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Fractional Optimization Problems

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Abstract. We consider fractional maximization and minimization problems with an arbitrary feasible set, with a convex function in the numerator, and with a concave function in the denominator. These problems have many applications in economics and engineering. It has been shown that both of kind of problems belongs to a class of global optimization problems. These problems can be treated as quasiconvex maximization and minimization problems under certain conditions. For such problems we use the approach developed earlier. The approach based on the special Global Optimality Conditions according to the Global Search Theory proposed by A. S. Strekalovsky. For the case of convex feasible set, we reduce the original minimization problem to pseudoconvex minimization problem showing that any local solution is global. On this basis, two approximate numerical algorithms for fractional maximization and minimization problems are developed. Successful computational experiments have been done on some test problems with a dimension up to 1000 variables.

Keywords: fractional maximization, fractional minimization, global optimality conditions, approximation set.

1. Introduction

In this paper we consider the fractional optimization problems of the following types:

$$\max_{x\in D} \frac{f(x)}{g(x)}\ , \tag{1.1}$$

$$\min_{x\in D} \frac{f(x)}{g(x)}\ , \tag{1.2}$$

where $D \subset R^n$ is a subset, and $f(x)$ is convex, $g(x)$ is concave on D, $f(x)$ and $g(x)$ are positive on D. We call these problems as the fractional optimization problems. Problems (1.1)-(1.2) have many applications in economics and engineering. For instance, problems such as minimization of average cost function [7] and minimizing the ratio between the amount of resource wasted and used on the production plan belong to a class of fractional programming.

The most well-known and studied class of fractional programming is the linear fractional programming class. When D is convex then well known existing methods for solving problem (1.2) are variable transformation [8], nonlinear programming approach [5], and parametric approach [3]. The variable transformation method reduces problem (1.2) to convex programming for the case

$$D = \{x \in S \subset R^n \mid h(x) \leqslant 0\}$$

with $h : R^n \to R^m$ a convex vector-valued function and S a convex set.

Theorem 1. *[5] Problem (1.2) can be reduced to convex programming*

$$\min\{tf(t^{-1}y) \mid th(t^{-1}y) \leq 0,\ tg(t^{-1}y),\ t^{-1}y \in S,\ t > 0\} \tag{1.3}$$

applying the transformation

$$y = xt \text{ and } t = \frac{1}{g(x)}\ .$$

Moveover, if (y^*, t^*) *solves problem (1.2) then* $x^* = t^{-1}y^*$ *solves (1.2).*

One of the most popular strategies for fractional programming is the parametric approach which considers the class of optimization problems associated with problem (1.2) given by

$$\inf_{x \in D}\{f(x) - \lambda g(x)\} \tag{1.4}$$

with $\lambda \in R$.

Introduce the function $F(\lambda)$ as follows

$$F(\lambda) = \min_{x \in D}\{f(x) - \lambda g(x)\}.$$

Lemma 1. *[3] If D is a compact set then*

(a) The function $F : R \to R$ *is concave, continuous and strictly increasing.*

(b) The optimal solution λ^* *to (1.2) is finite and* $F(\lambda^*) = 0$.

(c) $F(\lambda) = 0$ *implies that* $\lambda = \lambda^*$.

(d) $\lambda^* = \dfrac{f(x^*)}{g(x^*)} = \min_{x \in D} \dfrac{f(x)}{g(x)}$.

2. Fractional Maximization and Global Optimality Conditions

Consider the fractional maximization problem:

$$\max_{x \in D} \left\{ \varphi(x) = \frac{f(x)}{g(x)} \right\}, \tag{2.1}$$

where $f, g : D \to R$ are differentiable functions, D is a convex subset in R^n, and $f(x)$ is convex on D and $g(x)$ is concave on D, $f(x) > 0$ and $g(x) > 0$ for all $x \in D$.

Introduce the level set of the function $\varphi(x)$ for a given $C > 0$.

$$L(\varphi, C) = \{x \in D | \ \varphi(x) \le C\}.$$

Lemma 2. *The set $L(\varphi, C)$ is convex.*

Proof. Since $g(x) > 0$ on D, then $\varphi(x) \le C \quad \forall x \in D$ can be written as follows:

$$f(x) - Cg(x) \le 0 \qquad \forall x \in D.$$

Clearly, a set defined by

$$M = \{x \in D | \ f(x) - Cg(x) \le 0\}$$

is convex which implies convexity of $L(\varphi, C)$.

Definition 1. *A function $f : \ D \to R$ is said to be quasiconvex on D if*

$$f(\alpha x + (1 - \alpha)y) \le \max\{f(x), f(y)\}$$

hold for all $x, y \in D$ and $\alpha \in [0, 1]$.

Lemma 3. *[4] The function $f(x)$ is quasiconvex on D if and only if the set $L(f, C)$ is convex for all $C \in R$.*

Then it is clear that the function $\varphi(x)$ is quasiconvex on D. The optimality conditions for a quasiconvex maximization problem were given in [4]. Applying this result to problem (2.1), we obtain the following proposition.

The optimality condition for problem (1.1) will be formulated as follows [4].

Theorem 2. *Let z be a solution to problem (2.1), and let $E_C(\varphi) = \{y \in R^n | \ \varphi(y) = C\}$.*
Then

$$\langle \varphi'(y), x - y \rangle \le 0 \tag{2.2}$$

for all $y \in E_{\varphi(z)}(\varphi)$ and $x \in D$.

If in addition $\varphi'(y) \ne 0$ holds for all $y \in E_{\varphi(z)}(\varphi)$, then condition (2.2) is sufficient for $z \in D$ to be a global solution to problem (2.1).

Condition (2.2) can be simplified as:

$$\sum_{i=1}^{n}\left\{\frac{\partial f(y)}{\partial x_i}g(y)-\frac{\partial g(y)}{\partial x_i}f(y)\right\}\left(\frac{x_i-y_i}{g^2(y)}\right)\le 0$$

for all $y\in E_{\varphi(z)}(\varphi)$ and $x\in D$.

These global optimality conditions are based on the Global Search Theory developed by A.S. Strekalovsky [10].

Algorithm and Approximation Set

Definition 2. *The set $A(z)$ defined for a given m by*

$$A_z^m=\{y^1,\ y^2,\ \ldots,\ y^m|\ y^i\in E_{\varphi(z)}(\varphi)\cap D,\ i=1,2,\ldots,m\}$$

is called an approximation set.

Lemma 4. *If there are a point $y^i\in A_z^m$ and a feasible point $z\in D$ such that*

$$\langle\varphi'(y^j),\ u^j-y^j\rangle>0$$

then $\varphi(u^j)>\varphi(z)$, where $\langle\varphi'(y^j),u^j\rangle=\max_{x\in D}\langle\varphi'(y^j),x\rangle$.

Proof. By the definition of u^j, we have

$$\max_{x\in D}\langle f'(y^j),x-y^j\rangle=\langle f'(y^j),u^j-y^j\rangle.$$

Since f is convex,

$$f(u)-f(v)\ge\langle f'(v),u-v\rangle$$

holds for all $u,v\in R^n$. Therefore, the assumption in the lemma implies that

$$f(u^i)-f(z)=f(u^i)-f(y^i)\ge\langle f'(y^i),u^i-y^i\rangle>0.$$

Now we can construct an algorithm for solving problem (1.1) approximately.

Algorithm MAX

Step 1. Choose $x^k\in D,\ k:=0.$ $z^k=\text{argloc}\max_{x\in D}\varphi(x)$, and m is given.
Step 2. Construct an approximation set $A^m_{z^k}$ at z^k.
Step 3. Solve Linear programming problems:

$$\max_{x\in D}\langle\varphi'(y^i),x\rangle\ ,\ i=1,2,\ldots,m$$

Let u^i be solutions to above problems:

$$\langle\varphi'(u^i),x\rangle=\max_{x\in D}\langle\varphi'(y^i),x\rangle,\ i=1,2,\ldots,m.$$

Step 4. Compute η_k :

$$\eta_k = \max_{1\le i\le m} \langle \varphi'(y^i), u^i - y^i\rangle = \langle \varphi'(y^j), u^j - y^j\rangle.$$

Step 5. If $\eta_k > 0$ then $x^{k+1} := u^j$, $k := k+1$ and go to Step 1.
Step 6. Terminate, z^k is an approximate global solution.

Lemma 5. *If $\eta_k > 0$ for all $k = 0, 1, \ldots$, then the sequence $\{z^k\}$ constructed by the Algorithm MAX is a relaxation sequence, i.e,*

$$f(z^{k+1}) > f(z^k),\ k = 0, 1, \ldots.$$

The proof follows from Lemma 4.

3. Fractional Minimization and Global optimality conditions

Consider the fractional minimization problem

$$\min_{x\in D} \left\{ \varphi(x) = \frac{f(x)}{g(x)} \right\}\ , \tag{3.1}$$

where, $D \subset R^n$ is an arbitrary compact set, $f, g:\ R^n \to D$ are differentiable functions, $f(x)$ is convex, $g(x)$ is concave on D, f and g are positive on D.

As we have shown in section 2, the function $\varphi(x)$ is quasiconvex on D. Now we can apply global optimality conditions in [4] to problem (3.1) as follows.

Theorem 3. *Let z be a global solution to problem (3.1), and let*

$$E_C(\varphi) = \{y \in R^n \mid \varphi(y) = C\}.$$

Then

$$\langle \varphi'(x), x - y\rangle \ge 0 \ \ \textit{for all } y \in E_{\varphi(z)}(\varphi) \textit{ and } x \in D. \tag{3.2}$$

If, in addition

$$\lim_{||x||\to\infty} \varphi(x) = +\infty \textit{ and } \varphi'(x + \alpha\varphi'(x)) \ne 0$$

holds for all $x \in D$ and $\alpha \ge 0$, then condition (3.2) becomes sufficient.

The optimality condition (3.2) can be written as follows:

$$\sum_{i=1}^{n} \left\{ \frac{\partial f(x)}{\partial x_i} g(x) - \frac{\partial g(x)}{\partial x_i} f(x) \right\} \left(\frac{x_i - y_i}{g^2(x)} \right) \ge 0$$

for all $y \in E_{\varphi(z)}(\varphi)$ and $x \in D$.

Definition 3. *Let Q be a subset of R^n. A differentiable function $h: Q \to R$ is pseudoconvex at $y \in Q$ if*

$$h(x) - h(y) < 0 \text{ which implies } h'(y)(x-y) < 0 \quad \forall x \in Q.$$

A function $h(.)$ is pseudoconvex on Q if it is pseudoconvex at each point $y \in Q$.

Lemma 6. *Let D be a convex set in R^n. Let $f: D \to R$ be convex, differentiable and positive; Let $g: D \to R$ be concave, differentiable and positive. Then the function $\varphi(x) = \dfrac{f(x)}{g(x)}$ is pseudoconvex.*

Proof. Take any point $y \in D$. Introduce the function $\psi: D \to R$ as follows:

$$\psi(x) = f(x)g(y) - g(x)f(y).$$

Since $g(y) > 0$ and $f(y) > 0$, $\psi(x)$ is convex and differentiable. Clearly, $\psi(y) = 0$. It is obvious that

$$\varphi(y) > \varphi(x) \text{ which is equivalent to } \psi(y) > \psi(x).$$

Since $\psi(.)$ is convex and differentiable, then we have

$$0 > \psi(x) - \psi(y) \geq \langle \psi'(y), x-y \rangle.$$

Taking into account that

$$\frac{\langle \psi'(y), x-y \rangle}{[g'(y)]^2} = \frac{\langle f'(y)g(y) - g'(y)f'(y), x-y \rangle}{[g'(y)]^2} = \langle \varphi'(y), x-y \rangle,$$

we obtain implications

$$\varphi(y) > \varphi(x), \text{ hence we have } \langle \varphi'(y), x-y \rangle < 0$$

which prove the assertion.

Lemma 7. *Let D be a convex set. Then any local minimizer x^* of $\varphi(x)$ on D is also a global minimizer.*

Proof. On the contrary, assume that x^* is not a global minimizer. Then there exists a point $u \in D$ such that

$$\varphi(x^*) > \varphi(u). \tag{3.3}$$

Since D is a convex set,

$$x^* + \alpha(u - x^*) = \alpha u + (1-\alpha)x^* \in D \quad \forall \alpha : 0 < \alpha < 1.$$

By Taylor's expansion, we have

$$\varphi(x^* + \alpha(u - x^*)) = \varphi(x^*) + \alpha\langle\varphi'(x^*), u - x^*\rangle + o(\alpha||u - x^*||),$$

where $\lim_{\alpha\to 0+} \dfrac{o(\alpha||u - x^*||}{\alpha} = 0$. Since x^* is a local minimizer of $\varphi(.)$ on D, there exists $0 < \alpha^* < 1$ so that

$$\varphi(x^* + \alpha(u - x^*)) - \varphi(x^*) > 0 \qquad \forall\alpha: \ 0 < \alpha < \alpha^*,$$

which implies

$$\langle\varphi'(x^*), u - x^*\rangle > 0.$$

Since $\varphi(.)$ is pseudoconvex, $\langle\varphi'(x^*), u - x^*\rangle > 0$ implies that $\varphi(u) > \varphi(x^*)$ contradicting (3.3) $\varphi(x^*) > \varphi(u)$. This completes the proof.

Lemma 7 allows us to apply gradient methods for solving problem (3.1).

We provide with an algorithm of conditional gradient method [2].

Algorithm MIN

Step 1. Choose an arbitrary feasible point $x^0 \in D$ and *set* $k := 0$.
Step 2. Solve the linear programming

$$\langle\varphi'(x^k), \bar{x}_k\rangle = \min_{x\in D}\langle\varphi'(x^k), \bar{x}\rangle.$$

Let $\bar{x}_k$ be a solution to the above problem.
Step 3. Compute η_k :

$$\eta_k = \langle\varphi'(x^k), \bar{x}_k - x^k\rangle.$$

Step 4. If $\eta_k = 0$ then x^k is a solution.
Step 5. $x^{k+1} = x^k(\alpha_k), \ x^k(\alpha) = x^k + \alpha(\bar{x}_k - x^k), \ \alpha \in [0, 1],$

$$f(x^k(\alpha_k)) = \min_{\alpha\in[0,1]} f(x^k(\alpha)).$$

Step 6. Set $k := k + 1$ and go to Step 2.

The convergence of the Algorithm MIN is given below.

Theorem 4. *[2] The sequence $\{x^k, \ k = 0, 1, \dots\}$ generated by Algorithm MIN is a minimizing sequence, i.e,*

$$\lim_{k\to\infty} \varphi(x^k) = \min_{x\in D} \varphi(x).$$

4. Fractional optimization test problems

In order to implement numerically the proposed Algorithm MAX, we consider the problem of the following type:

$$\max_{x\in D}\left\{\varphi(x)=\frac{\langle Ax,x\rangle+\langle b,x\rangle+k}{\langle Cx,x\rangle+\langle d,x\rangle+e}\right\},$$

where $D=\{x\in R^n|\ \alpha_i\le x_i\le\beta_i,\ i=1,2,\dots,n\}$, $k=4000, e=6000000$. Elements of the approximation set are defined as:

$$y^i=z^k+\alpha h^i,\ i=1,2,\dots,m,$$

where z^k is a local solution found by the conditional gradient method starting from an arbitrary feasible point $x^k\in D$. Vectors h^i are generated randomly. Parameter α can be found from the equation $\varphi(y^i)=\varphi(z^k)$ in the following:

$$\alpha=\frac{\langle(\varphi(z^k)C-A)h^i,z^k\rangle+\langle(\varphi(z^k)C-A)z^k+\varphi(z^k)d-b-\varphi(z^k)e,h^i\rangle}{\langle(\varphi(z^k)C-A)h^i,h^i\rangle}.$$

The following problems have been solved numerically on MATLAB by the proposed Algorithm MAX and in all cases the global solutions are found.

Consider problem (3.1) for the quadratic case:

$$\min_{x\in D}\left\{\frac{f(x)}{g(x)}=\frac{\langle Ax,x\rangle+\langle b,x\rangle}{\langle Cx,x\rangle+\langle d,x\rangle+e}\right\}$$

where $D=\{x\in R^n\mid Bx\le l\}$ is compact, A and C are matrices such that $A_{n\times n}>0,\ C_{n\times n}<0, f(x)>0$ and $g(x)>0$ on D.

The algorithm of conditional gradient method is the following..

The problem (3.1) with the following data have been solved numerically on MATLAB based on Algorithm MIN.

Constraints of problems (2.1) and (3.1) are given as follows.

Problem 1.

$$A=\begin{pmatrix}1&-2&-1\\-1&3&0\\4&1&2\end{pmatrix},\quad C=\begin{pmatrix}2&1&3\\0&2&1\\-1&1&-1\end{pmatrix},\quad b=\begin{pmatrix}3\\2\\4\end{pmatrix},\quad d=\begin{pmatrix}2\\1\\1\end{pmatrix}.$$

$D_1=\{1\le x_1\le 3,\ 2\le x_2\le 5,\ 1\le x_3\le 4\}$;
$D_2=\{x\in R^n\mid \mathbb{Q}x\leqslant q\}$, where

$$\mathbb{Q}=\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix},\qquad q=\begin{pmatrix}3&2&1\end{pmatrix}.$$

Problem 2.

$$A=\begin{pmatrix}1&1&1&1\\1&2&1&1\\1&1&3&1\\1&1&1&4\end{pmatrix},\ C=\begin{pmatrix}-2&1&1&2\\1&-1&-1&-2\\1&-1&-3&1\\2&-2&1&-9\end{pmatrix},\ b=\begin{pmatrix}2\\-2\\3\\-4\end{pmatrix},\ d=\begin{pmatrix}1\\-2\\3\\-1\end{pmatrix}.$$

$D_1=\{-2\le x_1\le 2,-1\le x_2\le 4,-1\le x_3\le 5,-3\le x_4\le 1\}$;
$D_2=\{x\in R^n \mid \mathbb{Q}x\leqslant q\}$, where

$$\mathbb{Q}=\begin{pmatrix}4&3&2&1\\0&3&2&1\\0&0&2&1\\0&0&0&1\end{pmatrix},\qquad q=\begin{pmatrix}4&3&2&1\end{pmatrix}.$$

Problem 3.

$$A=\begin{pmatrix}n&n-1&n-2&...&2&1\\n-1&n&n-1&...&3&2\\n-2&n-1&n&...&4&3\\...&...&...&...&...&...\\...&...&...&...&...&...\\2&3&4&...&n&n-1\\1&2&3&...&n-1&n\end{pmatrix},\ C=\begin{pmatrix}-1&-1&-1&....&-1&-1\\-1&-2&-1&....&-1&-1\\-1&-1&-3&....&-1&-1\\...&...&...&...&...&...\\...&...&...&...&...&...\\-1&-1&-1&....&1-n&-1\\-1&-1&-1&....&-1&-n\end{pmatrix},$$

$$b=(n,n-1,n-2,...,3,2,1),\qquad d=(1,2,3,...,n-2,n-1,n).$$

$D_1=\{\ x\in R^n|\quad -1\leqslant x_i\leqslant 10,\quad i=1,2,...,n\}$;
$D_2=\{x\in R^n \mid \mathbb{Q}x\leqslant q\}$, where

$$\mathbb{Q}=\begin{pmatrix}1&1&1&...&1&1\\0&1&1&...&1&1\\0&0&1&...&1&1\\...&...&...&...&...&...\\...&...&...&...&...&...\\0&0&0&...&1&1\\0&0&0&...&0&1\end{pmatrix},\qquad q=\begin{pmatrix}n\\n-1\\n-2\\...\\...\\2\\1\end{pmatrix}.$$

problems	const-raints	max			min		
		Initial value	Global value	time /sec /	Initial value	Global value	time /sec/
problem 1	D1	0.4166	0.6566	0.063	0.4166	0.4167	0.0047
	D2	0.6684	0.6810	0.064	0.0667	0.0666	0.058
problem 2	D1	0.3500	3.6667	0.076	0.3500	0.1433	0.0773
	D2	0.6713	0.7155	0.071	0.6713	0.6686	0.356
problem 3	D1	0.0036	1.4800	1.037	0.0141	2.4224e-20	0.882
n=50	D2	0.0141	0.0178	1.297	0.0808	0.0667	1.205
problem 3	D1	0.0282	14.6461	6.259	0.1121	9.5365e-18	2.220
n=100	D2	0.1121	0.1649	6.571	0.1789	0.0667	5.951
problem 3	D1	0.2241	168.6475	16.131	0.8982	1.6110e-5	50.5362
n=200	D2	0.8982	1.3340	57.683	0.9653	0.7788	66.114
problem 3	D1	3.5008	916.4216	147.373	14.5133	3.9996	1482.09
n=500	D2	14.513	21.6828	1854.412	14.5829	7.1792	2217.02
problem 3	D1	28.4096	734.3228	982.559	133.4068	50.5366	3492.972
n=1000	D2	133.4068	199.7074	7494.645	133.4868	100.2541	6897.254

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Дробные задачи оптимизации

Аннотация. В статье рассматриваются дробные задачи оптимизации на максимум и на минимум на произвольном допустимом множестве с выпуклой функцией в числителе и вогнутой функцией в знаменателе, имеющие много приложений в экономике и технике. Показано, что оба типа задач относятся к классу задач глобальной оптимизации. При определенных условиях эти задачи можно исследовать как задачи квазивыпуклой максимизации и минимизации. С этой целью используется разработанный ранее подход. Этот подход базируется на специальных условиях глобальной оптимальности, построенных в соответствии с теорией глобального поиска А. С. Стрекаловского. Для случая выпуклого допустимого множества исходная дробная задача минимизации сводится к псевдовыпуклой задаче минимизации, в которой всякое локальное решение является глобальным. На этой основе разработаны два приближенных численных алгоритма для решения дробных задач оптимизации на максимум и на минимум. Проведены вычислительные эксперименты по решению ряда тестовых задач рассматриваемых классов размерности до 1000 переменных.

Ключевые слова: дробная максимизация, дробная минимизация, условия глобальной оптимальности, аппроксимирующее множество.

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