt \quad \text{for } 0 \leq t_1 \leq T_*, \quad (3.7)$$

where C_1 is a positive constant depending only on ρ_a, ρ_g, a, φ, κ, and L and

$$F(t) = a\psi(u(t, s(t))) - \rho_a \varphi(s(t))\dot{s}(t) - \frac{a\rho_a}{2}\varphi(s(t))^2 - \frac{\rho_g}{2a}\varphi(s(t))|\dot{s}(t)|^2$$
$$+ \frac{a\rho_g}{3}\varphi(s(t))^3 + \rho_g \varphi(s(t))^2 \dot{s}(t) \quad \text{for } 0 \leq t \leq T_*.$$

Furthermore, for the second term in the right hand side of (3.7) we can get

$$
\begin{aligned}
-\rho_g \int_0^{t_1} \dot{s}(t) \int_{s(t)}^L \frac{L-x}{L-s(t)} u_t(t) u_x(t) dx dt = \\
= -\kappa \int_0^{t_1} \dot{s}(t) \int_{s(t)}^L \frac{L-x}{L-s(t)} u_{xx}(t) u_x(t) dx dt = \\
= -\kappa \int_0^{t_1} \frac{\dot{s}(t)}{(L-s(t))} \int_{s(t)}^L (L-x) \frac{1}{2} \frac{\partial}{\partial x} |u_x(t)|^2 dx dt = \\
= -\kappa \int_0^{t_1} \frac{\dot{s}(t)}{2(L-s(t))} \int_{s(t)}^L |u_x(t)|^2 dx dt + \\
+ \frac{\kappa}{2} \int_0^{t_1} \dot{s}(t) |u_x(t, s(t))|^2 dt \quad \text{for } 0 \le t_1 \le T_*.
\end{aligned}
$$

By substituting this equation and (1.1) into (3.7) we have

$$
\begin{aligned}
\frac{\kappa^2}{2\rho_g} \int_0^{t_1} \int_{s(t)}^L |u_{xx}(t)|^2 dx dt + \frac{\kappa}{2} \int_{s(t_1)}^L |u_x(t_1)|^2 dx + F(t_1) \le \\
\le \frac{\kappa}{2} \int_{s_0}^L |u_{0x}|^2 dx + F(0) + C_2 \int_0^{t_1} (|\dot{b}(t)|^2 + 1) dt \text{ for } t_1 \in [0, T_*], \quad (3.8)
\end{aligned}
$$

where C_2 is a positive constant depending only on ρ_a, ρ_g, a, φ, κ and L.

Here, it is easy to see that

$$
\int_{s(t)}^L |u_x(t)|^2 dx \le 2L^2 \int_{s(t)}^L |u_{xx}(t)|^2 dx + 2L |u_x(t, s(t))|^2 \quad \text{for } t \in [0, T_*]. \tag{3.9}
$$

On account of (3.8) and (3.9) we see that

$$
\begin{aligned}
\frac{\kappa^2}{4\rho_g L^2} \int_0^{t_1} \int_{s(t)}^L |u_x(t)|^2 dx dt + \frac{\kappa}{2} E(t_1) + F(t_1) \le \\
\le \frac{\kappa}{2} E(0) + F(0) + C_3 \int_0^{t_1} (|\dot{b}(t)|^2 + 1) dt \quad \text{for } t_1 \in [0, T_*],
\end{aligned}
$$

where C_3 is a positive constant.

By regarding $t_0 \in [0, T_*]$ as the initial time the above argument implies that

$$
\begin{aligned}
\kappa_0 \int_{t_0}^{t_1} E(t) dt + \frac{\kappa}{2} E(t_1) + F(t_1) \le \\
\le \frac{\kappa}{2} E(t_0) + F(t_0) + C_3 \int_{t_0}^{t_1} (|\dot{b}(t)|^2 + 1) dt \quad \text{for } 0 \le t_0 \le t_1 \le T_*.
\end{aligned}
$$

Then, the absolutely continuity of E on $[0, T_*]$ guarantees that

$$\kappa_0 E(t) + \frac{\kappa}{2}\frac{d}{dt}E(t) + \frac{d}{dt}F(t_1) \le C_3(|\dot{b}(t)|^2 + 1) \quad \text{for a.e. } t \in [0, T_*]. \tag{3.10}$$

We multiply (3.10) by $e^{\kappa_0 t}$ and have

$$\frac{d}{dt}(e^{\kappa_0 t}(E(t)+F(t))) \le C_3 e^{\kappa_0 t}(|\dot{b}(t)|^2+1)+\kappa_0 e^{\kappa_0 t}F(t) \quad \text{for a.e. } t \in [0, T_*]$$

so that there exists a positive constant M_1 such that

$$E(t) \le e^{-\kappa_0 t}E(0) + M_1 \quad \text{for } t \in [0, T_*].$$

Thus we have obtained (3.2). □

Next, we show the uniform estimate for the free boundary s:

Lemma 3. *(cf. [6, Lemma 3.3]) Let $\{s, u\}$ be a solution of $P(b, s_0, u_0)$ on $[0, T_*]$ and M_* be a positive constant satisfying $\int_{s(t)}^L |u_x(t)|^2 dx \le M_*$ for $0 \le t \le T_*$, and put*

$$s_*(M_*) = L - \left(\frac{d_0}{2(\sqrt{M_*} + C_\varphi\sqrt{L})}\right)^2.$$

If $s(t) \ge s_*(M_*)$ for some $t \in [0, T_*]$, then $\dot{s}(t) < 0$.

Proof. Assume that $s(t) \ge s_*(M_*)$ for some $t \in [0, T_*]$. Then we observe that

$$\begin{aligned}
\dot{s}(t) &= a(u(t, s(t)) - \varphi(s(t))) = \\
&= a(u(t, s(t)) - b(t) + b(t) - b_* + b_* - \varphi(L) + \varphi(L) - \varphi(s(t))) \le \\
&\le a\left(\sqrt{L - s(t)}(\int_{s(t)}^L |u_x(t)|^2 dx)^{1/2} - d_0 + C_\varphi(L - s(t))\right) \le \\
&\le a\left(\sqrt{L - s_*(M_*)}(\sqrt{M_*} + C_\varphi\sqrt{L}) - d_0\right) \le -\frac{ad_0}{2}.
\end{aligned}$$

This is a conclusion of this lemma. □

Lemma 4. *(cf. [6, Proposition 3.4]) Let $\{s, u\}$ be a solution of $P(b, s_0, u_0)$ on $[0, T_*]$. Then it holds that*

$$s(t) \le \max\{s_0, s_*(M_*)\} \text{ for any } t \in [0, T_*]. \tag{3.11}$$

Proof. First, we assume that $s_0 > s_*(M_*)$. Then, since s is continuous on $[0, T_*]$, one of the following cases must occur:

(case 1) $s(t) \ge s_*(M_*) \quad$ for $t \in [0, T_*]$.

(case 2) there exists $t_0 \in (0, T_*]$ such that

$$s \geq s_*(M_*) \text{ on } [0, T_0] \text{ and } s < s_*(M_*) \text{ on } (t_0, t_0 + \delta') \text{ for some } \delta' > 0.$$

In case 1, by Lemma 3 we have $\dot{s} < 0$ on $[0, T_*]$ so that $s(t) \leq s(0)$ for $t \in [0, T_*]$. Thus, we get (3.11). In case 2, we suppose that there exists $t_1 > t_0$ such that $s(t_1) > s_*(M_*)$. Then we can take $t_2 \in (t_0, t_1)$ satisfying $s(t_2) > s_*(M_*)$ and $\dot{s}(t_2) > 0$, since $s \in C^1([0, T_*])$. This contradicts to Lemma 3. Hence, (3.11) holds.

If $s_0 \leq s_*(M_*)$, then we can obtain (3.11) in the similar way. This lemma is proved. □

4. Existence of periodic solution

The aim of this section is to prove Theorem 1. First, we recall the continuous dependence of data for a solution of P(b, s_0, u_0).

Lemma 5. *(cf. [6, Lemma 4.3]) Assume (A1) – (A3). For any $M > 0$ and $\delta > 0$ let $(s_{0i}, u_{0i}) \in K(M, \delta)$, $i = 1, 2$. Here, Theorem 2 implies the existence of a solution $\{s_i, u_i\}$ of P(b, s_{0i}, u_{0i}) on $[0, T_*]$ for each i. Here, by using (2.1) we define $\tilde{u}_i$ and $\tilde{u}_{0i}$ from u_i and u_{0i}, respectively, for each i. Then there exists a positive constant M_2 such that*

$$\begin{aligned} &|s_1(t) - s_2(t)|^2 + |\tilde{u}_1(t) - \tilde{u}_2(t)|_H^2 + \int_0^t |\tilde{u}_{1y} - \tilde{u}_{2y}|_{L^2(0,1)}^2 d\tau \\ \leq\ & M_2 \left(|s_{01} - s_{02}|^2 + |\tilde{u}_{01} - \tilde{u}_{02}|_{L^2(0,1)}^2 \right) \quad \text{for } 0 \leq t \leq T_*. \end{aligned}$$

Proof of Theorem 1. Let $M > 0$ such that

$$M_1 \leq (1 - e^{-\kappa_0 T_*}) M,$$

where M_1 is the positive constant obtained in Lemma 2. Moreover, we can take $\delta > 0$ such that $s_*(M) \leq 1 - \delta$, where $s_*(M)$ is defined in Lemma 3. Then, Theorem 2 implies that for $(s_0, u_0) \in K(M, \delta)$ P(b, s_0, u_0) has a unique solution $\{s, u\}$ on $[0, T_*]$. Here, we extend $u(T_*)$ to the function on $[0, L]$ by (2.7). Thanks to Lemma 2 we have

$$\int_0^L |u_x(T_*)|^2 dx \leq e^{-\kappa_0 T_*} \int_0^L |u_{0x}|^2 dx + M_1.$$

Easily, we see that $\int_0^L |u_x(T_*)|^2 dx \leq M$. From Lemma 4 it follows that $s(T_*) \leq \delta$. These facts indicate $\Lambda : K(M, \delta) \to K(M, \delta)$.

Clearly, Lemma 5 leads to the continuity of Λ in $\mathbb{R} \times L^2(0, L)$ and $K(M, \delta)$ is convex and compact in the topology of $\mathbb{R} \times L^2(0, L)$. Hence, the theorem is a direct consequence of the Schauder fixed point theorem. □

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Существование периодических решений в одномерной задаче со свободной границей, описывающей адсорбционные явления

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Аннотация. Рассматривается процесс сушки и смачивания в пористой среде для создания математической модели карбонизации цемента. Предполагается, что данный процесс характеризуется ростом воздушной зоны и диффузией влаги в воздушной зоне. При данном предположении предлагается одномерная задача со свободной границей, описывающая адсорбционные явления в пористой среде. Задача со свободной границей заключается в нахождении кривой, представляющей воздушную зону и относительную влажность воздушной зоны. Также устанавливаются существование, единственность и поведение решений на бесконечности. Также, улучшая метод равномерных оценок, показывается существование периодического решения задачи. Кроме этого, в доказательстве применяется метод расширения. Эта идея является весьма важной и новой, поскольку значение уровня влажности на свободной границе неизвестно.

Ключевые слова: задача со свободной границей, периодические решения.

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