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Kernel Determination Inverse Problem in the Moore–Gibson–Thompson Equation

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Abstract: We investigate an inverse problem of determining the kernel function $g(t)$ in the third-order Moore–Gibson–Thompson (MGT) equation. First, the solution to the direct initial–boundary value problem for a homogeneous MGT equation is constructed using the Fourier spectral method. Then, based on this solution and the overdetermination condition, a Volterra-type integral equation is derived to study the inverse problem. We prove global existence and uniqueness results for the solutions of the considered problems. In addition, stability estimates for the solution of the direct and inverse problem are obtained.

Keywords: MGT equation, initial-boundary value problem, integral equation, inverse problem, Banach principle

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Научная статья

Обратная задача определения ядра в уравнении Мура – Гибсона – Томпсона

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Аннотация: Исследуется обратная задача определения функции ядра $g(t)$ в уравнении Мура – Гибсона – Томпсона (MGT) третьего порядка. Сначала с помощью спектрального метода Фурье строится решение прямой начально-краевой задачи для однородного уравнения МГТ. Затем на основе полученного решения и условия переопределения выводится интегральное уравнение вольтерровского типа для исследования обратной задачи. Доказываются теоремы глобального существования и единственности решений рассматриваемых задач. Кроме того, получены оценки устойчивости решений прямой и обратной задач.

Ключевые слова: уравнение МГТ, начально-краевая задача, интегральное уравнение, обратная задача, принцип Банаха

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1. Introduction

Inverse problems for integro-differential equations are some of the most important mathematical problems in science and engineering. They have wide application in computed tomography, biology, mineral exploration, acoustics, communication theory, signal processing, medical imaging and other fields. Fundamentals of the theory of inverse problems for mathematical physics equations were developed in the works [12; 17].

Inverse convolution kernel determination problems for linear parabolic and hyperbolic integro-differential equations of the second order were considered in a number of papers (see, [5; 7; 20]). In [5; 20] the authors have proved the local existence and global uniqueness of identification problems, which focus on recovering unknown convolution kernels in hyperbolic integro-differential equations. In [7] it is studied the global solvability of two kernel determination problems in the integro-differential equation of rigid heat conductor. For close inverse problems related to other models, we refer to [2; 3; 6; 8; 9; 19; 21]. In [3], the authors studied the correctness of inverse problem for the multidimensional Moore-Gibson-Thompson equation. Using the Fourier method, the theorem of unique solvability of the generalized solution in the specified class of integrable functions is proved.

The MGT equation is derived from the modeling of high frequency ultrasound waves. It is the third order time dynamics that takes into account the finite speed of thermal signals propagation. In a more general case, it has the form:

$$\tau u_{ttt} + \alpha_1 u_{tt} + \alpha_2 Au + \alpha_3 Au_t - \int_0^t g(t-s)A\omega(x,s)ds = 0, \tag{1.1}$$

where τ, α_1, α_2, α_3 are physical parameters, the convolution kernel g exhibiting *memory effect* and A is a positive self-adjoint operator on a Hilbert space H. The form of ω classifies the memory into the following three types $u(x,s)$, $u_t(x,s)$, $\lambda u(x,s) + u_t(x,s)$. In the work [16], it is considered for the derivation of the MGT equation and the list of references therein. These equations are the subject of several recent research due to their wide range of applications in the medical and industrial fields of high intensity ultrasound waves.

The linearized MGT equation is in abstract form and it has a simple prototype where $A = -\Delta$. In [14], the well-posedness of linearized MGT equation and the uniform decay of its energy is studied under proper functional setting and initial-boundary conditions.

The inverse problem for MGT (1.1) with $A = -\Delta$, $g = 0$, $\tau = 1$, α_i, $i = 2,3$ are constants, but α_1 being a function of x was treated in [18], who proved uniqueness of determining the coefficient $\alpha_1(x) - \frac{\alpha_3}{\alpha_2^2}$ and also showed stability by Carleman estimates.

The main equation considered in this work (1.1) contains an integral term of convolution type. The Cauchy problems for the most general equations with an integral operator of convolution type in Banach spaces were studied in [11]. A new approach to constructing generalized solutions of the equations is proposed.

In the works [1,4], the authors investigate the inverse problems of determining kernel function $g(t)$ and three unknown parameters by using the Laplace transform. The research method in these papers is not universal, but uses the fact that the direct problem is considered with the Neumann boundary conditions.

In this paper, we propose a method for studying the determination of the convolution kernel in MGT equation using the Fourier spectral method and the method of integral equations.

We consider the MGT equation (1.1), when $\tau = \alpha_1 = \alpha_2 = \alpha_3 = 1$ and $A = -\Delta$:

$$u_{ttt} + u_{tt} - \Delta u_t - \Delta u + \int_0^t g(t-\tau)\Delta u(x,\tau)d\tau = 0, \tag{1.2}$$

in the domain $D = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^n$, where, $T > 0$ is arbitrary fixeds number, with initial

$$u(x,0) = a(x),\ u_t(x,0) = b(x),\ u_{tt}(x,0) = c(x),\ x \in \overline{\Omega}, \tag{1.3}$$

and boundary conditions

$$u = 0, \qquad \partial\Omega \times (0,T). \tag{1.4}$$

We use the notation Y_T to denote the following spaces:

$$Y_T := \left\{u | u_{ttt}, \Delta u, \Delta u_t \in C(D),\ \text{and}\ u, u_t, u_{tt} \in C(\overline{D})\right\}.$$

The problem of determining a function $u(x,t) \in Y_T$ that satisfies (1.2)–(1.4) with known functions $a(x)$, $b(x)$, $c(x)$ and $g(t)$ will be called the *direct problem.*

In the *inverse problem,* it is required to determine the function $g(t)$ using overdetermination conditions about the solution of the direct problem (1.2)–(1.4):

$$u(x_0,t) = h(t),\ x_0 \in \Omega. \tag{1.5}$$

Throughout this article, the functions $a(x)$, $b(x)$, $c(x)$ and $h(t)$ are assumed to satisfy the following conditions:
(K1): $a(x) \in C^{\left[\frac{n}{2}\right]+5}(\overline{\Omega})$, and $a = \Delta a = ... = \Delta^{\left[\frac{n+8}{4}\right]}a = 0$ on $\partial\Omega$;
(K2): $b(x) \in C^{\left[\frac{n}{2}\right]+4}(\overline{\Omega})$, and $b = \Delta b = ... = \Delta^{\left[\frac{n+6}{4}\right]}b = 0$ on $\partial\Omega$;
(K3): $c(x) \in C^{\left[\frac{n}{2}\right]+3}(\overline{\Omega})$, and $c = \Delta c = ... = \Delta^{\left[\frac{n+4}{4}\right]}c = 0$ on $\partial\Omega$;
(K4): $h(t) \in C^4[0,T]$, $(\Delta a)(x_0) \neq 0$.

The rest of this paper is organized as follows. In Section 2, we present our investigations on the direct problem (1.2)–(1.4). We find differential properties and stability estimates for the solution of this problem. Section 3 devoted to the formulation and proof of the main results of this work, which are the theorems of global existence and uniqueness and conditional stability of inverse problem solution. Section 4 contains the conclusion.

2. Investigation of direct problem

In this section, we will study the problem (1.2)–(1.4). We shall prove the existence and uniqiness of a classic solution of problem.

It is known that the spectral problem

$$\begin{cases} \Delta X + \lambda X = 0, & \text{in } \Omega, \\ X = 0, & \text{on } \partial\Omega, \end{cases}$$

corresponding to problem (1.2)–(1.4) has the following properties (see [10, pp. 354-355],):

1) Every eigenvalue of $-\Delta$ is real and positive;

2) If each eigenvalue is repeated according to its (finite) multiplicity, then the sequence $\{\lambda_k\}_{k=1}^{\infty}$ satisfies

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots, \text{ and } \lim_{k\to\infty} \lambda_k = \infty;$$

3) There exists an orthonormal basis $\{X_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$ consisting of eigenfunctions of $-\Delta$.

The solution to problem (1.2)–(1.4) is sought in the form:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) X_k(x), \tag{2.1}$$

where $u_k(t) = \big(u(\cdot,t), X_k\big)_{L^2(\Omega)}$.

Applying the method of separation of variables to determine the desired coefficients $u_k(t)$, $k = 1, 2, ...$ of the function $u(x,t)$ from (1.2)–(1.4), we get

$$u_k'''(t) + u_k''(t) + \lambda_k u_k'(t) + \lambda_k u_k(t) = \lambda_k \int\limits_0^t g(t-\tau) u_k(\tau) d\tau. \tag{2.2}$$

From the (1.3) initial conditions, we have

$$A_k := u_k(0) = \big(a, X_k\big)_{L^2(\Omega)}, \quad B_k := u_k'(0) = \big(b, X_k\big)_{L^2(\Omega)}, \tag{2.3}$$

$$C_k := u_k''(0) = \big(c, X_k\big)_{L^2(\Omega)}. \tag{2.4}$$

Let us consider (2.2) as a third-order ordinary differential equation with the right-hand side of the form $\lambda_k \int\limits_0^t g(t-\tau) u_k(\tau) d\tau$ with the Cauchy data (2.3), (2.4). Solving this problem for each fixed $k \geq 1$, it is easy to conclude that it is equivalent to the following Volterra-type integral equation:

$$u_k(t) = \frac{1}{\lambda_k + 1}\left[\lambda_k e^{-t} + \sqrt{\lambda_k}\sin\left(\sqrt{\lambda_k} t\right) + \cos\left(\sqrt{\lambda_k} t\right)\right] A_k + \frac{F_k(t)}{\lambda_k + 1} C_k$$
$$+ \frac{1}{\sqrt{\lambda_k}} \sin\left(\sqrt{\lambda_k} t\right) B_k - \frac{\lambda_k}{\lambda_k + 1} \int_0^t F_k(t-\tau) \int_0^\tau g(\tau - s) u_k(s) ds d\tau, \tag{2.5}$$

where $F_k(t) = e^{-t} + \frac{1}{\sqrt{\lambda_k}} \sin\left(\sqrt{\lambda_k}t\right) - \cos\left(\sqrt{\lambda_k}t\right)$.

Further tabs in this section will be devoted to obtaining sufficient conditions for given functions $a(x)$, $b(x)$, $c(x)$ and kernel $g(t)$, under which the integral equation (2.5) has a unique classical solution.

Estimating functions $u_k(t)$, for $t \in [0, T]$, we get the following integral inequalities

$$|u_k(t)| \leq \left(1 + \frac{1}{\sqrt{\lambda_k}}\right) |A_k| + \frac{1}{\sqrt{\lambda_k}} |B_k| + \frac{2}{\lambda_k} |C_k| + 2\|g\| \int_0^t (t-s)|u_k(s)|ds,$$

where $\|g\| = \max\limits_{0 \leq t \leq T} |g(t)|$. Applying the Gronwall lemma, we obtain

$$|u_k(t)| \leq \left[\left(1 + \frac{1}{\sqrt{\lambda_k}}\right) |A_k| + \frac{1}{\sqrt{\lambda_k}} |B_k| + \frac{2}{\lambda_k} |C_k|\right] e^{2\|g\|T^2}. \quad (2.6)$$

Calculating the first derivative of (2.5), we have

$$u_k'(t) = \frac{1}{\lambda_k + 1} \left[-\lambda_k e^{-t} + \lambda_k \cos\left(\sqrt{\lambda_k}t\right) - \sqrt{\lambda_k} \sin\left(\sqrt{\lambda_k}t\right)\right] A_k + \frac{F_k'(t)}{\lambda_k + 1} C_k$$
$$+ \cos\left(\sqrt{\lambda_k}t\right) B_k - \frac{\lambda_k}{\lambda_k + 1} \int_0^t F_k'(t-\tau) \int_0^\tau g(\tau - s) u_k(s) ds d\tau. \quad (2.7)$$

Using (2.6) and (2.7), we find an estimate for $u_k'(t)$:

$$\left|u_k'(t)\right| \leq 2|A_k| + |B_k| + \left(\frac{1}{\lambda_k} + \frac{1}{\sqrt{\lambda_k}}\right) |C_k|$$
$$+ \|g\|T^2 \left[\left(1 + \frac{1}{\sqrt{\lambda_k}}\right) |A_k| + \frac{1}{\sqrt{\lambda_k}} |B_k| + \frac{2}{\lambda_k} |C_k|\right] e^{2\|g\|T^2}. \quad (2.8)$$

Calculating now the second derivative of (2.5) and taking into account (2.6) and (2.8), we obtain an estimate for $u_k''(t)$:

$$\left|u_k''(t)\right| \leq \left(1 + \sqrt{\lambda_k}\right) |A_k| + \sqrt{\lambda_k} |B_k| + \left(1 + \frac{1}{\lambda_k}\right) |C_k|$$
$$+ \|g\|T^2 \left[\left(1 + \frac{1}{\sqrt{\lambda_k}}\right) |A_k| + \frac{1}{\sqrt{\lambda_k}} |B_k| + \frac{2}{\lambda_k} |C_k|\right] e^{2\|g\|T^2}. \quad (2.9)$$

Estimate for $u_k'''(t)$ is obtained by the equation (2.5) using (2.6)–(2.9) and it has the form

$$\left|u_k'''(t)\right| \leq (1 + \lambda_k) |A_k| + \lambda_k |B_k| + \left(\frac{1}{\lambda_k} + \sqrt{\lambda_k}\right) |C_k|$$
$$+ \lambda_k \|g\| T \left(1 + \left(\frac{1}{\lambda_k} + \sqrt{\lambda_k}\right) T\right)$$
$$\times \left[\left(1 + \frac{1}{\sqrt{\lambda_k}}\right) |A_k| + \frac{|B_k|}{\sqrt{\lambda_k}} + \frac{2}{\lambda_k} |C_k|\right] e^{2\|g\|T^2}. \quad (2.10)$$

Now, using inequalities (2.6)–(2.10), we focus on the existence of a unique solution to problem (1.2)–(1.5).

Formally termwise differentiating the series in formula (2.1), we get the following series:

$$u_{tt}=\sum_{k=1}^{\infty}u_k''(t)X_k(x),\ \ \Delta u=-\sum_{k=1}^{\infty}\lambda_k u_k(t)X_k(x), \tag{2.11}$$

$$u_{ttt}=\sum_{k=1}^{\infty}u_k'''(t)X_k(x),\ \ \Delta u_t=-\sum_{k=1}^{\infty}\lambda_k u_k'(t)X_k(x). \tag{2.12}$$

Obviously, if we show that series (2.12) are uniform convergence, then series (2.1) and (2.11) are also uniform convergence. Hence, we obtain the conditions for the series (2.12) to be uniform convergents.

From (2.12), we get the following estimaties

$$\begin{aligned}
&\left|\sum_{k=1}^{\infty}\lambda_k u_k' X_k(x)\right| \\
&\quad\le C_1\sum_{k=1}^{\infty}\lambda_k\left[|A_k|+|B_k|+\frac{|C_k|}{\sqrt{\lambda_k}}\right]\left(1+T^2\|g\|e^{2\|g\|T^2}\right)|X_k(x)| \\
&\quad= C_2\sum_{k=1}^{\infty}|X_k(x)|\lambda_k^{-\frac{1}{2}\left(\left[\frac{n}{2}\right]+1\right)}\left[|A_k|\lambda_k^{\frac{1}{2}\left(\left[\frac{n}{2}\right]+3\right)}+|B_k|\lambda_k^{\frac{1}{2}\left(\left[\frac{n}{2}\right]+3\right)}\right] \\
&\qquad+C_2\sum_{k=1}^{\infty}\lambda_k^{-\frac{1}{2}\left(\left[\frac{n}{2}\right]+1\right)}|X_k(x)|\cdot|C_k|\lambda_k^{\frac{1}{2}\left(\left[\frac{n}{2}\right]+2\right)} \\
&\qquad\le C_2\left(\sum_{k=1}^{\infty}\lambda_k^{-\left(\left[\frac{n}{2}\right]+1\right)}|X_k(x)|^2\right)^{\frac{1}{2}}\times \\
&\left[\left(\sum_{k=1}^{\infty}\lambda_k^{\left[\frac{n}{2}\right]+3}|A_k|^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\lambda_k^{\left[\frac{n}{2}\right]+3}|B_k|^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\lambda_k^{\left[\frac{n}{2}\right]+2}|C_k|^2\right)^{\frac{1}{2}}\right],
\end{aligned}$$

where C_1 is positive number and $C_2=C_1\left(1+T^2\|g\|e^{2\|g\|T^2}\right)$. According to Lemma 1 in [13], the first series on the right-hand side of this relation converges. Indeed, let conditions (K1)-(K3) hold. Then, according to Bessel's inequality, we have:

$$\left|\sum_{k=1}^{\infty}\lambda_k u_k'(t)X_k(x)\right|\le C\left[\|\Delta^{\varepsilon_0+1}a\|_{L^2(\Omega)}+\|\Delta^{\varepsilon_0+1}b\|_{L^2(\Omega)}+\|\Delta^{\varepsilon_0+1/2}c\|_{L^2(\Omega)}\right],$$

where $\varepsilon_0=\frac{1}{2}\left[\frac{n}{2}\right]+\frac{1}{2}$. The proved assertion implies the next theorem.

Theorem 1. *Let $a(x)$, $b(x)$, $c(x)$ satisfies conditions of (K1)-(K3) and $g(t) \in C[0,T]$, then problem (1.2)–(1.4) has a unique solution, which is defined by series (2.1) and belongs to class $u(x,t) \in Y_T$.*

Let $\widetilde{u}_k$ be solution of the integral equation (2.5) corresponding to the functions $\widetilde{A}_k$, $\widetilde{B}_k$, $\widetilde{C}_k$, and $\widetilde{g}$. Let us derive an estimate for the norm of the difference between the solution of the original integral equations (2.5) and the solution of this equation with perturbed functions $\widetilde{A}_k$, $\widetilde{B}_k$, $\widetilde{C}_k$, and $\widetilde{g}$. After making obvious estimates, we have

$$|\widehat{u}_k| \le \left[\left(1+\frac{1}{\sqrt{\lambda_k}}\right)\left|\widehat{A}_k\right| + \frac{1}{\sqrt{\lambda_k}}\left|\widehat{B}_k\right| + \frac{2}{\lambda_k}\left|\widehat{C}_k\right|\right] e^{2\|\widetilde{g}\|T^2} + \left[2\,|A_k| + \frac{1}{\sqrt{\lambda_k}}\,|B_k| + \frac{2}{\lambda_k}\,|C_k|\right] T^2 e^{2(\|g\|+\|\widetilde{g}\|)T^2}\|\widehat{g}\|, \tag{2.13}$$

$$\left|\widehat{u}_k'\right| \le 2\left|\widehat{A}_k\right| + \left|\widehat{B}_k\right| + \frac{2}{\sqrt{\lambda_k}}\left|\widehat{C}_k\right| + \left[\sqrt{\lambda_k}\,|A_k| + |B_k| + \frac{|C_k|}{\sqrt{\lambda_k}}\right] T^2 e^{2\|g\|T^2}\,\|\widehat{g}\| + \left[\sqrt{\lambda_k+1}\left|\widehat{A}_k\right| + \left|\widehat{B}_k\right| + \frac{3}{\sqrt{\lambda_k}}\left|\widehat{C}_k\right|\right] T^2 e^{2\|g\|T^2}\|\widetilde{g}\| + \left[\sqrt{\lambda_k+1}\,|A_k| + |B_k| + \frac{4}{\sqrt{\lambda_k}}\,|C_k|\right] T^4 e^{2(\|g\|+\|\widetilde{g}\|)T^2}\|\widetilde{g}\|\|\widehat{g}\|, \tag{2.14}$$

where $\widehat{u}_k = u_k - \widetilde{u}_k$, $\widehat{A}_k = A_k - \widetilde{A}_k$, $\widehat{B}_k = B_k - \widetilde{B}_k$, $\widehat{C}_k = C_k - \widetilde{C}_k$, $\widehat{g}_k = g_k - \widetilde{g}_k$. Indeed, the expression (2.13) is stability estimate for the solution to the integral equation (2.5). The uniqueness of this solution follows from (2.13).

As required for the analysis of the inverse problem, we show that the function $u(x,t)$ has a fourth-order derivative with respect to t. Differentiating (2.5) four times and using (2.1), we obtain:

$$u_{tttt}(x,t) = \sum_{k=1}^{\infty} u_k^{(IV)}(t)X_k(x) = \sum_{k=1}^{\infty}\left(\lambda_k^{3/2}\sin\left(\sqrt{\lambda_k}t\right)B_k - g(t)\lambda_k A_k\right)X_k(x) + \sum_{k=1}^{\infty}\frac{\lambda_k}{\lambda_k+1}\left(e^{-t} + \lambda_k^{3/2}\sin\left(\sqrt{\lambda_k}t\right) + \lambda_k\cos\left(\sqrt{\lambda_k}t\right)\right)A_kX_k(x) + \sum_{k=1}^{\infty}\frac{1}{\lambda_k+1}F_k^{(IV)}(t)C_kX_k(x) + \int_0^t g(t-s)\sum_{k=1}^{\infty}\lambda_k\big(u_k(s)-u_k'(s)\big)X_k(x)\,ds - \int_0^t\int_0^\tau g(\tau-s)\sum_{k=1}^{\infty}\frac{\lambda_k}{\lambda_k+1}F_k^{(IV)}(t-\tau)u_k(s)X_k(x)\,ds\,d\tau. \tag{2.15}$$

From (2.15), we get

$$\left|\sum_{k=1}^{\infty} u_k^{(IV)}(t)X_k(x)\right| \le C_3\sum_{k=1}^{\infty}\left(\lambda_k^2|A_k| + \lambda_k^{3/2}|B_k| + \lambda_k|C_k|\right)|X_k(x)|.$$

where C_3 is positive number depending on $\|g\|$ and T. Using conditions (K1)–(K3), Lemma 1 in [13], and Bessel's inequality, we obtain the following estimate:

$$\left|\sum_{k=1}^{\infty} u_k^{(IV)}(t)X_k(x)\right| \leq C\left[\|\Delta^{\varepsilon_0+2}a\|_{L^2(\Omega)}+\|\Delta^{\varepsilon_0+3/2}b\|_{L^2(\Omega)}+\|\Delta^{\varepsilon_0+1}c\|_{L^2(\Omega)}\right].$$

Therefore, the relation $u_{tttt} \in C(\overline{D})$ is valid.

3. The existence and uniqueness theorem for the inverse problem

Now, to study the main problem (1.2)–(1.5), we consider the following auxiliary inverse initial and boundary value problem.

Lemma 1. *Let (K1)-(K4) be held and that the following compatibility conditions are fulfilled*

$$\begin{aligned} a(x_0) = h(0),\ b(x_0) = h'(0),\ c(x_0) = h''(0),\\ (\Delta a)(x_0) + (\Delta b)(x_0) - c(x_0) = h'''(0). \end{aligned} \tag{3.1}$$

Then the problem of finding a solution to (1.2)–(1.5) is equivalent to the problem of determining functions $u(x,t) \in Y_T$, $g(t) \in C[0,T]$ satisfying (1.2)–(1.4) and

$$h^{(IV)}(t) = \sum_{k=1}^{\infty} u_k^{(IV)}(t)X_k(x_0),\ t \in [0,T]. \tag{3.2}$$

Remark 1. By Lemma 1, we know that problem (1.2)–(1.4), (3.2) is an equivalent form of the original inverse problem (1.2)–(1.5). Therefore, in the following sections, we will discuss problem (1.2)–(1.4), (3.2), rather than the original one.

Proof. The solution $\{u(x,t), g(t)\}$ of our inverse problem (1.2)–(1.5) is also a solution to problem (1.2)–(1.4) in $Y_T \times C[0,T]$. Therefore, we should show only (3.2). In Section 2, we proved that the solution to the direct problem (1.2)–(1.4) is given in the form of the series (2.1). Therefore, set $x = x_0$ in (2.1), the procedure yields

$$u(x_0,t) = \sum_{k=1}^{\infty} u_k(t)X_k(x_0), \quad t \in [0,T]. \tag{3.3}$$

Using equality (3.3) and condition (1.5), we derive the following relation:

$$h(t) = \sum_{k=1}^{\infty} u_k(t)X_k(x_0), \quad t \in [0,T]. \tag{3.4}$$

By (K1)-(K4), differentiating the equality (3.3) four times, we obtain the equality (3.2).

Now, assume that $\{u, g\}$ satisfies (1.2)–(1.4), (3.2). To prove that $\{u, g\}$ is a solution to the inverse problem (1.2)–(1.5), it suffices to show that $\{u, g\}$ satisfies (1.5).

Differentiating (2.1) four times, we obtain (2.15). Setting $x = x_0$ in (2.15), we get

$$u_{tttt}(x_0, t) = \sum_{k=1}^{\infty} u_k^{(IV)}(t) X_k(x_0), \ t \in [0, T]. \tag{3.5}$$

Then subtracting (3.2) into (3.5), and using (3.1), we have that $y(t) = u(x_0, t) - h(t)$ satisfies

$$\begin{cases} y^{(IV)}(t) = 0, \quad t > 0, \\ y(0) = y'(0) = y''(0) = y'''(0) = 0. \end{cases} \tag{3.6}$$

The problem (3.6) has only a trivial solution. Then, $y(t) \equiv 0$, $0 \le t \le T$ i.e., the condition (1.5) is satisfied. □

We rewrite equality (3.2) in the following form:

$$\begin{aligned} g(t) = {} & \frac{1}{A_0} h^{(IV)}(t) + \frac{1}{A_0} \sum_{k=1}^{\infty} \frac{1}{\lambda_k + 1} (\lambda_k \sqrt{\lambda_k} \, (\lambda_k (A_k + B_k) + B_k + C_k) \, X_k(x_0) \\ & + \frac{1}{A_0} \sum_{k=1}^{\infty} \frac{1}{\lambda_k + 1} \left[(\lambda_k A_k + C_k) e^{-t} + \lambda_n^2 (A_n - C_k) \cos \sqrt{\lambda_k} t \right] X_k(x_0) \\ & + \frac{1}{A_0} \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k + 1} X_k(x_0) \int_0^t g(\tau) \left[u_k'(t - \tau) - u_k(t - \tau) \right] d\tau \\ & - \frac{1}{A_0} \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k + 1} X_k(x_0) \int_0^t \int_0^{\tau} u_k(s) g(\tau - s) F_k^{(IV)}(t - \tau) ds d\tau, \end{aligned} \tag{3.7}$$

where $A_0 = -\sum\limits_{k=1}^{\infty} \lambda_k A_k X_k(x_0) = -(\Delta a)(x_0)$.

According to the introduced notation for the vector function $\vartheta(x, t) = (\vartheta_1, \vartheta_2, \vartheta_3) = (u(x, t), u_t(x, t), g(t))$, we write this system in the operator form

$$\vartheta = \Lambda \vartheta, \tag{3.8}$$

where $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ is defined the following equalities:

$$\Lambda_1 \vartheta = \vartheta_1^0(x, t) - \sum_{k=1}^{\infty} \frac{\lambda_k X_k(x)}{\lambda_k + 1} \int_0^t F_k(t - \tau) \int_0^{\tau} \vartheta_{1k}(s) \vartheta_3(\tau - s) ds d\tau, \tag{3.9}$$

$$\Lambda_2\vartheta = \vartheta_2^0(x,t) - \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k+1} A_k X_k(x) \int_0^t \vartheta_3(\tau) F_k(t-\tau) d\tau$$

$$- \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k+1} X_k(x) \int_0^t F_k(t-\tau) \int_0^\tau \vartheta_{2k}(s)\vartheta_3(\tau-s) ds d\tau, \qquad (3.10)$$

$$\Lambda_3\vartheta = \vartheta_3^0(t) - \frac{1}{A_0} \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k+1} X_k(x_0) \int_0^t \vartheta_3(\tau) \left[\vartheta_{2k} - \vartheta_{1k}\right](t-\tau) d\tau$$

$$- \frac{1}{A_0} \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k+1} X_k(x_0) \int_0^t \int_0^\tau \vartheta_{1k}(s)\vartheta_3(\tau-s) F_k^{(IV)}(t-\tau) ds d\tau. \qquad (3.11)$$

The following notations were introduced in the equalities (3.9)–(3.11):

$$\vartheta_1^0(x,t) = \sum_{k=1}^{\infty} \frac{X_k(x)}{\lambda_k+1} \left[\lambda_k e^{-t} + \sqrt{\lambda_k} \sin\left(\sqrt{\lambda_k} t\right) + \cos\left(\sqrt{\lambda_k} t\right)\right] A_k$$

$$+ \sum_{k=1}^{\infty} \frac{X_k(x)}{\sqrt{\lambda_k}} \left[\sin\left(\sqrt{\lambda_k} t\right) B_k + \frac{1}{\lambda_k+1} F_k C_k\right],$$

$$\vartheta_2^0(x,t) = \sum_{k=1}^{\infty} \frac{X_k(x)}{\lambda_k+1} \left[-\lambda_k e^{-t} + \lambda_k \cos\left(\sqrt{\lambda_k} t\right) - \sqrt{\lambda_k} \sin\left(\sqrt{\lambda_k} t\right)\right] A_k$$

$$+ \sum_{k=1}^{\infty} \frac{X_k(x)}{\sqrt{\lambda_k}} \left[\sqrt{\lambda_k} \cos\left(\sqrt{\lambda_k} t\right) B_k + \frac{1}{\lambda_k+1} F_k' C_k\right],$$

$$\vartheta_3^0(t) = \frac{1}{A_0} h^{(IV)}(t) - \frac{1}{A_0} \sum_{k=1}^{\infty} \frac{\lambda_k\sqrt{\lambda_k}}{\lambda_k+1} \left(\lambda_k(A_k+B_k) + B_k + C_k\right) \sin\sqrt{\lambda_k} t X_k(x_0)$$

$$- \frac{1}{A_0} \sum_{k=1}^{\infty} \frac{1}{\lambda_k+1} \left[(\lambda_k A_k + C_k) e^{-t} + \lambda_k^2 (A_k - C_k) \cos\sqrt{\lambda_k} t\right] X_k(x_0).$$

The main result of this work is the following theorem:

Theorem 2. *Let the conditions of Lemma 1 are satisfied. Then the operator equations (3.8) has the unique solution in domain D for arbitrary fixed $T > 0$.*

Proof. Let $C_\sigma(\overline{D})$, $(\sigma \geq 0)$ be the Banach space of continuous functions with the ordinary norm, denoted by $\|\cdot\|_\sigma$,

$$\|\vartheta\|_\sigma = \max\left[\sup_{(x,t)\in\overline{D}} \left|\vartheta_1(x,t)e^{-\sigma t}\right|, \sup_{(x,t)\in\overline{D}} \left|\vartheta_2(x,t)e^{-\sigma t}\right|, \sup_{t\in[0,T]} \left|\vartheta(t)e^{-\sigma t}\right|\right].$$

C_σ with $\sigma = 0$ is the usual space of continuous functions.

Note that the weight norm $\|\cdot\|_\sigma$ is equivalent to the usual norm $\|\cdot\|$:

$$\|\cdot\|_\sigma \le \|\cdot\| \le \|\cdot\|_\sigma e^{\sigma T}. \quad \sigma > 0$$

The convolution operator is commutative and invariant with respect to multiplication by $e^{-\sigma t}$:

$$e^{-\sigma t}\left(k_1 * k_2\right)(t) = \left(e^{-\sigma t}k_1(t)\right) * \left(e^{-\sigma t}k_2(t)\right).$$

The last formula implies the estimation $\|k_1 * k_2\|_\sigma \le \|k_1\|_\sigma \|k_2\|_\sigma T$. Then, we have

$$\|k_1 * k_2\|_\sigma \le \frac{1}{\sigma}\|k_1\|\|k_2\|_\sigma \le \frac{1}{\sigma}\|k_1\|\|k_2\|, \ \sigma > 0, \tag{3.12}$$

using the result of [20].

In the space $C_\sigma[0,T]$, by $B(\vartheta^0,\rho)$ we denote the ball with the center ϑ^0 and radius ρ, i.e.

$$B(\vartheta^0,\rho) := \left\{\vartheta(x,t): \ \|\vartheta - \vartheta^0\|_\sigma \le \rho\right\}.$$

The fact that the function $\vartheta(x,t)$ belongs to the ball $B(\vartheta^0,\rho)$ implies the inequality

$$\|\vartheta\|_\sigma \le \|\vartheta_0\| + \rho. \tag{3.13}$$

We introduce the following notation:

$$\|a\|_{C^{[\frac{n}{2}]+5}(\overline{\Omega})} = a_0; \ \|b\|_{C^{[\frac{n}{2}]+4}(\overline{\Omega})} = b_0; \ \|c\|_{C^{[\frac{n}{2}]+3}(\overline{\Omega})} = c_0.$$

Let us verify the first condition of a fixed point argument. After making obvious estimates from (3.9)–(3.11) based on (3.13) for $\vartheta_i \in B(\vartheta^0,\rho)$, $i = 1,2,3$, we get

$$\|\Lambda_1\vartheta - \vartheta_1^0\|_\sigma = \sup_{(x,t)\in\overline{D}}\left|\sum_{k=1}^{\infty} X_k \int_0^t e^{-\sigma t}F_k(t-\tau)\int_0^\tau \vartheta_{1k}(s)\vartheta_3(\tau-s)ds d\tau\right|$$

$$\le \frac{2}{\sigma}\|\vartheta_3\|_\sigma \sup_{(x,t)\in\overline{D}}\left|\sum_{k=1}^{\infty} X_k(x)\|\vartheta_{1k}\|\right| \le \frac{M_1}{\sigma}\|\vartheta_3\|_\sigma\left[a_0 + b_0 + c_0\right]e^{2\|\vartheta_3\|T^2}$$

$$\le \frac{M_1}{\sigma}\left(\rho + \|\vartheta^0\|\right)\left[a_0 + b_0 + c_0\right]e^{2\left(\rho + \|\vartheta^0\|\right)T^2} := \frac{\sigma_1}{\sigma}.$$

If we choose σ as $\sigma \ge (1/\rho)\sigma_1$, then it satisfied the first condition of contractive mapping for Λ_1.

Now, we carry out the estimations for Λ_2 :

$$\|\Lambda_2\vartheta - \vartheta_2^0\|_\sigma = \sup_{(x,t)\in\overline{D}}\left|\sum_{k=1}^{\infty}\frac{\lambda_k e^{-\sigma t}}{\lambda_k + 1}A_k X_k(x)\int_0^t \vartheta_3(\tau)F_k(t-\tau)d\tau\right|$$

$$+\sup_{(x,t)\in\overline{D}}\left|\sum_{k=1}^{\infty}\frac{\lambda_k e^{-\sigma t}}{\lambda_k+1}X_k(x)\int_0^t F_k(t-\tau)\int_0^\tau \vartheta_{2k}(s)\vartheta_3(\tau-s)dsd\tau\right|$$

$$\leq \frac{M_2}{\sigma}\left(\rho+\|\vartheta^0\|\right)\left(1+(\rho+\|\vartheta^0\|)e^{2\left(\rho+\|\vartheta^0\|\right)T^2}+e^{2\left(\rho+\|\vartheta^0\|\right)T^2}\right):=\frac{\sigma_2}{\sigma}.$$

Choosing σ to satisfy the inequality $\sigma \geq (1/\rho)\sigma_2$ implies that $\Lambda_2\vartheta \in B(\vartheta^0,\rho)$.

Similarly, for Λ_3, we have

$$\|\Lambda_3\vartheta-\vartheta_3^0\|_\sigma \leq \sup_{t\in[0,T]}\left|\frac{e^{-\sigma t}}{A_0}\sum_{k=1}^{\infty}X_k(x_0)\int_0^t \vartheta_3(\tau)\left[\vartheta_{2k}-\vartheta_{1k}\right](t-\tau)d\tau\right|$$

$$+\sup_{t\in[0,T]}\left|\frac{e^{-\sigma t}}{A_0}\sum_{k=1}^{\infty}X_k(x_0)\int_0^t\int_0^\tau \vartheta_{1k}(s)\vartheta_3(\tau-s)F_k'(t-\tau)dsd\tau\right|$$

$$\leq \frac{M_3}{\sigma}\left(\rho+\|\vartheta^0\|\right)\left(1+(\rho+\|\vartheta^0\|)e^{2\left(\rho+\|\vartheta^0\|\right)T^2}+e^{2\left(\rho+\|\vartheta^0\|\right)T^2}\right):=\frac{\sigma_3}{\sigma},$$

where, M_1, M_2, M_3 are the positive constants whose values depend on a_0, b_0 and c_0. Consequently, if we choose $\sigma \geq (1/\rho)\sigma_3$, then $\Lambda_3\vartheta \in B(\vartheta^0,\rho)$.

As a result, we conclude that if σ satisfy the conditions

$$\sigma > \max\{\sigma_1,\sigma_2,\sigma_3\},$$

then operator Λ maps $B(\vartheta^0;\rho)$ into itself, i.e. $\Lambda\vartheta \in B(\vartheta^0;\rho)$.

Second, we check the second condition of contractive mapping. In accordance with (3.9) for the first component of operator Λ we get

$$\left\|\Lambda_1\vartheta-\Lambda_1\widetilde{\vartheta}\right\|_\sigma \leq \sup_{(x,t)\in\overline{D}}\left|\sum_{k=1}^{\infty}X_k(x_0)\int_0^t e^{-\sigma t}F_k(t-\tau)\int_0^\tau\left(\vartheta_{1k}\vartheta_3-\widetilde{\vartheta}_{1k}\widetilde{\vartheta}_3\right)dsd\tau\right|.$$

The integrands in the integral can be estimated as follows:

$$\left\|\vartheta_i\vartheta_j-\widetilde{\vartheta}_i\widetilde{\vartheta}_j\right\|_\sigma \leq \left\|\vartheta_i-\widetilde{\vartheta}_i\right\|_\sigma\|\vartheta_j\|_\sigma+\left\|\widetilde{\vartheta}_j\right\|_\sigma\left\|\vartheta_j-\widetilde{\vartheta}_j\right\|_\sigma \leq 2(\|\vartheta^0\|+\rho)\left\|\vartheta-\widetilde{\vartheta}\right\|_\sigma.$$

Taking into account (2.13) and (2.14), we get the following

$$\left\|\Lambda_1\vartheta-\Lambda_1\widetilde{\vartheta}\right\|_\sigma \leq \frac{M_4}{\sigma}e^{2(\|\vartheta^0\|+\rho)T^2}\left(1+\frac{T^2}{2}e^{2(\|\vartheta^0\|+\rho))T^2}\right)\|\vartheta-\widetilde{\vartheta}\|_\sigma;$$

where M_4 is a positive constant. It is obviously, if we choose σ as

$$\sigma > \sigma_4 = M_4 e^{2(\|\vartheta^0\|+\rho)T^2}\left(1+\frac{T^2}{2}e^{2(\|\vartheta^0\|+\rho))T^2}\right),$$

then, it is satisfied the second condition of contractive mappings for Λ_1.

The second component of Λ can be estimated in the following form:

$$\left\|\Lambda_2\vartheta - \Lambda_2\widetilde{\vartheta}\right\|_\sigma \leq \sup_{(x,t)\in\overline{D}} \left| \sum_{k=1}^{\infty} A_k X_k(x) \int_0^t e^{-\sigma t} \left(\vartheta_3 - \widetilde{\vartheta}_3\right) F_k(t-\tau) d\tau \right|$$

$$+ \sup_{(x,t)\in\overline{D}} \left| \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k+1} X_k(x) \int_0^t e^{-\sigma t} F_k(t-\tau) \int_0^\tau \left(\vartheta_{2k}\vartheta_3 - \widetilde{\vartheta}_{2k}\widetilde{\vartheta}_3\right) ds d\tau \right|$$

$$\leq \frac{M_5}{\sigma}(\|\vartheta^0\|+\rho)e^{2(\|\vartheta^0\|+\rho)T^2}\left[\frac{T^2}{2} + \frac{T^4}{4}e^{2(\|\vartheta^0\|+\rho)T^2}(\|\vartheta^0\|+\rho)\right]\|\vartheta-\widetilde{\vartheta}\|_\sigma,$$

where M_5 is a positive constant Now we choose number σ so that the expression at $\|\vartheta-\widetilde{\vartheta}\|_\sigma$ becomes less than 1; i.e., the following inequality

$$\sigma > \sigma_5 = M_5(\|\vartheta^0\|+\rho)e^{2(\|\vartheta^0\|+\rho)T^2}\left[\frac{T^2}{2} + \frac{T^4}{4}e^{2(\|\vartheta^0\|+\rho)T^2}(\|\vartheta^0\|+\rho)\right].$$

Then, $\|\Lambda_2\vartheta - \Lambda_2\widetilde{\vartheta}\|_\sigma \leq q\|\vartheta-\widetilde{\vartheta}\|_\sigma$, $(q<1)$. Similarly, for Λ_3, we have

$$\left\|\Lambda_3\vartheta - \Lambda_3\widetilde{\vartheta}\right\|_\sigma \leq \sup_{t\in[0,T]} \left| \frac{e^{-\sigma t}}{A_0} \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k+1} X_k(x_0) \int_0^t \left[\vartheta_3\vartheta_{1k} - \widetilde{\vartheta}_3\widetilde{\vartheta}_{1k}\right] d\tau \right|$$

$$+ \sup_{t\in[0,T]} \left| \frac{e^{-\sigma t}}{A_0} \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k+1} X_k(x_0) \int_0^t \left[\vartheta_3\vartheta_{2k} - \widetilde{\vartheta}_3\widetilde{\vartheta}_{2k}\right] d\tau \right|$$

$$+ \sup_{t\in[0,T]} \left| \frac{e^{-\sigma t}}{A_0} \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k+1} X_k(x_0) \int_0^t \int_0^\tau \left[\vartheta_3\vartheta_{1k} - \widetilde{\vartheta}_3\widetilde{\vartheta}_{1k}\right] F_k'(t-\tau) ds d\tau \right|$$

$$\leq \frac{M_6}{\sigma} e^{2(\|\vartheta^0\|+\rho)T^2}\left[1 + \frac{T^2}{2}(\|\vartheta^0\|+\rho)e^{2(\|\vartheta^0\|+\rho)T^2}\right]^2 \|\vartheta-\widetilde{\vartheta}\|_\sigma,$$

where M_6 is a positive constant whose value depends on a_0, b_0 and c_0. We will choose σ the following inequality

$$\sigma > \sigma_6 = M_6 e^{2(\|\vartheta^0\|+\rho)T^2}\left[1 + \frac{T^2}{2}(\|\vartheta^0\|+\rho)e^{2(\|\vartheta^0\|+\rho)T^2}\right]^2.$$

From these estimates it is clear that if σ is chosen from condition $\sigma > \sigma_6$, then, $\|\Lambda_3\vartheta - \Lambda_3\widetilde{\vartheta}\|_\sigma \leq q\|\vartheta-\widetilde{\vartheta}\|_\sigma$, $(q<1)$.

As result, we conclude that if σ is taken from conditions $\sigma > \max\limits_{1\leq i\leq 6} \sigma_i$, then the operator Λ is a contraction in the ball $B(g_0; \|g_0\|)$. However, in accordance with the Banach theorem [15], the operator Λ has unique fixed point in the ball $B(\vartheta^0;\rho)$ i.e., there exists a unique solution of equation (3.8). $\square$

Based on Theorem 2 and Lemma 1, the following theorem on the existence and uniqueness of the solution of the inverse problem (1.2)–(1.5) is obtained.

Theorem 3. *Let (K1)–(K4) and the compatibility conditions (3.1) be fulfilled. Then, for any $T > 0$, there exists a unique solution $u(x,t) \in Y_T$, $g(t) \in C[0,T]$ to the inverse problem (1.2)–(1.5).*

Let T be a fixed number. Consider the set $\Gamma(\nu_0)$ ($\nu_0 > 0$ is some fixed number) of the given functions ($a(x)$, $b(x)$, $c(x)$, $h(t)$) for which all conditions from Theorem 2 are fulfilled and $\max\{a_0, b_0, c_0, h_0\} \leq \nu_0$. We denote by $Q(\nu_1)$ the set of functions $g(t)$, that for some $T > 0$ satisfy the following condition

$$\|g\| \leq \nu_1, \ \nu_1 > 0.$$

Theorem 4. *Let $(a,b,c,h) \in \Gamma(\nu_0)$, $(\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{h}) \in \Gamma(\nu_0)$ and $g \in Q(\nu_1)$, $\widetilde{g} \in Q(\nu_1)$ are met, then there exists number $T > 0$ that the solution of the inverse problem (1.2)–(1.5) satisfies the following stability estimate:*

$$\|g - \widetilde{g}\| \leq \rho^* \|h(t) - \widetilde{h}(t)\|_{C^4[0,T]} +$$

$$+\rho^* \left(\|a - \widetilde{a}\|_{C^{[\frac{n}{2}]+5}(\overline{\Omega})} + \|b - \widetilde{b}\|_{C^{[\frac{n}{2}]+4}(\overline{\Omega})} + \|c - \widetilde{c}\|_{C^{[\frac{n}{2}]+3}(\overline{\Omega})} \right), \qquad (3.14)$$

where the constant ρ^* depends only on T, ν_0, ν_1.

Proof. To prove this theorem, using (3.7) we write down the equations for $\widetilde{g}(t)$ and compose the difference $\widehat{g} = g(t) - \widetilde{g}(t)$. Then after estimating this expression and using estimates $\widehat{u}_k(t)$, $u_k(t)$, we obtain following estimates

$$\begin{aligned}
|\widehat{g}(t)| &\leq \frac{1}{A_0} \|\widehat{h}\|_{C^4[0,T]} + \frac{1}{A_0} \sum_{k=1}^{\infty} \left(\lambda_k + \lambda_k \sqrt{\lambda_k} + 1 \right) \left| \widehat{A}_k \right| \\
&+ \frac{1}{A_0} \sum_{k=1}^{\infty} \left[(\lambda_k + 1) \sqrt{\lambda_k} \left| \widehat{B}_k \right| + \left(\lambda_k + \sqrt{\lambda_k} + \frac{1}{\lambda_k} \right) \left| \widehat{C}_k \right| \right] \\
&+ \frac{1}{A_0} \|\widehat{g}\| \, T \sum_{k=1}^{\infty} \left[\lambda_k |\widehat{u}_k| + \lambda_k |\widehat{u}'_k| + \left(\lambda_k^2 + \lambda_k + 1 \right) \frac{T}{2} |\widehat{u}_k| \right] \\
&+ \frac{1}{A_0} \int_0^t |\widehat{g}(\tau)| \left(\sum_{k=1}^{\infty} \lambda_k |u'_k| + \lambda_k |u_k| + (\lambda_k^2 + \lambda_k + 1) T |u_k| \right) d\tau \\
&\leq \rho_0 \left(\|\widehat{h}\|_{C^4[0,T]} + \|\widehat{a}\|_{C^{[\frac{n}{2}]+5}(\overline{\Omega})} + \|\widehat{b}\|_{C^{[\frac{n}{2}]+4}(\overline{\Omega})} + \|\widehat{c}\|_{C^{[\frac{n}{2}]+3}(\overline{\Omega})} \right) \\
&+ \rho_1 \int_0^t |\widehat{g}(\tau)| \, d\tau,
\end{aligned}$$

where $\widehat{h} = h - \widetilde{h}$, $\widehat{a} = a - \widetilde{a}$, $\widehat{b} = b - \widetilde{b}$, $\widehat{c} = c - \widetilde{c}$, and ρ_0, ρ_1 depends only on T, ν_0, ν_1. Applying the Gronwall's lemma here, we get the estimate:

$$|\widehat{g}(t)| \leq \rho_0 \left(\|\widehat{h}\|_{C^4[0,T]} + \|\widehat{a}\|_{C^{[\frac{n}{2}]+5}(\overline{\Omega})} + \|\widehat{b}\|_{C^{[\frac{n}{2}]+4}(\overline{\Omega})} + \|\widehat{c}\|_{C^{[\frac{n}{2}]+3}(\overline{\Omega})} \right) e^{2\rho_1 t^2}.$$

This inequality implies the estimate (3.14), if we set $\rho^* = \rho_0 e^{2\rho_1 T^2}$. $\square$

From Theorem 4 follows also the next assertion on uniqueness in whole for solution to the inverse problem.

Theorem 5. *Let the functions $(g(t), \widetilde{g}(t)) \in C[0,T]$ and a, b, c, h and $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{h}$ have the same meaning as the Theorem 4. If in this $a = \widetilde{a}$, $b = \widetilde{b}$, $c = \widetilde{c}$, $h = \widetilde{h}$ for $t \in [0,T]$, then $g(t) = \widetilde{g}(t)$.*

4. Conclusion

In this work, inverse problem was considered for determining the kernel $g(t)$ included in the equation (1.2) with by using additional condition (1.5) of the solution of problem with the initial and boundary conditions (1.3), (1.4). Sufficient conditions for given functions are obtained, under which the inverse problem has unique solutions for a sufficiently small interval. Interesting problems arise in the case when, simultaneously in addition to the convolution kernel $g(t)$, some of parameters α_i, $i = 1, 2, 3$, that depends on the spatial variable in equation (1.2) is also to be determined. Then, it is clear that two additional conditions must be specified, so far such a problem is open.

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