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On Certain Subclasses of Analytic Functions with Varying Arguments of Coefficients

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Abstract. In this paper we introduce and study the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$ of analytic functions with varying arguments of coefficients. We obtain coefficients inequalities, distortion theorems involving fractional calculus, radii of close to convexity, starlikeness and convexity and square root transformation for functions in the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$. Finally, integral convolution for functions in this class are considered.

Keywords: analytic functions, Hadamard product, fractional calculus operators, varying arguments of coefficients, square root transformation, integral convolution.

1. Introduction

Let $\mathcal{A}_n$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \ (n \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathcal{S}_n$ be the subclass of all functions in $\mathcal{A}_n$, which are univalent in $\mathbb{U}$. Let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the subclasses of starlike and convex functions of order α $(0 \le \alpha < 1)$. We note that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$, the subclasses

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of starlike and convex functions (see, for example, Srivastava and Owa [8]). Also, let $\mathcal{R}(\alpha)$ $(1 \leq \alpha < 2)$ denote the subclass of $\mathcal{A}_1$ satisfies the inequality

$$\Re\left\{f'(z)\right\} < \alpha. \tag{1.2}$$

The class $\mathcal{R}(\alpha)$ $(1 \leq \alpha < 2)$ was studied by Uralegaddi et al. [9].

Many essentially equivalent definitions of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (cf., e.g. [1], [3], [5] and [6]). We find it to be convenient to recall here the following definitions which were used recently by Owa [3] and by Srivastava and Owa [7].

Definition 1. *The fractional integral of order δ is defined, for a function $f(z)$, by*

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\delta}} d\zeta \ (\delta > 0),$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$plane containing the origin and the multiplicity of $(z-\zeta)^{\delta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 2. *The fractional derivative of order δ is defined, for a function $f(z)$, by*

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\delta}} d\zeta \ (0 \leq \delta < 1),$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$plane containing the origin and the multiplicity of $(z-\zeta)^{-\delta}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 3. *Under the hypotheses of Definition 2, the fractional derivative of order $k+\delta$ is defined by*

$$D_z^{k+\delta} f(z) = \frac{d^k}{dz^k} D_z^{\delta} f(z) \ (0 \leq \delta < 1; \ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

In this paper, we define the following subclass of $\mathcal{A}_n$ as follows.

Definition 4. *A function $f(z) \in \mathcal{A}_n$ is said to be in the class $\mathcal{R}_\delta(n, \lambda, \alpha)$ if*

$$\Re\left\{\Gamma(2-\delta) z^{\delta-1} \left[(1-\lambda) D_z^{\delta} f(z) + \lambda z D_z^{\delta+1} f(z)\right]\right\} < \alpha$$

$$(1 \leq \alpha < 2; \ 0 \leq \lambda \leq 1; \ 0 \leq \delta < 1; \ \alpha + \delta < 2; \ z \in \mathbb{U}). \tag{1.3}$$

In [4], Silverman introduced and studied the univalent functions with varying arguments of coefficients, as follows:

Definition 5. *[4] We say that a function $f(z)$ of the form (1.1) is in the class $\mathcal{V}(\theta_k)$ if $f(z) \in \mathcal{S}$ (the class of analytic and univalent functions in $\mathbb{U}$) and $\arg(a_k) = \theta_k$ for all k $(k \geq 2)$. Further, if there exists a real number η such that*

$$\theta_k + (k-1)\eta \equiv \pi \,(mod\ 2\pi)\,, \tag{1.4}$$

then $f(z)$ is said to be in the class $\mathcal{V}(\theta_k, \eta)$. The union of $\mathcal{V}(\theta_k, \eta)$ taken over all possible sequences $\{\theta_k\}$ and all possible real numbers η is denoted by $\mathcal{V}$.

Silverman [4] used the concept of varying arguments of the coefficients to introduce and study the class $\mathcal{V}^*(\alpha)$, which is a subclass of $\mathcal{V}$ consisting of starlike functions of order α.

For $\eta = 0$, we obtain the class $\mathcal{T}_n$ consisting of functions $f(z)$ with negative coefficients.

Using the concept of varying arguments of coefficients in univalent functions, we introduce the following subclass.

Definition 6. *Let $\mathcal{VR}_{\delta,\eta}(n, \lambda, \alpha)$ denote the subclass of $\mathcal{V}$ consisting of functions $f(z) \in \mathcal{R}_\delta(n, \lambda, \alpha)$.*

We note that:

(i) $\mathcal{VR}_{0,\eta}(n, \lambda, \alpha) = \mathcal{VR}_\eta(n, \lambda, \alpha) =$
$$= \left\{ f(z) \in \mathcal{A}_n : \Re\left[(1-\lambda)\tfrac{f(z)}{z} + \lambda f'(z)\right] < \alpha \right\};$$

(ii) $\mathcal{VR}_{0,\eta}(n, 0, \alpha) = \mathcal{V}\mathbb{R}_\eta(n, \alpha) = \left\{ f(z) \in \mathcal{A}_n : \Re\left(\tfrac{f(z)}{z}\right) < \alpha \right\};$

(iii) $\mathcal{VR}_{0,\eta}(n, 1, \alpha) = \mathcal{VR}_\eta(n, \alpha) = \{ f(z) \in \mathcal{A}_n : \Re\, f'(z) < \alpha \};$

(iv) $\mathcal{VR}_{0,0}(n, 0, \alpha) = \mathbb{R}(n, \alpha) = \left\{ f(z) \in \mathcal{T}_n : \Re\left(\tfrac{f(z)}{z}\right) < \alpha \right\};$

(v) $\mathcal{VR}_{0,0}(n, 1, \alpha) = \mathcal{R}(n, \alpha) = \{ f(z) \in \mathcal{T}_n : \Re\, f'(z) < \alpha \};$

(vi) $\mathcal{VR}_{0,0}(1, 1, \alpha) = \mathcal{R}(1, \alpha) = \mathcal{R}(\alpha)$ (see Uralegaddi et al. [10]).

2. Coefficient estimates for the class $\mathcal{VR}_{\delta,\eta}(n, \lambda, \alpha)$

Unless otherwise mentioned, we assume throughout this paper that

$1 \leq \alpha < 2,\ 0 \leq \lambda \leq 1,\ 0 \leq \delta < 1,\ \alpha + \delta < 2,\ k \geq n+1,\ n \in \mathbb{N}$ and $z \in \mathbb{U}$.

Theorem 1. *Let the function $f(z)$ be given by (1.1). Then $f(z) \in \mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{\Gamma(k+1-\delta)} |a_k| \leq \alpha + \lambda\delta - 1. \tag{2.1}$$

Proof. Assume that the condition (2.1) holds, then it is sufficient to show the inequality (1.3) holds. We find that

$$\begin{aligned}
&\left|\Gamma(2-\delta)z^{\delta-1}\left[(1-\lambda)D_z^{\delta}f(z)+\lambda z D_z^{\delta+1}f(z)\right]-(1-\lambda\delta)\right| \\
&= \left|\sum_{k=n+1}^{\infty} \frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{\Gamma(k+1-\delta)} a_k z^{k-1}\right|.
\end{aligned}$$

Since $f(z) \in \mathcal{V}$, then $f(z) \in \mathcal{V}(\theta_k,\eta)$ for some sequence $\{\theta_k\}$ and a real number η such that

$$\theta_k + (k-1)\eta \equiv \pi \,(mod\ 2\pi)\,.$$

Let $z = re^{i\eta}$, we have

$$\begin{aligned}
&\left|\Gamma(2-\delta)z^{\delta-1}\left[(1-\lambda)D_z^{\delta}f(z)+\lambda z D_z^{\delta+1}f(z)\right]-(1-\lambda\delta)\right| \\
&= \left|\sum_{k=n+1}^{\infty} \frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{\Gamma(k+1-\delta)} |a_k|\, e^{i[\theta_k+(k-1)\eta]} r^{k-1}\right| \\
&= \left|-\sum_{k=n+1}^{\infty} \frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{\Gamma(k+1-\delta)} |a_k|\, r^{k-1}\right| \\
&\leq \sum_{k=n+1}^{\infty} \frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{\Gamma(k+1-\delta)} |a_k| \\
&\leq \alpha + \lambda\delta - 1.
\end{aligned}$$

This shows that the function

$$\Phi(z) = \Gamma(2-\delta)z^{\delta-1}\left[(1-\lambda)D_z^{\delta}f(z)+\lambda z D_z^{\delta+1}f(z)\right],$$

lies in a circle which is centered at $w = 1-\lambda\delta$ with radius $\alpha+\lambda\delta-1$, hence the inequality (1.3) holds. Conversely, assume that

$$\Re\left\{\Gamma(2-\delta)z^{\delta-1}\left[(1-\lambda)D_z^{\delta}f(z)+\lambda z D_z^{\delta+1}f(z)\right]\right\} < \alpha,$$

or, equivalently

$$\Re\left\{1-\lambda\delta+\sum_{k=n+1}^{\infty} \frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{\Gamma(k+1-\delta)} a_k z^{k-1}\right\} < \alpha.$$

Letting $z \to 1^-$, we obtain the required result and hence the proof is completed. ☐

Corollary 1. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$.*
Then

$$|a_k| \leq \frac{(\alpha+\lambda\delta-1)\,\Gamma(k+1-\delta)}{\Gamma(2-\delta)\Gamma(k+1)\,[1+\lambda(k-\delta-1)]}.$$

The result is sharp for the function

$$f(z) = z + \frac{(\alpha+\lambda\delta-1)\,\Gamma(k+1-\delta)}{\Gamma(2-\delta)\Gamma(k+1)\,[1+\lambda(k-\delta-1)]} e^{i\theta_k} z^k. \tag{2.2}$$

3. Distortion theorems involving fractional calculus for the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$

Theorem 2. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$, then for $\mu > 0$ and $z \in \mathbb{U}$, we have*

$$\left|D_z^{-\mu} f(z)\right| \leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left[1 + \frac{(\alpha+\lambda\delta-1)\,\Gamma(2+\mu)\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+2+\mu)\,[1+\lambda(n-\delta)]}\,|z|^n\right], \tag{3.1}$$

and

$$\left|D_z^{-\mu} f(z)\right| \geq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left[1 - \frac{(\alpha+\lambda\delta-1)\,\Gamma(2+\mu)\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+2+\mu)\,[1+\lambda(n-\delta)]}\,|z|^n\right]. \tag{3.2}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z + \frac{(\alpha+\lambda\delta-1)\,\Gamma(2+\mu)\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+2+\mu)\,[1+\lambda(n-\delta)]} e^{i\theta_{n+1}} z^{n+1}. \tag{3.3}$$

Proof. It is easy to see from Theorem 1 that

$$\frac{\Gamma(2-\delta)\Gamma(n+2)\,[1+\lambda(n-\delta)]}{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)} \sum_{k=n+1}^{\infty} |a_k|$$

$$\leq \sum_{k=n+1}^{\infty} \frac{\Gamma(2-\delta)\Gamma(k+1)\,[1+\lambda(k-\delta-1)]}{(\alpha+\lambda\delta-1)\,\Gamma(k+1-\delta)} |a_k| \leq 1.$$

Hence

$$\sum_{k=n+1}^{\infty} |a_k| \leq \frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+2)\,[1+\lambda(n-\delta)]}.$$

Let

$$\begin{aligned} F(z) &= \Gamma(2+\mu)z^{-\mu}D_z^{-\mu}f(z) \\ &= z+\sum_{k=n+1}^{\infty}\frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+1+\mu)}a_k z^k. \end{aligned} \tag{3.4}$$

Since

$$0<\frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+1+\mu)}\leq\frac{\Gamma(n+2)\Gamma(2+\mu)}{\Gamma(n+2+\mu)},$$

then

$$\begin{aligned} |F(z)| &\leq \left|z+\sum_{k=n+1}^{\infty}\frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+1+\mu)}a_k z^k\right| \\ &\leq |z|+\sum_{k=n+1}^{\infty}\frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+1+\mu)}|a_k|\,|z|^{n+1} \\ &\leq |z|+\frac{\Gamma(n+2)\Gamma(2+\mu)}{\Gamma(n+2+\mu)}\sum_{k=n+1}^{\infty}|a_k|\,|z|^{n+1} \\ &\leq |z|+\frac{(\alpha+\lambda\delta-1)\,\Gamma(2+\mu)\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+2+\mu)\,[1+\lambda(n-\delta)]}|z|^{n+1}, \end{aligned}$$

and

$$|F(z)|\geq|z|-\frac{(\alpha+\lambda\delta-1)\,\Gamma(2+\mu)\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+2+\mu)\,[1+\lambda(n-\delta)]}|z|^{n+1},$$

which proves the inequalities (3.1) and (3.2). Since each of equalities in (3.1) and (3.2) is satisfied by the function $f(z)$ given by (3.3), the proof is thus completed. □

Theorem 3. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$, then for $0\leq\mu<1$ and $z\in\mathbb{U}$, we have*

$$|D_z^{\mu}f(z)|\leq\frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left[1+\frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\,[1+\lambda(n-\delta)]}|z|^{n}\right], \tag{3.5}$$

and

$$|D_z^{\mu}f(z)|\geq\frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left[1-\frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\,[1+\lambda(n-\delta)]}|z|^{n}\right]. \tag{3.6}$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z+\frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\,[1+\lambda(n-\delta)]}e^{i\theta_{n+1}}z^{n+1}. \tag{3.7}$$

Proof. It is easy to see from Theorem 1 that

$$\frac{\Gamma(2-\delta)\Gamma(n+1)\left[1+\lambda(n-\delta)\right]}{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}\sum_{k=n+1}^{\infty}k\left|a_k\right|$$

$$\leq\sum_{k=n+1}^{\infty}\frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{(\alpha+\lambda\delta-1)\,\Gamma(k+1-\delta)}\left|a_k\right|\leq 1.$$

Hence

$$\sum_{k=n+1}^{\infty}k\left|a_k\right|\leq\frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\left[1+\lambda(n-\delta)\right]}. \tag{3.8}$$

Let

$$\begin{aligned} G(z) &= \Gamma(2-\mu)z^{\mu}D_z^{\mu}f(z) \\ &= z+\sum_{k=n+1}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k+1-\mu)}a_k z^k. \end{aligned}$$

Since

$$1<\frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k+1-\mu)}<k,$$

Then

$$\begin{aligned} |G(z)| &\leq \left|z+\sum_{k=n+1}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k+1-\mu)}a_k z^k\right| \\ &\leq |z|+\sum_{k=n+1}^{\infty}\frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+1+\mu)}\left|a_k\right||z|^{n+1} \\ &\leq |z|+\sum_{k=n+1}^{\infty}k\left|a_k\right||z|^{n+1} \\ &\leq |z|+\frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\left[1+\lambda(n-\delta)\right]}|z|^{n+1}, \end{aligned}$$

and

$$|G(z)|\geq|z|-\frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\left[1+\lambda(n-\delta)\right]}|z|^{n+1},$$

which proves the inequalities (3.5) and (3.6). Since each of equalities in (3.5) and (3.6) is satisfied by the function $f(z)$ given by (3.7), the proof is thus completed. □

Putting $\mu=0$ in Theorem 3, we obtain the following corollary.

Corollary 2. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$, then for $z \in \mathbb{U}$, we have*

$$|f(z)| \le |z| + \frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\,[1+\lambda(n-\delta)]}\,|z|^{n+1},$$

and

$$|f(z)| \ge |z| - \frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\,[1+\lambda(n-\delta)]}\,|z|^{n+1}.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z + \frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\,[1+\lambda(n-\delta)]} e^{i\theta_{n+1}} z^{n+1}.$$

Theorem 4. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$, then for $\mu > 0$ and $z \in \mathbb{U}$, we have*

$$\left|D_z^{1-\mu} f(z)\right| \le$$
$$\le \frac{|z|^{\mu}}{\Gamma(2+\mu)}\left[(1+\mu) + \frac{(\alpha+\lambda\delta-1)\,(n+1+\mu)\Gamma(2+\mu)\Gamma(n+2-\mu)}{\Gamma(2-\delta)\Gamma(n+2+\mu)\,[1+\lambda(n-\delta)]}\,|z|^{n}\right],$$

and

$$\left|D_z^{1-\mu} f(z)\right| \ge \max\left\{0, \frac{|z|^{\mu}}{\Gamma(2+\mu)}\right.$$
$$\left.\times\left[(1-\mu) - \frac{(\alpha+\lambda\delta-1)\,(n+1-\mu)\Gamma(2+\mu)\Gamma(n+2-\mu)}{\Gamma(2-\delta)\Gamma(n+2+\mu)\,[1+\lambda(n-\delta)]}\,|z|^{n}\right]\right\},$$

Proof. Differentiating both sides of (3.4), we have

$$\begin{aligned}&\Gamma(2+\mu) z^{-\mu} D_z^{1-\mu} f(z) - \mu\Gamma(2+\mu) z^{-\mu-1} D_z^{-\mu} f(z)\\ &= 1 + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+1+\mu)} k a_k z^{k-1}.\end{aligned} \tag{3.9}$$

Hence, we obtain the required result from (3.1), (3.2), (3.8) and (3.9). This completes the proof of Theorem 4. □

4. Radii of close-to-convexity, starlikeness and convexity

Theorem 5. *Let the function $f(z)$ defined by (1.1) be in the class $VR_{\delta,\eta}(n,\lambda,\alpha)$. Then $f(z)$ is close-to-convex of order β $(0 \le \beta < 1)$ in $|z| \le r_1$, where*

$$r_1 = \inf_{k \ge n+1} \left\{ \frac{\Gamma(2-\delta)\Gamma(k+1)\,[1+\lambda(k-\delta-1)]}{(\alpha+\lambda\delta-1)\,\Gamma(k+1-\delta)} \left(\frac{1-\beta}{k}\right)\right\}^{\frac{1}{k-1}}. \tag{4.1}$$

The result is sharp, the extremal function given by (2.2).

Proof. We must show that

$$\left|f'(z)-1\right| \leq 1-\beta \ for \ |z| \leq r_1,$$

where r_1 is given by (4.1). Indeed we find from (1.1) that

$$\left|f'(z)-1\right| \leq \sum_{k=n+1}^{\infty} ka_k\,|z|^{k-1}.$$

Thus

$$\left|f'(z)-1\right| \leq 1-\beta,$$

if

$$\sum_{k=n+1}^{\infty}\left(\frac{k}{1-\beta}\right)|a_k|\,|z|^{k-1} \leq 1. \tag{4.2}$$

But by using Theorem 1, (4.2) will be true if

$$\left(\frac{k}{1-\beta}\right)|z|^{k-1} \leq \frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{(\alpha+\lambda\delta-1)\,\Gamma(k+1-\delta)}.$$

Then

$$|z| \leq \left\{\frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{(\alpha+\lambda\delta-1)\,\Gamma(k+1-\delta)}\left(\frac{1-\beta}{k}\right)\right\}^{\frac{1}{k-1}}.$$

This ends the proof. □

Theorem 6. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$.* Then $f(z)$ is starlike of order β $(0 \leq \beta < 1)$ in $|z| \leq r_2$, where

$$r_2 = \inf_{k \geq n+1}\left\{\frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{(\alpha+\lambda\delta-1)\,\Gamma(k+1-\delta)}\left(\frac{1-\beta}{k-\beta}\right)\right\}^{\frac{1}{k-1}}. \tag{4.3}$$

The result is sharp, with the extremal function given by (2.2).

Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\beta \ for \ |z| \leq r_2,$$

where r_2 is given by (4.3). Indeed we find from (1.1) that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{\sum\limits_{k=n+1}^{\infty}(k-1)a_k\,|z|^{k-1}}{1-\sum\limits_{k=n+1}^{\infty}a_k\,|z|^{k-1}}.$$

Thus

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \beta,$$

if

$$\sum_{k=n+1}^{\infty} \left(\frac{k-\beta}{1-\beta} \right) a_k \, |z|^{k-1} \leq 1. \tag{4.4}$$

But by using Theorem 1, (4.4) will be true if

$$\left(\frac{k-\beta}{1-\beta} \right) |z|^{k-1} \leq \frac{\Gamma(2-\delta)\Gamma(k+1)\,[1+\lambda(k-\delta-1)]}{(\alpha+\lambda\delta-1)\,\Gamma(k+1-\delta)}.$$

Then

$$|z| \leq \left\{ \frac{\Gamma(2-\delta)\Gamma(k+1)\,[1+\lambda(k-\delta-1)]}{(\alpha+\lambda\delta-1)\,\Gamma(k+1-\delta)} \left(\frac{1-\beta}{k-\beta} \right) \right\}^{\frac{1}{k-1}}.$$

This completes the proof of Theorem 6. □

Using similar arguments to those in the proof of the Theorem 6, we obtain the following corollary.

Corollary 3. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$. Then $f(z)$ is convex of order β $(0 \leq \beta < 1)$ in $|z| \leq r_3$, where*

$$r_3 = \inf_{k \geq n+1} \left\{ \frac{\Gamma(2-\delta)\Gamma(k)\,[1+\lambda(k-\delta-1)]}{(\alpha+\lambda\delta-1)\,\Gamma(k+1-\delta)} \left(\frac{1-\beta}{k-\beta} \right) \right\}^{\frac{1}{k-1}}.$$

The result is sharp, with the extremal function given by (2.2).

5. Square root transformation for the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$

Definition 7. **[2]** *If $f(z) \in \mathcal{S}$ and $h(z) = \sqrt{f(z^2)}$, then $h(z) \in \mathcal{S}$ and $h(z) = z + \sum_{k=n+1}^{\infty} c_{2k-1} z^{2k-1}$ $(z \in \mathbb{U})$. The function $h(z)$ is called a square root transformation.*

Theorem 7. *If $f(z) \in \mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha), (\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta) < \Gamma(2-\delta)\Gamma(n+1)\,[1+\lambda(n-\delta)]$ and $h(z)$ be the square root transformation of $f(z)$, then*

$$r\sqrt{1 - \frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\,[1+\lambda(n-\delta)]} r^{2n}}$$

$$\leq |h(z)| \leq$$

$$r\sqrt{1+\frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\,[1+\lambda(n-\delta)]}r^{2n}}.$$

The result is sharp for the function

$$f(z)=z+\frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\,[1+\lambda(n-\delta)]}e^{i\theta_n}z^{2n}$$

Proof. In view of Corollary 2, we have

$$r^2-\frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\,[1+\lambda(n-\delta)]}r^{2(n+1)}$$

$$\leq \left|f(z^2)\right| \leq$$

$$r^2+\frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\,[1+\lambda(n-\delta)]}r^{2(n+1)}.$$

We find that

$$\begin{aligned}|h(z)| &= \sqrt{|f(z^2)|}\\ &\leq r\sqrt{1+\frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\,[1+\lambda(n-\delta)]}r^{2n}},\end{aligned}$$

and

$$|h(z)|\geq r\sqrt{1-\frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+1)\,[1+\lambda(n-\delta)]}r^{2n}}.$$

This completes the proof of Theorem 7. □

6. Integral convolution for the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$

Let $f_j(z)$ $(j=1,2)$ be defined by

$$f_j(z)=z+\sum_{k=n+1}^{\infty}a_{k,j}z^k, \tag{6.1}$$

then, the integral convolution of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1\circledast f_2)(z)=z+\sum_{k=n+1}^{\infty}\tfrac{a_{k,1}a_{k,2}}{k}z^k=(f_2\circledast f_1)(z).$$

Theorem 8. *Let $f_j(z)$ $(j=1,2)$ defined by (6.1) be in the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$.*

Then $(f_1 \circledast f_2)(z) \in \mathcal{VR}_{\delta,\eta}(n,\lambda,\zeta)$, *where*

$$\zeta = 1 - \lambda\delta + \frac{(\alpha+\lambda\delta-1)^2\Gamma(n+2-\delta)}{(n+1)\,\Gamma(2-\delta)\Gamma(n+2)\,[1+\lambda(n-\delta)]}.$$

The result is sharp for the functions $f_j(z)$ $(j=1,2)$ *given by*

$$f_j(z) = z + \frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+2)\,[1+\lambda(n-\delta)]} e^{i\theta_{n+1}} z^{n+1} \ (j=1,2). \qquad (6.2)$$

Proof. We need to find the largest ζ such that

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(2-\delta)\Gamma(k+1)\,[1+\lambda(k-\delta-1)]}{(\zeta+\lambda\delta-1)\Gamma(k+1-\delta)} \frac{|a_{k,1}|\,|a_{k,2}|}{k} \le 1.$$

Since $f_j(z) \in \mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$ $(j=1,2)$, we readily see that

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(2-\delta)\Gamma(k+1)\,[1+\lambda(k-\delta-1)]}{(\alpha+\lambda\delta-1)\Gamma(k+1-\delta)} |a_{k,1}| \le 1,$$

and

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(2-\delta)\Gamma(k+1)\,[1+\lambda(k-\delta-1)]}{(\alpha+\lambda\delta-1)\Gamma(k+1-\delta)} |a_{k,2}| \le 1.$$

By the Cauchy Schwarz inequality we have

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(2-\delta)\Gamma(k+1)\,[1+\lambda(k-\delta-1)]}{(\alpha+\lambda\delta-1)\Gamma(k+1-\delta)} \sqrt{|a_{k,1}|\,|a_{k,2}|} \le 1. \qquad (6.3)$$

Thus it is sufficient to show that

$$\frac{\Gamma(2-\delta)\Gamma(k+1)\,[1+\lambda(k-\delta-1)]}{(\zeta+\lambda\delta-1)\Gamma(k+1-\delta)} \frac{|a_{k,1}|\,|a_{k,2}|}{k} \le$$

$$\frac{\Gamma(2-\delta)\Gamma(k+1)\,[1+\lambda(k-\delta-1)]}{(\alpha+\lambda\delta-1)\Gamma(k+1-\delta)} \sqrt{|a_{k,1}|\,|a_{k,2}|},$$

or, equilvalently, that

$$\sqrt{|a_{k,1}|\,|a_{k,2}|} \le k \left(\frac{\zeta+\lambda\delta-1}{\alpha+\lambda\delta-1} \right).$$

Hence, in light of the inequality (6.3), it is sufficient to prove that

$$\frac{(\alpha+\lambda\delta-1)\Gamma(k+1-\delta)}{\Gamma(2-\delta)\Gamma(k+1)\,[1+\lambda(k-\delta-1)]} \le k \left(\frac{\zeta+\lambda\delta-1}{\alpha+\lambda\delta-1} \right). \qquad (6.4)$$

It follows from (6.4) that

$$\zeta \geq 1-\lambda\delta+\frac{(\alpha+\lambda\delta-1)^2\Gamma(k+1-\delta)}{k\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}.$$

Now defining the function $H(k)$ by

$$H(k)=1-\lambda\delta+\frac{(\alpha+\lambda\delta-1)^2\Gamma(k+1-\delta)}{k\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]},$$

we see that $H(k)$ is an decreasing function of k. Therefore, we conclude that

$$\zeta \geq H(n+1)=1-\lambda\delta+\frac{(\alpha+\lambda\delta-1)^2\Gamma(n+2-\delta)}{(n+1)\,\Gamma(2-\delta)\Gamma(n+2)\left[1+\lambda(n-\delta)\right]},$$

which evidently completes the proof of Theorem 8. □

Using similar arguments to those in the proof of Theorem 8, we obtain the following theorem.

Theorem 9. *Let $f_1(z)$ defined by (6.1) be in the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$ and $f_2(z)$ defined by (6.1) be in the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\gamma)$. Then $(f_1 \circledast f_2)(z) \in \mathcal{VR}_{\delta,\eta}(n,\lambda,\xi)$, where*

$$\xi=1-\lambda\delta+\frac{(\alpha+\lambda\delta-1)(\gamma+\lambda\delta-1)\Gamma(n+2-\delta)}{(n+1)\,\Gamma(2-\delta)\Gamma(n+2)\left[1+\lambda(n-\delta)\right]}.$$

The result is sharp for the functions $f_j(z)$ $(j=1,2)$ given by

$$f_1(z)=z+\frac{(\alpha+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+2)\left[1+\lambda(n-\delta)\right]}e^{i\theta_{n+1}}z^{n+1}.$$

and

$$f_2(z)=z+\frac{(\gamma+\lambda\delta-1)\,\Gamma(n+2-\delta)}{\Gamma(2-\delta)\Gamma(n+2)\left[1+\lambda(n-\delta)\right]}e^{i\theta_{n+1}}z^{n+1}.$$

Theorem 10. *Let $f_j(z)$ $(j=1,2)$ defined by (6.1) be in the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$. Then the function*

$$h(z)=z+\sum_{k=n+1}^{\infty}\frac{\left(|a_{k,1}|^2+|a_{k,2}|^2\right)}{k}z^k,$$

belongs to the class $\mathcal{VR}_{\delta,\eta}(n,\lambda,\chi)$, where

$$\chi=1-\lambda\delta+\frac{2(\alpha+\lambda\delta-1)^2\Gamma(n+2-\delta)}{(n+1)\,\Gamma(2-\delta)\Gamma(n+2)\left[1+\lambda(n-\delta)\right]}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by (6.2).

Proof. By using Theorem 1, we obtain

$$\sum_{k=n+1}^{\infty} \left\{ \frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{(\alpha+\lambda\delta-1)\Gamma(k+1-\delta)} \right\}^2 |a_{k,1}|^2 \leq$$

$$\left\{ \sum_{k=n+1}^{\infty} \frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{(\alpha+\lambda\delta-1)\Gamma(k+1-\delta)} |a_{k,1}| \right\}^2 \leq 1, \tag{6.5}$$

and

$$\sum_{k=n+1}^{\infty} \left\{ \frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{(\alpha+\lambda\delta-1)\Gamma(k+1-\delta)} \right\}^2 |a_{k,2}|^2 \leq$$

$$\left\{ \sum_{k=n+1}^{\infty} \frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{(\alpha+\lambda\delta-1)\Gamma(k+1-\delta)} |a_{k,2}| \right\}^2 \leq 1, \tag{6.6}$$

It follows from (6.5) and (6.6) that

$$\sum_{k=n+1}^{\infty} \frac{1}{2} \left\{ \frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{(1-\lambda\delta-\alpha)\Gamma(k+1-\delta)} \right\}^2 \left(|a_{k,1}|^2 + |a_{k,2}|^2 \right) \leq 1.$$

Therefore, we need to find the largest χ such that

$$\frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{k(\chi+\lambda\delta-1)\Gamma(k+1-\delta)} \leq$$

$$\frac{1}{2} \left\{ \frac{\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}{(\alpha+\lambda\delta-1)\Gamma(k+1-\delta)} \right\}^2,$$

that is

$$\chi \geq 1 - \lambda\delta + \frac{2(\alpha+\lambda\delta-1)^2\Gamma(k+1-\delta)}{k\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]}.$$

Now defining the function $I(k)$ by

$$I(k) = 1 - \lambda\delta + \frac{2(\alpha+\lambda\delta-1)^2\Gamma(k+1-\delta)}{k\Gamma(2-\delta)\Gamma(k+1)\left[1+\lambda(k-\delta-1)\right]},$$

we see that $I(k)$ is an decreasing function of k. Therefore, we conclude that

$$\zeta \geq I(n+1) = 1 - \lambda\delta + \frac{2(\alpha+\lambda\delta-1)^2\Gamma(n+2-\delta)}{(n+1)\,\Gamma(2-\delta)\Gamma(n+2)\left[1+\lambda(n-\delta)\right]},$$

which evidently ends the proof. □

Remark 1. For different choices of δ, η and λ, we will obtain new results for different choices mentioned in the introduction.

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Некоторые подклассы аналитических функций с переменными аргументами в коэффициентах

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Аннотация. В настоящей работе вводится и изучается класс $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$ аналитических функций с переменными аргументами в коэффициентах. Получены неравенства на коэффициенты, теоремы искажения с использованием дробного исчисления, радиусы почти выпуклости, звездности и выпуклости и извлечение квадратного корня для функций из класса $\mathcal{VR}_{\delta,\eta}(n,\lambda,\alpha)$. Рассмотрен интеграл свертки для функций в этом классе.

Ключевые слова: аналитические функции; произведение Адамара; операторы дробного исчисления; переменные аргументы коэффициентов; извлечение квадратного корня; интегральная свертка

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