Серия «Математика»
2026. Т. 57. С. 143—152

Онлайн-доступ к журналу:
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ИЗВЕСТИЯ
Иркутского
государственного
университета

Research article

УДК 510.67:512.57
MSC 03C60
DOI https://doi.org/10.26516/1997-7670.2026.57.143

Some Model-Theoretic Properties of the Class of T-Pseudofinite S-Acts

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Abstract: The structure $\mathfrak{M}$ in a language L is called pseudofinite if every sentence in a language L true in $\mathfrak{M}$ has a finite model. A model $\mathfrak{M}$ of a theory T in a language L is called T-pseudofinite if every sentence in a language L true in $\mathfrak{M}$ has a finite model, which is a model of the theory T. In this work we prove that for any theory T the class of all pseudofinite (T-pseudofinite) models of this theory is axiomatizable. A (left) S–act over monoid S is a set A upon which S acts unitarily on the left. If T is the theory of all S-acts, then completeness, model completeness, and totally categoricalness of the class of all T-pseudofinite S-acts are equivalent to the following condition: every T-pseudofinite S-act is a coproduct of one-element S-acts. If S is a finite monoid or a commutative Noetherian and Artinian monoid then the theory of every S-act (T-pseudofinite S-act) is stable iff S is a linearly ordered monoid, where T is the theory of all S-acts.

Keywords: T-pseudofinite structure, act over monoid, axiomatizable theory, complete theory, stable theory

Acknowledgements: This work was supported by the Ministry of Science and Higher Education (agreement № 075-02-2026-1312 of 20.02.2026).

For citation: Stepanova A. A., Efremov E. L., Chekanov S. G. Some Model-Theoretic Properties of the Class of T-Pseudofinite S-Acts. *The Bulletin of Irkutsk State University. Series Mathematics*, 2026, vol. 57, pp. 143–152.
https://doi.org/10.26516/1997-7670.2026.57.143

Научная статья

Некоторые теоретико-модельные свойства класса T-псевдоконечных S-полигонов

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Аннотация: Структура $\mathfrak{M}$ языка L называется псевдоконечной, если каждое предложение языка L, истинное в $\mathfrak{M}$, имеет конечную модель. Модель $\mathfrak{M}$ теории T языка L называется T-псевдоконечной, если каждое предложение языка L, истинное в $\mathfrak{M}$, имеет конечную модель, являющуюся моделью теории T. Доказываеся, что для любой теории T класс всех псевдоконечных (T-псевдоконечных) моделей этой теории аксиоматизируем. (Левый) S-полигон над моноидом S — это множество A, на котором S действует унитарно слева. Если T — теория всех S-полигонов, то полнота, модельная полнота и тотальная категоричность класса всех T-псевдоконечных S-полигонов эквивалентны следующему условию: каждый T-псевдоконечный S-полигон является копроизведением одноэлементных S-полигонов. Если S — конечный моноид или коммутативный нётеров и артинов моноид, то теория каждого S-полигона (T-псевдоконечного S-полигона) стабильна тогда и только тогда, когда S — линейно упорядоченный моноид, где T — теория всех S-полигонов.

Ключевые слова: T-псевдоконечная структура, полигон над моноидом, аксиоматизируемая теория, полная теория, стабильная теория

Благодарности: Работа поддержана Министерством науки и высшего образования (соглашение № 075-02-2026-1312 от 20.02.2026).

Ссылка для цитирования: Stepanova A. A., Efremov E. L., Chekanov S. G. Some Model-Theoretic Properties of the Class of T-Pseudofinite S-Acts // Известия Иркутского государственного университета. Серия Математика. 2026. Т. 57. С. 143–152.
https://doi.org/10.26516/1997-7670.2026.57.143

1. Introduction

The L-structure $\mathfrak{M}$ is called pseudofinite if every sentence in language L true in $\mathfrak{M}$ has a finite model. More precisely, a structure $\mathfrak{M}$ is pseudofinite if it is elementarily equivalent to an ultraproduct of finite structures. One of the foundational developments in the area was James Ax's work [1], particularly his characterization of pseudofinite fields. Subsequently, the theory of pseudofinite fields received great development (see, for example, [4; 5]). A large review of the theory of models of finite and pseudofinite groups is presented in [10]. The results related to the model theory of finite and pseudofinite rings can be found in [2]. In [11], [12], the structure of pseudofinite acyclic graphs is studied. In [6], the authors of this work considered the issues of pseudofiniteness of connected unars without cyclics. In [7], the concept of T-pseudofiniteness for a theory T of a language L was introduced. A model $\mathfrak{M}$ of a theory T in a language L is called T-pseudofinite if every sentence in a language L true in $\mathfrak{M}$ has a finite model, which is a model of the theory T.

In this work we consider the issues of aximatizability of the class of all pseudofinite (T-pseudofinite) models of arbitrary theory T, of completeness and stability of the theory T of all S-acts.

2. Preliminaries

Let us recall some definitions and facts from model theory and model theory of S-acts.

Theorem 1. [7] *Let T be a theory of language L, and $\mathfrak{M}$ be a model of T. Then $\mathfrak{M}$ is a T-pseudofinite structure if and only if $\mathfrak{M}$ is elementary equivalent to the ultraproduct of finite models of the theory T.*

Let S be a monoid, and 1 be unity in S. A structure $\langle A; s\rangle_{s\in S}$ in the language $L_S = \{s \mid s \in S\}$ is called *a (left) S-act* if $s_1(s_2a) = (s_1s_2)a$ and $1a = a$ for all $s_1, s_2 \in S$ and $a \in A$. An S-act $\langle A; s\rangle_{s\in S}$ is denoted by ${}_SA$.

Theorem 2. [8] *Every coproduct of finite S-acts is a T-pseudofinite S-act, where T is the theory of all S-acts.*

Theorem 3. *(Los's theorem [3]) Let $\{\mathfrak{M}_i \mid i \in I\}$ be a set of L-structures, D be an ultrafilter on I, $\mathfrak{M} = \prod_{i\in I} \mathfrak{M}_i/D$ be the ultraproduct, $\Phi(x_1, \dots, x_n)$ be the formula of the language L, and $m_1, \dots, m_n \in \prod_{i\in I} \mathfrak{M}_i$. Then*

$$\mathfrak{M} \models \Phi(m_1/D, \dots, m_n/D) \Leftrightarrow \{i \in I \mid \mathfrak{M}_i \models \Phi(m_1(i), \dots, m_n(i))\} \in D.$$

3. Axiomatizability

A class of L-structures K for a first order language L is *axiomatizable* or *elementary* if there is a set of sentences Π in L such that an L-structure A lies in K if and only if every sentence in Π is true in A.

Theorem 4. [9] *A class K of L-structures is axiomatizable if and only if it is closed under an elementary equivalence and ultraproducts.*

Let P be some property of an L-structure. An L-structure $\mathfrak{M}$ is called a *P-structure* if $\mathfrak{M}$ satisfies property P. A model $\mathfrak{M}$ of a theory T in a language L is called a *P-model* of T if $\mathfrak{M}$ is a P-structure. An L-structure $\mathfrak{M}$ is called a *pseudo-P-structure* if every sentence in L that is true in $\mathfrak{M}$ has a P-model. Note that a pseudofinite (T-pseudofinite) structure is a pseudo-P-structure where the property of P is the property of being a finite structure (a finite model of the theory T).

Theorem 5. *Let P be some property of L-structure, K be the class of all pseudo-P-structure in a language L. Then K is an axiomatizable class.*

Proof. Let $A \in K$, $A \equiv B$, and $B \models \Phi$, where Φ is a sentence of the language L. Then $A \models \Phi$. Since A is a pseudo-P-structure, there exists a P-structure C such that $C \models \Phi$. Therefore, B is a pseudo-P-structure, and the class K is closed under elementary equivalence.

Let $A_i \in K$ for all $i \in I$, D be a nonprincipal ultrafilter on I, $B = \prod_{i\in I} A_i/D$, and $B \models \Phi$, where Φ is a sentence of the language L. By Los's theorem, $J = \{i \in I \mid A_i \models \Phi\} \in D$, that is, $A_i \models \Phi$ for all $i \in J$. Let $i_0 \in J$. Since A_{i_0} is a pseudo-P-structure, there exists a P-structure C such that $C \models \Phi$. Therefore, B is a pseudo-P-structure, and the class K is closed under ultraproducts. Thus, by Theorem 4, K is the axiomatizable class. □

Corollary 1. *Let T be a theory of a language L, K be the class of all pseudofinite (T-pseudofinite) models of T. Then K is an axiomatizable class.*

4. Completeness

Let K be the class of L-structures for a first order language L. The class K is called *complete*, if the theory $\mathrm{Th}(K_\infty)$ of the infinite structures of this class is complete, that is, for every infinite $A, B \in K$ we have $A \equiv B$. The class K is called *model complete*, if the theory $\mathrm{Th}(K_\infty)$ of the infinite structures of this class is model complete, that is, for every infinite $A, B \in K$ such that $A \subseteq B$ we have $A \preceq B$. The class K of the structures is called *totally categorical*, if the theory $\mathrm{Th}(K)$ of the structures of this class is totally categorical, that is, every structures from K of the same cardinality are isomorphic.

To prove Theorem 7 we will need Lindström theorem.

Theorem 6. *(Lindström's theorem* [9]*) If T is $\forall\exists$-axiomatizable categorical in a countable cardinality consistent theory in a language L, then T is model complete.*

A congruence α on a non-singleton L-structure A is called *proper* if $\alpha \neq A^2$.

Theorem 7. *Let T be a theory of all S-acts, K be the class of all T-pseudofinite S-acts. Then the following properties are equivalent:*

(1) the class K is complete;

(2) the class K is model complete;

(3) the class K is totally categorical;

(4) every T-pseudofinite S-act is a coproduct of one-element S-acts;

(5) if α is a proper congruence of an S-act ${}_SS$, then the S-act ${}_SS/\alpha$ is infinite.

Proof. If $|S| = 1$ then Theorem is true. Let $|S| > 1$. By ${}_S\Theta$ we denote a one-element S-act.

Let us prove (1) $\Longrightarrow$ (5) and (2) $\Longrightarrow$ (5). Assume the contrary, i.e. there exists a congruence α of the S-act ${}_SS$ such that $1 < |S/\alpha| < \omega$. By Theorem

2, $\coprod_{i\in\omega} {}_S\Theta_i \in K$ and ${}_SS/\alpha \sqcup \coprod_{i\in\omega} {}_S\Theta_i \in K$. Since $\coprod_{i\in\omega} {}_S\Theta_i \subset {}_SS/\alpha \sqcup \coprod_{i\in\omega} {}_S\Theta_i$ and (1) or (2) are satisfied, then $\coprod_{i\in\omega} {}_S\Theta_i \equiv {}_SS/\alpha \sqcup \coprod_{i\in\omega} {}_S\Theta_i$. We have ${}_SS/\alpha \sqcup \coprod_{i\in\omega} {}_S\Theta_i \models \exists x \exists y(x = sy \wedge x \neq y)$, where s is some element of S. Then $\coprod_{i\in\omega} {}_S\Theta_i \models \exists x \exists y(x = sy \wedge x \neq y)$. A contradiction.

Let us prove (5) $\Longrightarrow$ (4). Assume the contrary, i.e. there exists an S-act ${}_SA \in K$ and $a \in A$ such that $|Sa| > 1$. Then ${}_SA \models \exists x \exists y(x = sy \wedge x \neq y)$ for some $s \in S$. Since ${}_SA \in K$ and K are a class of T-pseudofinite S-acts, we have ${}_SB \models \exists x \exists y(x = sy \wedge x \neq y)$, where ${}_SB$ is a finite S-act. Consequently, ${}_SB \models \exists x(x = sb \wedge x \neq b)$ for some $b \in B$. By α we denote the congruence ${}_SS$ such that ${}_SSb \cong {}_SS/\alpha$. Thus, there exists a congruence α of the S-act ${}_SS$ such that $1 < |S/\alpha| < \omega$. A contradiction.

Implication (4) $\Longrightarrow$ (3) is obvious.

Implication (3) $\Longrightarrow$ (1) follows from [9], proposition 5.6.1.

Let us prove (4) $\Longrightarrow$ (2). The class K is axiomatizable by the set of axioms $\{\forall x(sx = x) \mid s \in S\}$. Therefore, the class K is an $\forall\exists$-axiomatizable class. Moreover, by (3), it is a countable categorical class. Then, by Lindström's theorem, K is a model complete class. $\square$

5. Stability

Let T be a complete theory. For a given infinite cardinal κ, T is called *κ-stable* if for every set A of cardinality κ in a model of T, the set $S(A)$ of complete types over A also has cardinality κ. T is called *stable* if it is κ-stable for some infinite cardinal κ. T is called *unstable* if T is not stable.

Theorem 8. [14] *A complete theory is unstable if and only if there exists a formula $\Phi(\bar{x}, \bar{y})$ on $2n$ variables, a model $\mathcal{A}$ of the theory T, and $\bar{a}_i \in A^n$, $i \in \omega$, such that for any i, j, $i \neq j$,*

$$i < j \iff \mathcal{A} \models \Phi(\bar{a}_i, \bar{a}_j).$$

Everywhere below T is a theory of all pseudofinite S-acts. A monoid S is called a *stabilizer* (a *T-$\mathcal{PF}$-stabilizer*) if the theory of every S-act (T-pseudofinite S-act respectively) is stable. Note that any stabilizer is a T-$\mathcal{PF}$-stabilizer. An S-act ${}_SA$ is called *linearly ordered* if the set of all subacts of ${}_SA$ is linearly ordered by inclusion. A monoid S is called *linearly ordered* if the S-act ${}_SS$ is linearly ordered.

Theorem 9. [13] *A monoid S is a stabilizer if and only if S is a linearly ordered monoid.*

Lemma 1. *If S is a T-$\mathcal{PF}$-stabilizer then every cyclic finite S-act is linearly ordered.*

Proof. Suppose that S is T-$\mathcal{PF}$-stabilizer and there is a cyclic finite S-act ${}_SA = {}_SSa$ which is not linearly ordered, that is, there exist $b, c \in A$ such that $Sb \not\subseteq Sc \subseteq Sa$ and $Sc \not\subseteq Sb \subseteq Sa$. Then $b = ta$ and $c = sa$ for some $t, s \in S$. Let $K = \{\langle i, j\rangle \mid j \leqslant i < \omega\}$ and $\preccurlyeq$ be the lexicographic order on K. By ${}_SA_{ij}$ we denote a copy of ${}_SA$ ($\langle i, j\rangle \in K$) such that $A_{ij} \cap A_{kl} = \varnothing$ if $\langle i, j\rangle \neq \langle k, l\rangle$; by d_{ij} we denote a copy of an element $d \in A$ in A_{ij}. We will write ${}_SB_{ij}$ for an S-act $(\bigcup\limits_{\langle k,l\rangle \preccurlyeq \langle i,j\rangle} {}_SA_{kl})/\theta_{ij}$, where θ_{ij} is a congruence on $\bigcup\limits_{\langle k,l\rangle \preccurlyeq \langle i,j\rangle} {}_SA_{kl}$ generated by the set

$$\{\langle b_{uv}, b_{uw}\rangle \mid \langle u, v\rangle, \langle u, w\rangle \preccurlyeq \langle i, j\rangle\} \cup \{\langle c_{uv}, c_{wv}\rangle \mid \langle u, v\rangle, \langle w, v\rangle \preccurlyeq \langle i, j\rangle\}.$$

By ${}_SC$ we denote the S-act $\prod_{\langle i,j\rangle \in K} {}_SB_{ij}/D$ where D is a nonprincipal ultrafilter on K. Since ${}_SC$ is an ultraproduct of finite S-acts then by Theorem 2, ${}_SC$ is a T-pseudofinite S-act. Let $\bar{a}_{ij}/D$, $\bar{b}_i/D$, $\bar{c}_j/D$ be the elements of ${}_SC$ such that $\bar{a}_{ij}(\langle k, l\rangle) = a_{ij}/\theta_{kl}$, $\bar{b}_i(\langle k, l\rangle) = b_{ij}/\theta_{kl}$, $\bar{c}_j(\langle k, l\rangle) = c_{ij}/\theta_{kl}$ for all $\langle k, l\rangle \succcurlyeq \langle i, j\rangle$. Write $\varphi(x, y)$ to abbreviate the formula $\exists z(x = tz \wedge y = sz)$. Note that $\bar{b}_i/D = t\bar{a}_{ij}/D$ and $\bar{c}_j/D = s\bar{a}_{ij}/D$ for any $\langle i, j\rangle \in K$. Moreover,

$${}_SC \models \varphi(\bar{b}_i/D, \bar{c}_j/D) \Leftrightarrow i \geqslant j.$$

By Theorem 8, this contradicts the stability of $Th({}_SC)$.

Everywhere below T is th theory of all S-acts.

Theorem 10. *Let S be a finite monoid. Then the following conditions are equivalent:*

(1) S is a stabilizer;

(2) S is a T-$\mathcal{PF}$-stabilizer;

(3) S is a linearly ordered monoid.

Proof. The implication (1) $\Longrightarrow$ (2) is obvious. Since S is a finite monoid, the implication (2) $\Longrightarrow$ (3) follows from Lemma 1. The implication (3) $\Longrightarrow$ (1) follows from Theorem 9.

For a complete theory T of a language L, by $\mathfrak{M}$ we denote a rather large and rich model of the theory T, which we call a monster model, since we assume that all considered models of the theory T are its elementary submodels. A complete theory T of S-acts is called *stationary* [13] if for any ${}_SA \models T$ and $a, b \in \mathfrak{M} \setminus A$

$$a \in Sb \Longrightarrow A \cap Sa = A \cap Sb.$$

Theorem 11. [13] *Each stationary theory is stable.*

Proposition 1. *Let for any $s, t \in S$ there exist $r_1, \ldots, r_n \in S$ such that for any cyclic finite S-act ${}_SSa$ we have $sa = r_ita$ or $ta = r_isa$ for some $1 \leq i \leq n$. Then S is a T-$\mathcal{PF}$-stabilizer.*

Proof. Let the conditions of Proposition be hold and ${}_SA$ be T-pseudofinite S-act. By Theorem 11, it suffices to prove that $\mathrm{Th}({}_SA)$ is stationary. If ${}_SA$ is a finite S-act, then this is obviously true. Let $|A| \geq \omega$. Suppose ${}_SB \equiv {}_SA$, $a, b \in \mathfrak{M} \setminus B$, $a = sb$. It is clear that $B \cap Sa \subseteq B \cap Sb$. Let $c = tb \in B$. Note that ${}_SB$ is T-pseudofinite S-act. By Theorem 1, ${}_SB \equiv \prod_{i\in I} C_i/D$, where ${}_SC_i$ are the finite S-acts and D in nonprincipal ultrafilter on I. By the conditions of Proposition, for any $i \in I$ and $c_i \in C_i$ we have ${}_SSc_i \models \Phi(c_i)$, where

$$\Phi(x) \leftrightharpoons \bigvee_{j=1}^{n} (sx = r_j tx \vee tx = r_j sx).$$

Therefore, ${}_SC_i \models \forall x\Phi(x)$ for any $i \in I$. By Los's theorem, ${}_SB \models \forall x\Phi(x)$. Hence $\mathfrak{M} \models \forall x\Phi(x)$. In particular, there exists r_j, $1 \leq j \leq n$, such that $sb = r_j tb$ or $tb = r_j sb$. Since $c = tb \in B$ and $a = sb \notin B$, we have $c = r_j a$. Thus, $\mathrm{Th}({}_SA)$ is stationary, i.e. S is a T-$\mathcal{PF}$-stabilizer.

A monoid S is called *Noetherian,* respectively *Artinian,* if it satisfies the ascending chain condition on ideals, respectively if it satisfied the descending chain condition on ideals.

Theorem 12. *Let S be a commutative Noetherian and Artinian monoid. Then the following conditions are equivalent:*

(1) S is a stabilizer;

(2) S is a T-$\mathcal{PF}$-stabilizer;

(3) S is a linearly ordered monoid.

Proof. The implication (1) $\Longrightarrow$ (2) is obvious. The implication (3) $\Longrightarrow$ (1) follows from Theorem 9.

Let us prove the implication (2) $\Longrightarrow$ (3). Suppose that S is not a linearly ordered monoid. Then there are ideals Sa_0 and Sb_0 such that $Sa_0 \not\subseteq Sb_0$ and $Sb_0 \not\subseteq Sa_0$. Since S is an Noetherian monoid, there exists a maximal ideal $Sa_1 \supseteq Sa_0$ such that $b_0 \notin Sa_1$. Similarly, there exists a maximal ideal $Sb_1 \supseteq Sb_0$ such that $a_1 \notin Sb_1$. There exists a maximal ideal $Sa_2 \supseteq Sa_1$ such that $b_1 \notin Sa_2$. And so on. Thus, we have two ascending chains $Sa_0 \subseteq Sa_1 \subseteq \ldots$ and $Sb_0 \subseteq Sb_1 \subseteq \ldots$ of ideals of the monoid S. But S is an Noetherian monoid. Therefore, there exists $n \in \omega$ such that $Sa_i = Sa_{i+1}$ and $Sb_i = Sb_{i+1}$ for all $i \geq n$. Let $Sa = Sa_n$ and $Sb = Sb_n$. Since S is an Artinian monoid, there exists a minimal ideal Sc such that $Sa \subseteq Sc$ and $Sb \subseteq Sc$. We introduce the notation: $U_v = \{u \in S | Su = Sv\}$. Note that $Ssu_1 = Ssu_2$ for any $u_1, u_2 \in U_v$ and $s \in S$. The partition $Sc = U_a \cup U_b \cup U_c \cup U$ defines an equivalence relation θ on Sc, where $U = Sc \setminus (U_a \cup U_b \cup U_c)$. We prove that θ is a congruence of ${}_SSc$. In view of the above remark, it suffices to show that $su_1 \theta su_2$ for any $u_1, u_2 \in U$ and $s \in S$. Let $u \in U$ and $s \in S$. We prove that $su \in U$. Since $Su \subset Sc$ and $Ssu \subseteq Su$ then $Ssu \subset Sc$ and $su \notin U_c$. Let $su \in U_a$. Then $Ssu = Sa$.

Since $Su \neq Sa$, then $Sa \subset Su$. If $Sb \subset Su$, then due to the minimality of Sc we have $Su = Sc$, a contradiction. Consequently, $b \notin Su$. But Sa is a maximal ideal such that $b \notin Sa$. A contradiction. Similarly, $su \notin U_b$. Therefore, $su \in U$. Thus, θ is a congruence of ${}_SSc$. The S-act ${}_SSc/\theta$ is finite. Let us prove that ${}_SSc/\theta$ is not a linearly ordered S-act. Indeed, if, for example, $sa/\theta = b/\theta$, then $Sb = Ssa$ and $b \in Sa$, a contradiction. By Lemma 1, S is not a T-$\mathcal{PF}$-stabilizer. A contradiction.

Example 1. Let Q_1, Q_2 be disjoint copies of the set of rational numbers Q, $q_1 \in Q_1$, $q_2 \in Q_2$ be copies of the element $q \in Q$, $S = \{1\} \cup (Q_1 \cup Q_2)/\theta$, where $\theta = \{(0_1, 0_2), (0_2, 0_1)\} \cup \{(a, a) \mid a \in S\}$. We equip S with a binary operation defined as follows: $q_i/\theta \cdot q'_j/\theta = 0_1/\theta$, $q_i/\theta \cdot q'_i/\theta = (q \cdot q')_i/\theta$ for any $i, j \in \{1, 2\}$, $i \neq j$, $q, q' \in Q$, 1 being the identity element. Then S is a commutative Noetherian and Artinian monoid; S is not a stabilizer and is not a T-$\mathcal{PF}$-stabilizer.

Proof. Since $Sq_i = Q_i$ for any $i \in \{1, 2\}$ and $q \in Q \setminus \{0\}$, then S is a Noetherian and Artinian monoid. Since S is is not a linearly ordered monoid, then, by Theorem 12 S is not a stabilizer and is not a T-$\mathcal{PF}$-stabilizer.

Example 2. Let G is an infinite multiplicative group, $S = G \cup \{a_1, a_2\}$. Equip S with a binary operation defined as follows: a_1, a_2 are the right zeros of S; $a_1x = a_1$ and $a_2x = a_2$ for any $x \in G$. Then S is a Noetherian and Artinian non-commutative monoid; S is not a stabilizer and is not a T-$\mathcal{PF}$-stabilizer.

Proof. Since $Sg = S$ and $Sa_i = \{a_i\}$ for any $i \in \{1, 2\}$ and $g \in G$, then S is a Noetherian and Artinian monoid. Since S is not a linearly ordered monoid, then, by Theorem 9, S is not a stabilizer. Let θ be the congruence ${}_SS$ generated by G. Then ${}_SS/\theta$ is a cyclic finite non-linearly ordered S-act. Therefore, by Lemma 1, S is not a T-$\mathcal{PF}$-stabilizer.

Example 3. Let S be the multiplicative monoid ω of natural numbers. Then S is a commutative Noetherian monoid, not an Artinian monoid; S is not a stabilizer and is not a T-$\mathcal{PF}$-stabilizer.

Proof. Since S is not a linearly ordered monoid, then, by Theorem 9, S is not a stabilizer. The partition $S = \{1\} \cup \{2\} \cup \{3\} \cup (S \setminus \{1, 2, 3\})$ defines a congruence θ on ${}_SS$. The S-act ${}_SS/\theta$ is finite not a linerly ordered S-act. By Lemma 1 and is not a T-$\mathcal{PF}$-stabilizer.

Example 4. Let G be an infinite group with respect to the operation $\cdot$, with unit 1 and without nontrivial subgroups of finite index; let G_i be pairwise distinct copies of G, $g_i \in G_i$ be copies of $g \in G$, where $0 \leq i \leq m$, $m > 0$; and let $S = \bigcup_{0 \leq i \leq m} G_i$. Equip S with a binary operation defined as follows: for all $a, b \in G$ and $i, j \in \{0, \ldots, m\}$, $a_i b_j = (a \cdot b)_{max\{i,j\}}$. Then the monoid S is not a stabilizer, but is a T-$\mathcal{PF}$-stabilizer. Moreover, S is a non-Artinian and non-Noetherian monoid.

Proof. It is easy to check that S is a monoid with unit 1_0. Since $Sg_m = \{g_m\}$ then S is not a linearly ordered monoid. By Theorem 9, S is not a stabilizer. Let θ be any congruence on ${}_SS$ such that the S-act ${}_SS/\theta$ is finite. Then $1_0/\theta \cap G_0$ is a subgroup of G_0. Since G is a group without nontrivial subgroups of finite index then $1_0/\theta \cap G_0 = G_0$. If $a, b \in G$ then for all $i > 0$ we have $a_i = (1_i \cdot a_0)\ \theta\ (1_i \cdot b_0) = b_i$, that is, $a_i/\theta = b_i/\theta$. Therefore, $S/\theta = \{1_i/\theta \mid 0 \leq i \leq m\}$, $1_i/\theta\ 1_j/\theta = 1_{max\{i,j\}}/\theta$, and $S1_i/\theta = \{1_i/\theta, \ldots, 1_m/\theta\}$. Thus, the conditions of Proposition 1 are true and S is a T-$\mathcal{PF}$-stabilizer.

6. Conclusion

In this work we prove that for any theory T the class of all pseudofinite (T-pseudofinite) models of T is aximatizable (Corollary 1). We consider the issues of completeness and stability of the theory T of all S-acts. Namely, it is proved that for a theory T of all S-acts, completeness, model completeness, and total categoricity of the class of all T-pseudofinite S-acts are equivalent to the following condition: every T-pseudofinite S-act is a coproduct of one-element S-acts (Theorem 7). We also prove that for a finite monoid and for a commutative Noetherian and Artinian monoid S, the theory of every S-act (T-pseudofinite S-act) is stable iff S is a linearly ordered monoid (Theorem 10, Theorem 12). We construct a commutative (non-commutative) Noetherian and Artinian monoid S that is not a stabilizer and is not a T-$\mathcal{PF}$-stabilizer (Example 1, Example 2); a commutative Noetherian monoid, not an Artinian monoid that is not a stabilizer and is not a T-$\mathcal{PF}$-stabilizer (Example 3); a T-$\mathcal{PF}$-stabilizer, but not a stabilizer (Example 4).

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Поступила в редакцию / Received 24.11.2025
Поступила после рецензирования / Revised 14.01.2026
Принята к публикации / Accepted 22.01.2026