Серия «Математика»
2009. Т. 2, № 2, С. 56—61

Онлайн-доступ к журналу:
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ИЗВЕСТИЯ
Иркутского государственного университета

УДК 510.53+510.67

Computable Limit Models

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Abstract. In the paper we construct a complete theory having exactly 6 countable models up to isomorphism, such that there are two limits over a powerful type models, one of which has a computable presentation and the other does not.

Keywords: computable model, small theory, prime model, limit model, Rudin-Keisler ordering.

This area of problems of constructive models theory is connected with the class of small theories.

Denote by $S(T)$ the set of all types (over $\varnothing$, i. e. without any parameters) consistent with the theory T.

Definition 1. *A complete theory T of countable language is* small*, if the set $S(T)$ is at most countable.*

In the class of small theories there are three well known subclasses: ω-categorical, ω_1-categorical and Ehrenfeucht theories.

Denote by $\omega(T)$ the number of countable models up to isomorphism of a complete theory T.

Definition 2. *A theory T is ω-categorical, if every two countable models of T are isomorphic, i. e. $\omega(T) = 1$.*

Definition 3. *A theory T is ω_1-categorical, if every two models of T of the power ω_1 are isomorphic.*

If a theory is ω_1-categorical not ω-categorical, than there are exactly ω non-isomorphic countable models of the theory. And these models form $(\omega + 1)$-chain (from the prime to the saturated model) with respect to elementary embeddings.

Definition 4. *A theory T is Ehrenfeucht theory, if $1 < \omega(T) < \omega$.*

By Vaught's and Ehrenfeucht's theorems the only impossible case is $\omega(T) = 2$.

In works [1]-[5] the authors justified the necessity of studying computable models spectra of small theories. All known examples of spectra are mentioned in these works. The problem of description of spectra of computable models is nontrivial in case of ω_1- not ω-categorical theories. And both the problems of description of spectra of computable models and spectra of decidable models are rather hard in case of Ehrenfeucht theories.

This article adds one more spectrum of computable models for the case of Ehrenfeucht theories to the list of known spectra.

Firstly we will give necessary definitiones, then formulate and prove the main theorem of the paper.

Definition 5. *A function f is said to be* limitwise monotonic *function if there exists a computable function $g(n, m)$ such that*
1. $g(n, m) \leqslant g(n, m + 1)$, *for all n and m,*
2. $lim_m g(n, m)$ *exists, for every n,*
3. $f(n) = lim_m g(n, m)$.

Definition 6. *A model $\mathfrak{M} \models T$ is said to be* quasi-prime*, if it is prime over realization of some type of the theory T, i. e. there is a type p of the theory T and a realization $\bar{a}$ of p in some model of T, such that $\langle \mathfrak{M}, \bar{a} \rangle$ is a prime model.*

We denote the class of (isomorphic) prime models over realizations of a type p by $\mathfrak{M}_p$, i. e.

$$\mathfrak{M}_p = \{\mathfrak{M}_{\overline{a}} \,|\, \langle \mathfrak{M}_{\overline{a}}, \overline{a} \rangle \text{ is a prime model of the theory } Th(\mathfrak{M}, \overline{a}), \text{ where } \mathfrak{M} \models p(\overline{a})\}.$$

While it is not necessary for our goals to distinguish models from its isomorphic copies, we will usually think about $\mathfrak{M}_p$ as about a model.

Definition 7. *A type p* does not exceed *a type q* under the Rudin-Keisler ordering *(p* is dominated by *q), if $\mathfrak{M}_q \models p$. Written $p \leqslant_{RK} q$.*

$p \sim_{RK} q \Leftrightarrow (p \leqslant_{RK} q \,\&\, q \leqslant_{RK} p)$.[1]

Denote by $RK(T)$ the set of all types of isomorphism $\mathfrak{M}_p$, throughout all $p \in S(T)$. $RK(T) = \{\mathfrak{M}_p \,|\, p \in S(T)\}$. This set is pre-ordered by the relation $\leqslant_{RK}$.

Definition 8. *A type p of a theory T is said to be* powerful *in the theory T if every model $\mathfrak{M}$ of T, realizing p, also realizes every type from $S(T)$. I. e. $\mathfrak{M} \models p \Rightarrow \mathfrak{M} \models S(T)$.*

Definition 9. *A model sequence $\mathfrak{M}_0 \preceq \mathfrak{M}_1 \preceq \ldots$ is said to be* elementary chain over a type *p, if $\mathfrak{M}_n \cong \mathfrak{M}_p$, for every $n \in \omega$.*

[1] When types p and q are in some relation with respect to RK-ordering, we say that models $\mathfrak{M}_p$ and $\mathfrak{M}_q$ are also in this relation.

Definition 10. *A model* $\mathfrak{M}$ *is said to be* limit over a type p *if* $\mathfrak{M} = \bigcup_{n\in\omega} \mathfrak{M}_n$, *for some elementary chain* $(\mathfrak{M}_n)_{n\in\omega}$ *over* p, *and* $\mathfrak{M} \not\cong \mathfrak{M}_p$. $\mathfrak{M}$ *is said to be a* limit *model if it is limit over some type.*

Consider $\widetilde{\mathbf{M}} \in RK(T)/\sim_{RK}$. Let $\widetilde{\mathbf{M}} = \{\mathfrak{M}_{p_0}, \ldots, \mathfrak{M}_{p_n}\}$. Denote by $IL(\widetilde{\mathbf{M}})$ the number of two by two non-isomorphic models each of which is a limit over some type p_i, $i \in \{0, \ldots, n\}$.

Lemma 1 (Sudoplatov [6])**.** *Every model of an Ehrenfeucht theory either quasi-prime or limit.*

Theorem 1 (Sudoplatov [6])**.** *The following conditions are equivalent:*
1. $\omega(T) < \omega$;
2. $|S(T)| = \omega$, $|RK(T)| < \omega$, $IL(\widetilde{\mathbf{M}}) < \omega$, *for any* $\widetilde{\mathbf{M}} \in RK(T)/\sim_{RK}$.

Definition 11. *Let* $\langle X; \leqslant\rangle$ *be a finite pre-ordered set with the least element* x_0 *and the greatest class* $\widetilde{x}_1$ *in ordered factor-set* $\langle X; \leqslant\rangle/\sim$ *(where* $x \sim y \Leftrightarrow x \leqslant y$ *and* $y \leqslant x$*),* $\widetilde{x}_0 \neq \widetilde{x}_1$. *Let* $F : X/\sim \to \omega$ *is a function, satisfying next properties* $F(\widetilde{x}_0) = 0$, $F(\widetilde{x}_1) > 0$, $F(\widetilde{y}) > 0$, *when* $|\widetilde{y}| > 1$. *The pair* (X, F) *is said to be* e-parameters[2]. *At that, the set* X *is said to be* the first e-parameter *and the function* F *—* the second e-parameter.

Definition 12. *A theory* T *is said to be* Ehrenfeucht theory with e-parameters (X, F) *if there exists an isomorphism* $\varphi : X \to RK(T)$*, and for any* $\widetilde{x} \in X/\sim$*, an equality* $IL(\varphi(\widetilde{x})) = F(\widetilde{x})$ *holds.*

Let T be an Ehrenfeucht theory with e-parameters (X, F).

Definition 13. Spectrum of computable models of Ehrenfeucht theory $\mathbb{SCM}(T)$ *is a pair* (Y, G)*, where* $Y = \{x \in X \mid$ *the element* x *corresponds to a computable model of the theory* $T\}$ *(corresponds — in terms of isomorphism* φ *from the previous definitione);* $DomF = DomG$*, (*$G(x) = m \Leftrightarrow$ *there exist exactly* m *computable limit non-isomorphic models of* T *over the model, corresponding to* x*).*

Now we go on to the main theorem of the paper.

Let (X, F) be e-parameters such that $X = \{x_0 < x_2 < x_1\}$, $F(x_2) = 0$, $F(x_1) = 3$, and $G : X \to \omega$ be a function, for which $G(x_0) = 0$, $G(x_2) = 0$, $G(x_1) = 2$ hold.

Theorem 2. *There exists a theory* T *satisfying the following statements:*
1. T *has e-parameters* (X, F);
2. $\omega(T) = 6$;
3. $\mathbb{SCM} = (\{x_0\}, G)$.

[2] From «parameters of Ehrenfeuchtness».

Proof. We note first of all, that the item 2 follows from the item 1 immediately.

We will use the technique of constructing of Ehrenfeucht theories from [4].

At first we need to define models $Q(n)$, for all cardinal $n \leqslant \omega$, and $Q_j(\omega)$, for $j \in \{0,1,2,3\}$.

Let $\mathbb{Q}$ be the set of rationals. The domain of the structure $Q(n)$ is

$$\{q \in \mathbb{Q} \,|\, 1 \leqslant q < 2\} \cup \{a_{q,1}, a_{q,2}, \dots, a_{q,n} \,|\, q \in \mathbb{Q}\},$$

where $\{a_{q,i} | q \in \mathbb{Q}, 1 \leqslant i \leqslant n\}$ is a set of new elements. The language of the model is $\langle \leqslant; f \rangle$, where $\leqslant$ is a binary relation and f is a unary function symbol. The relation $\leqslant$ and the function f are defined as follows. For all x and y we have $x \leqslant y$ if and only if $x, y \in Q$ and x is less then or equal to y as rational numbers. For all z and y define $f(z) = y$ if and only if for some rational number q, $y = q$ and $z \in \{a_{q,1}, \dots, a_{q,m}\}$ or $y = z = q$.

Let I be the structure obtained from $Q(\omega)$ by removing all elements, except the elements $1, a_{1,1}, \dots, a_{1,\omega}$, from the domain of $Q(\omega)$.

Let $Q_0(\omega)$ be the structure obtained from $Q(\omega)$ by adding to the domain the rational number 2 and the new elements $\{a_{2,1}, a_{2,2}, \dots, a_{2,\omega}\}$, define $f(2) = f(a_{2,1}) = f(a_{2,2}) = \dots = f(a_{2,\omega}) = 2$.

Let $Q_1(\omega)$ be the structure obtained from $Q_0(\omega)$ by removing the elements $1, a_{1,1}, \dots, a_{1,\omega}$ from the domain of $Q_0(\omega)$.

Let $Q_2(\omega)$ be the structure $Q(\omega)$.

And let $Q_3(\omega)$ be the structure obtained from $Q_2(\omega)$ by removing the elements $1, a_{1,1}, \dots, a_{1,\omega}$ from the domain of $Q_2(\omega)$.

If $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic copies of the structures $Q_i(n)$ and $Q_j(m)$, respectively, and $A \cap B = \varnothing$, then one can naturally define the isomorphism type of the structure $Q_i(n) + Q_j(m)$ as follows. The domain of the new structure is the set $A \cup B$. The relation $\leqslant$ in the new structure is the least partial ordering which contains the partial ordering of $\mathfrak{A}$, the partial ordering of $\mathfrak{B}$, and the relation $\{(x, y) | x \in A \;\&\; f^{\mathfrak{A}}(x) = x \;\&\; y \in B \;\&\; f^{\mathfrak{B}}(y) = y\}$. The unary function f in the new model is the union of the unary functions of the first and the second structures.

If $n_0, n_1, \dots, n_i, \dots$, $i < \omega$ is a sequence of cardinals less then or equal to ω, then as above we can define the structure

$$Q_{\lambda_0}(n_0) + Q_{\lambda_1}(n_1) + \dots + Q_{\lambda_1}(n_i) + \dots,$$

where $\lambda_i \in \{0,1,2,3\}$ for all $i \in \omega$.

Let S be a Δ^0_2-set which is not the range of a limitwise monotonic function. Examples of such kind of sets are independently constructed in [3] and [4]. There exists a computable function h such that, for all n $i(n) = \lim_s h(n, s)$ exists and $range(i) = S$. Consider the model

$$\mathfrak{A} = Q(i(0)) + Q(i(1)) + \dots + \dots + Q(1) + Q(0).$$

In other words, to get the model $\mathfrak{A}$ we replace in the ordering $\omega + \omega^\star$ every element k from ω by $Q(i(k))$, and every element m from $\omega^\star$ by $Q(m)$.

The theory T of the structure $\mathfrak{A}$ is desired.

It is not hard exercise to prove that every countable model of T is isomorphic to exactly one of the following models:

$$\mathfrak{A} = Q(i(0)) + Q(i(1)) + \ldots + \ldots + Q(1) + Q(0),$$
$$\mathfrak{A}_1 = Q(i(0)) + Q(i(1)) + \ldots + I + \ldots + Q(1) + Q(0),$$
$$\mathfrak{A}_2 = Q(i(0)) + Q(i(1)) + \ldots + Q_0(\omega) + \ldots + Q(1) + Q(0),$$
$$\mathfrak{A}_3 = Q(i(0)) + Q(i(1)) + \ldots + Q_1(\omega) + \ldots + Q(1) + Q(0),$$
$$\mathfrak{A}_4 = Q(i(0)) + Q(i(1)) + \ldots + Q_2(\omega) + \ldots + Q(1) + Q(0),$$
$$\mathfrak{A}_5 = Q(i(0)) + Q(i(1)) + \ldots + Q_3(\omega) + \ldots + Q(1) + Q(0),$$

Obviously, $\mathfrak{A}$ is a prime model. The model $\mathfrak{A}_1$ is prime over a realization of non-principal non-powerful type, saying that there exists an element with infinite number of f-preimages. The model $\mathfrak{A}_2$ is prime over a realization of a powerful type, saying that there exist at least two elements with infinite number of f-preimages. Models $\mathfrak{A}_3$, $\mathfrak{A}_4$ and $\mathfrak{A}_5$ are limit over the powerful type. This proves the item 1.

To finish the proof of the theorem we need to show that models $\mathfrak{A}$, $\mathfrak{A}_3$ and $\mathfrak{A}_5$ have computable presentations and the others do not.

Note that if the model $\mathfrak{A}_{-1} \leftrightharpoons Q(i(0)) + Q(i(1)) + \ldots$ would have computable presentation, than the set S would be the range of a limitwise monotonic function. This contradicts to the choice of S. (If it is not clear one can refer to [1] for detailed proof of the fact.) And the model $\mathfrak{A}_{-1}$ is definable in models $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_4$ by quantifier-free formula with one parameter, then these models also have no computable presentations.

Now we are going to prove that $\mathfrak{A}$ has computable presentation. We will construct it by steps.

Step 0. Consider the structure $\mathfrak{M}_0 = Q(g(0,0)) + Q(0)$.

Step n+1. Suppose the structure

$$\mathfrak{M}_n = Q(g(0,n)) + \ldots + Q(g(n,n)) + Q(k_{j_n}) + \ldots + Q(k_{j_1}) + Q(n) + \ldots + Q(0)$$

is already defined. Calculate $g(0, n+1), g(1, n+1), \ldots, g(n+1, n+1)$. Let t be the least number such that $g(t,n) \neq g(t, n+1)$. Find

$$k_{j_{n+1}} = max\{g(t,n), g(t+1,n), \ldots, g(n,n), k_{j_n}\}.$$

Extend the structure $Q(g(t,n)) + Q(g(t+1,n)) + \ldots + Q(g(n,n))$ to $Q(k_{j_{n+1}})$. And inserting the structure

$$Q(g(t, n+1)) + Q(g(t+1, n+1)) + \ldots + Q(g(n+1, n+1)) + Q(k_{j_{n+1}})$$

between structures $Q(g(0,n)) + Q(g(1,n)) + \ldots + Q(g(t-1,n))$ and

$$Q(k_{j_n}) + \ldots + Q(k_{j_1}) + Q(n+1) + Q(n) + \ldots + Q(0),$$

we get

$$\mathfrak{M}_{n+1} = Q(g(0,n+1)) + \ldots + Q(g(n+1,n+1)) + Q(k_{j_{n+1}}) + \ldots + Q(k_{j_1}) +$$

$$+Q(n+1) + \ldots + Q(0).$$

After ω steps we get the sequence of models $\mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \ldots$ It is easy to note that models $\cup_n \mathfrak{M}_n$ and $\mathfrak{A}$ are isomorphic. Thus $\mathfrak{A}$ has computable presentation.

Slight modification of the same argument works for models $\mathfrak{A}_3$ and $\mathfrak{A}_5$. If it is not clear, one can refer to [4] for the proof of the fact that the model $Q(i(0)) + Q(i(1)) + \ldots + Q_3(\omega)$ has computable presentation.

□

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