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On the Numerical Range and Numerical Radius of the Volterra Operator

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Abstract. In this paper, we investigated the numerical range and the numerical radius of the classical Volterra operator on the complex space $L^2[0,1]$. In particular, we determined the numerical range, the numerical radius of real and imaginary part of the Volterra operator.

Keywords: Volterra operator, numerical range, numerical radius.

1. Introduction

Let H be a complex Hilbert space equipped with the inner product $(\cdot,\cdot)$, which induces the norm $\|\cdot\|$. Denote by $B(H)$ the Banach algebra of bounded linear operators acting on H with the operator norm defined by

$$\|A\| = \sup_{\|x\|=1} \{\|Ax\| : x \in H\}, \;\; A \in B(H).$$

Recall that for an operator A the spectrum

$$\sigma(A) = \{\lambda \in \mathbb{C} : \text{such that } \; A - \lambda I \; \text{ is not invertible}\}$$

is a non-empty compact subset of the complex plane.

For a bounded linear operator A on a complex Hilbert space H, the numerical range $W(A)$ is the image of the unit sphere of H under the quadratic form $x \to (Ax, x)$ associated with the operator. More precisely,

$$W(A) = \{(Ax, x) : x \in H, \|x\| = 1\}.$$

It is well known that numerical range of an operator is convex (The Toeplitz-Hausdorf theorem) and spectrum is contained in the closure of its numerical range. Note that A is a self-adjoint if and only if $W(A) \subset \mathbb{R}$. The numerical radius of an operator A is defined by

$$\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\}$$

and the following inequalities

$$\frac{\|A\|}{2} \leq \omega(A) \leq \|A\|$$

hold (see [2]).

Denote by V the classical Volterra operator

$$(Vf)(x) = \int_0^x f(t)dt, \quad f \in L^2[0, 1].$$

The adjoint of the Volterra operator is

$$(V^*f)(x) = \int_x^1 f(t)dt.$$

The Volterra operator is compact and quasinilpotent. The Volterra operator pencils studies were made in several directions concerning it, see e.g. [5–8] and the references therein.

2. The Results

We will need the following theorem.

Theorem 1. *[1, page 268] If A is a bounded operator on H and $\theta \in [-\pi, \pi]$, put $\lambda_\theta = \max \sigma(B_\theta)$, where $B_\theta = \frac{1}{2}(e^{-i\theta}A + e^{i\theta}A^*) = B_\theta^*$. Then*

$$\overline{W(A)} = \bigcap_{\theta \in [-\pi, \pi]} H_\theta$$

where the half-space H_θ is defined by

$$H_\theta = \{z \in \mathbb{C} : \mathrm{Re}(e^{-i\theta}z) \leq \lambda_\theta\}.$$

Remark 1. According to Theorem 1 and under the assumption $\lambda_\theta \in C^1[-\pi,\pi]$, we have

$$x\cos\theta + y\sin\theta = \lambda_\theta$$

which is envelope curve. Because, if $0<\theta<\pi$, then $\sin\theta>0$ and $y \le \frac{\lambda_\theta}{\sin\theta} - x\cot\theta$. Similarly, if $-\pi<\theta<0$, then $\sin\theta<0$ and $y \ge \frac{\lambda_\theta}{\sin\theta} - x\cot\theta$.

Proposition 1. *The operator zV $(z\in\mathbb{C})$ is accretive on $L^2[0,1]$ if and only if $Rez \ge 0$ and $Imz = 0$.*

Proof. (If) Let zV $(z\in\mathbb{C})$ be accretive, that is

$$((zV+\bar{z}V^*)f,f) \ge 0 \ \text{ for all } \ f\in L^2[0,1].$$

We choose $f_k(x)=e^{ik\pi x}$ for all $k\in\mathbb{Z}$. Then

$$(Vf_k)(x) = \int_0^x e^{ik\pi t}dt = \frac{1}{ik\pi}(e^{ik\pi x}-1)$$

$$(Vf_k,f_k) = \frac{1}{ik\pi}\int_0^1 (e^{ik\pi t}-1)e^{-ik\pi t}dt = \frac{1}{ik\pi} + \frac{1-(-1)^k}{k^2\pi^2}$$

and

$$(Vf,f)+\bar{z}(V^*f,f) = \frac{2\mathrm{Im}z}{k\pi} + \frac{2(1-(-1)^k)\mathrm{Re}z}{k^2\pi^2} \ge 0$$

for all $k\in\mathbb{Z}$. Thus, $\mathrm{Re}z \ge 0$ and $\mathrm{Im}z=0$.

(Only if) Assume that $\mathrm{Re}z\ge 0$ and $\mathrm{Im}z=0$. Then

$$((zV+\bar{z}V^*)f,f) = z\left|\int_0^1 f(t)dt\right|^2 \ge 0.$$

□

Remark 2. The operator V is accretive.

Proposition 2. *The numerical range of V is the set lying between the curves*

$$\frac{1-\cos\varphi}{\varphi^2} \pm i\frac{\varphi-\sin\varphi}{\varphi^2}$$

where $\varphi\in[0,2\pi]$. (see [4], Problem 150)

Proof. The identity

$$\mathrm{Re}(Vf,f) = \frac{1}{2}\left|\int_0^1 f(t)dt\right|^2 \le \frac{1}{2}\int_0^1 |f(t)|^2dt$$

implies that $W(V)\subseteq\{z : 0\le \mathrm{Re}z \le \frac{1}{2}\}$. Put $A=V$ as in Theorem 1, then

$$\lambda_\theta = \frac{\sin\theta}{2\theta}.$$

(see [1, page 270])

Remark 1 implies that, envelope curve is

$$x\cos\theta + y\sin\theta = \frac{\sin\theta}{2\theta} \tag{2.1}$$

for $\theta \in [-\pi, \pi]$.

Moreover, (2.1) implies that the boundary of numerical range is

$$\begin{cases} x\cos\theta + y\sin\theta = \dfrac{\sin\theta}{2\theta} \\ -x\sin\theta + y\cos\theta = \dfrac{\theta\cos\theta - \sin\theta}{2\theta^2} \end{cases}$$

or

$$\begin{cases} x = \dfrac{\sin^2\theta}{2\theta^2} = \dfrac{1-\cos 2\theta}{(2\theta)^2} = \dfrac{1-\cos\varphi}{\varphi^2} \\ y = \dfrac{\theta - \sin\theta\cos\theta}{2\theta^2} = \dfrac{2\theta - \sin 2\theta}{(2\theta)^2} = \dfrac{\varphi - \sin\varphi}{\varphi^2} \end{cases}$$

for $\varphi = 2\theta \in [0, 2\pi]$.

If $z = (Vf, f) \in W(V)$ then $(V\bar{f}, \bar{f}) = \bar{z} \in W(V)$. We have

$$\begin{cases} x = \dfrac{1-\cos\varphi}{\varphi^2} \\ y = \pm\dfrac{\varphi - \sin\varphi}{\varphi^2} \end{cases}$$

where $\varphi \in [0, 2\pi]$. Put $f_\varphi(t) = e^{it\varphi}$, then

$$(Vf_\varphi)(x) = \frac{1}{i\varphi}(e^{i\varphi x} - 1);$$

$$(Vf_\varphi, f_\varphi) = -\frac{1}{\varphi^2}(i\varphi - 1 + e^{-i\varphi})$$

$$\begin{cases} x = \mathrm{Re}(Vf_\varphi, f_\varphi) = \dfrac{1-\cos\varphi}{\varphi^2} \\ y = \mathrm{Im}(Vf_\varphi, f_\varphi) = -\dfrac{\varphi - \sin\varphi}{\varphi^2} \end{cases}$$

Therefore, the numerical range of V is the set lying between the curves

$$\frac{1-\cos\varphi}{\varphi^2} \pm i\frac{\varphi - \sin\varphi}{\varphi^2}$$

where $\varphi \in [0, 2\pi]$. □

Proposition 3. *Let V be the Volterra operator on $L^2[0,1]$. Then*

i) $\omega(V) = \frac{1}{2}$. *(For on real space see [3, Theorem 6.1])*

ii) $W(ReV) = [0, \frac{1}{2}]$.

iii) $W(ImV) = [-\frac{1}{\pi}, \frac{1}{\pi}]$.

Proof. i). It is easy to see that

$$f(\varphi) = x^2 + y^2 = \frac{1}{\varphi^4}\left(2 - 2\cos\varphi - 2\varphi\sin\varphi + \varphi^2\right)$$

and

$$f^{'}(\varphi) = -\frac{4}{\varphi^5}\left(\varphi\cos\frac{\varphi}{2} - 2\sin\frac{\varphi}{2}\right)^2 \leq 0.$$

We have

$$\omega^2(V) = \sup_{0\leq\varphi\leq 2\pi} f(\varphi) =$$

$$= \lim_{\varphi\to 0} f(\varphi) = \lim_{\varphi\to 0}\left(\left(\frac{1-\cos\varphi}{\varphi^2}\right)^2 + \left(\frac{\varphi - \sin\varphi}{\varphi^2}\right)^2\right) = \frac{1}{4},$$

or

$$\omega(V) = \frac{1}{2}.$$

ii). ReV is self-adjoint, bounded convex subset of the real line. Note that

$$((\mathrm{Re}V)\cos\pi x, \cos\pi x) = 0, \text{ and } ((\mathrm{Re}V)1, 1) = \frac{1}{2}$$

and $\left\{0, \frac{1}{2}\right\} \in \partial W(\mathrm{Re}V)$. We have $W(\mathrm{Re}V) = \left[0, \frac{1}{2}\right]$.

iii). ImV is self-adjoint and $\max_{\varphi\in[0,2\pi]} \frac{\varphi - \sin\varphi}{\varphi^2} = \frac{1}{\pi}$ imply that

$$W(\mathrm{Im}V) = \left[-\frac{1}{\pi}, \frac{1}{\pi}\right].$$

□

Remark 3. From Proposition 3, we have

$$\omega(\mathrm{Re}V) = \frac{1}{2} \text{ and } \omega(\mathrm{Im}V) = \frac{1}{\pi}.$$

3. Conclusion

We investigated the numerical range and the numerical radius of the classical Volterra operator. In particular, we give new proof of the numerical range and the numerical radius of the classical Volterra operator on the complex space $L^2[0,1]$ (see [3, page 984], [4, page 113]).

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Числовая область и числовой радиус оператора Вольтерра

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Аннотация. Обозначим через V классический оператор Вольтерра на комплексном пространстве $L^2[0,1]$. Мы определили числовую область и числовой радиус классического оператора Вольтерра, при этом доказательства соответствующих утверждений отличаются от известных. В частности, определены числовая область и числовой радиус действительной (комплексной) части оператора Вольтерра.

Ключевые слова: Оператор Вольтерра, числовая область, числовой радиус.

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