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Concatenations of Linear Orderings and Their Rigidity Characteristics

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Abstract: Among algebraic operations the operation of concatenation for families of words and for orderings is broadly used in various fields of theoretical and applied mathematics, and informatics. It allows to structure to group and to modify information relative required sequences. Degrees of rigidity are considered as a measures of the mobility of a structure with respect to both the automorphism group and definable closures. Classifying structures with respect to rigidity characteristics allows us to determine the dynamic capabilities of the structure when certain elements are fixed. By examining and describing the degrees of rigidity of concatenations of linear orders, we can both classify these concatenations and establish their diversity. We introduce a series of notions relative degrees of rigidity varying their possibilities from absolute rigidity till nowhere almost rigidity, somewhere non-almost rigidity, and absolute finite non-rigidity. These variations of rigidity characteristics are both described and classified for concatenations of linear orderings. We represent a classification of absolutely rigid linear orderings, describe all linear orderings whose all rigidity characteristics equal 1, and describe possibilities of rigidity characteristics for arbitrary concatenations of countable linear orderings.

Keywords: linear ordering, concatenation, degree of rigidity

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Научная статья

Конкатенации линейных порядков и их характеристики жесткости

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Аннотация: Среди алгебраических операций операция конкатенации для семейств слов и порядков широко используется в различных областях теоретической и прикладной математики, а также информатики. Она позволяет структурировать, группировать и видоизменять информацию относительно требуемых последовательностей. Степени жесткости рассматриваются как меры подвижности структуры относительно группы автоморфизмов и определимых замыканий. Классификация структур по характеристикам жесткости позволяет определить динамические возможности структуры при фиксировании определенных элементов. Изучая и описывая степени жесткости конкатенаций линейных порядков, можно как классифицировать эти конкатенации, так и устанавливать их разнообразие. Вводится ряд понятий относительных степеней жесткости, варьируя их возможности от абсолютной жесткости до нигде почти жесткости, где-то не почти жесткости и абсолютной конечной нежесткости. Эти вариации характеристик жесткости описываются и классифицируются для конкатенаций линейных порядков. Дается классификация абсолютно жестких линейных порядков, описываются все линейные порядки, у которых все характеристики жесткости равны 1, а также показываются возможности характеристик жесткости для произвольных конкатенаций счетных линейных порядков.

Ключевые слова: линейный порядок, конкатенация, степень жесткости

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Forming a monoid, the operation of concatenation is broadly used in theoretical and applied mathematics, and informatics allowing to connect words of given alphabets, to represent semantical and syntactical information in various ways, and to transform it via finite automata [1; 2; 5; 7; 8].

In [10], four degrees of rigidity, existential and universal, semantical and syntactical, were introduced, allowing to estimate for given structures how far they are relative to identical automorphism groups and to definable closures of the empty set, for some or arbitrary sets of elements of given finite cardinality. These characteristics were studies for ordered structures [3], models of weakly o-minimal theories [4], abelian groups [6], and unars [9].

At the present paper, we study properties of concatenations of linear orderings with respect to their rigidity characteristics.

In Section 1, we remind these characteristics, extend the terminology for structures with respect to various rigidity values, including semantically/syntactically/absolutely rigid structures (i.e. structures with identical automorphism groups/all $\emptyset$-definable elements/both properties), somewhere/somehow, or everywhere/anyhow, almost/finitely semantically/syntactically rigid ones (with some rigid expansion by finitely many constants), nowhere/nohow almost/finitely semantically/syntactically rigid ones, somewhere non-almost/non-finitely semantically/syntactically rigid ones (without rigid expansions by finitely many constants). Here we also discuss some possibilities of concatenations of linear orderings.

We study and describe linear orderings with given tuples of rigidity characteristics in Section 2. Theorem 1 represents a classification of absolutely rigid linear orderings. In Theorem 2 we describe all linear orderings whose all rigidity characteristics equal 1. Theorem 3 describes possibilities of rigidity characteristics for arbitrary concatenations of countable linear orderings.

1. Preliminaries

Following [3; 10], for a set A in a structure $\mathcal{N}$, $\mathcal{N}$ is called *semantically A-rigid* or *automorphically A-rigid* if any A-automorphism $f \in \mathrm{Aut}(\mathcal{N})$ is identical. The structure $\mathcal{N}$ is called *syntactically A-rigid* if $N = \mathrm{dcl}(A)$. If $A = \emptyset$ we omit A- and say on semantically / syntactically rigid structures instead of A-rigid ones.

A structure $\mathcal{N}$ is called *$\forall$-semantically / $\forall$-syntactically n-rigid* (respectively, *$\exists$-semantically / $\exists$-syntactically n-rigid*), for $n \in \omega$, if $\mathcal{N}$ is semantically / syntactically A-rigid for any (some) $A \subseteq N$ with $|A| = n$.

The least n such that $\mathcal{N}$ is Q-semantically / Q-syntactically n-rigid, where $Q \in \{\forall, \exists\}$, is called the *$Q$-semantical / Q-syntactical degree of rigidity*, it is denoted by $\deg^{Q\text{-sem}}_{\mathrm{rig}}(\mathcal{N})$ and $\deg^{Q\text{-synt}}_{\mathrm{rig}}(\mathcal{N})$, respectively. Here if a set A produces the value of Q-semantical / Q-syntactical degree then we say that A *witnesses* that degree. If such n does not exists we put $\deg^{Q\text{-sem}}_{\rm rig}(\mathcal{N})=\infty$ and $\deg^{Q\text{-synt}}_{\rm rig}(\mathcal{N})=\infty$, respectively.

For a structure $\mathcal{N}$ we put:

$$\deg_4(\mathcal{M})\rightleftharpoons\left(\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M}),\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M}),\deg^{\forall\text{-sem}}_{\rm rig}(\mathcal{M}),\deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M})\right).$$

Definition 1. A structure $\mathcal{M}$ is called *absolutely rigid* if $\mathcal{M}$ is both semantically and syntactically rigid, i.e. $\deg_4(\mathcal{M})=(0,0,0,0)$.

If the value $(\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M})$, respectively $(\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})$, is natural then $\mathcal{M}$ is called *somewhere*, or *somehow*, *almost*, or *finitely*, *semantically / syntactically rigid*. If $(\deg^{\forall\text{-sem}}_{\rm rig}(\mathcal{M})$ or $(\deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M})$ is natural then we replace somewhere by everywhere, or somehow by anyhow, i.e. $\mathcal{M}$ is everywhere (anyhow) almost (finitely) semantically / syntactically rigid.

If the value $(\deg^{\exists\text{-sem}}_{\rm rig}(\mathcal{M})$, respectively $(\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})$, is equal to ∞ then $\mathcal{M}$ is *nowhere*, or *nohow*, *almost*, or *finitely*, *semantically / syntactically rigid*. If $(\deg^{\forall\text{-sem}}_{\rm rig}(\mathcal{M})$ or $(\deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M})$ is equal ∞ then we replace nowhere by somewhere and nohow by somehow with further negation, i.e. $\mathcal{M}$ is somewhere non-almost (non-finitely) semantically / syntactically rigid, or somehow non-almost (non-finitely) semantically / syntactically rigid.

Structures $\mathcal{M}$ with $\deg_4(\mathcal{M})=(\infty,\infty,\infty,\infty)$ are called *absolutely finitely nonrigid*.

Remark 1. By the definition any syntactically rigid structures are absolutely rigid, whereas there are semantically rigid structures which are not syntactically rigid. Indeed, syntactically rigid structures are characterized by Hausdorff topologies with isolated singletons covering the universe of the structures, and a semantically rigid structure admits both isolated and non-isolated singletons forming one-element orbits only, with respect to the automorphism group. Here $\deg_4(\mathcal{M})=(0,\lambda,0,\mu)$, $\lambda,\mu\in\omega\cup\{\infty\}$, $\lambda\leq\mu$, $\mu=0$ for $\lambda=0$.

Clearly each everywhere finitely semantically / syntactically rigid is somewhere finitely semantically / syntactically rigid but not vice versa, and each somewhere / everywhere finitely syntactically rigid structure is somewhere / everywhere finitely semantically rigid but not vice versa. Here $\deg_4(\mathcal{M})=(m,\lambda,n,\mu)$, $m,n\in\omega$, $m\leq n$, $n=0$ for $m=0$, $\lambda,\mu,\nu\in\omega\cup\{\infty\}$, $\lambda\leq\mu$, $\mu=0$ for $\lambda=0$.

Besides each nowhere almost semantically / syntactically rigid structure is somewhere non-almost semantically / syntactically rigid, and nowhere almost semantically rigid structure is nowhere almost syntactically rigid, and nowhere almost semantically rigid structures are absolutely finitely nonrigid. Here $\deg_4(\mathcal{M})=(m,\infty,n,\infty)$ or $\deg_4(\mathcal{M})=(m,\infty,\infty,\infty)$, $m,n\in\omega$, $m\leq n$, $n=0$ for $m=0$, or $\deg_4(\mathcal{M})=(\infty,\infty,\infty,\infty)$.

Below we consider rigidity characteristics for *concatenations*, or *sums*, $\mathcal{M}_1+\mathcal{M}_2$ of linearly ordered sets $\mathcal{M}_1=\langle M_1;\leq_1\rangle$ and $\mathcal{M}_2=\langle M_1;\leq_2\rangle$, i.e.

for the linearly ordered sets $\langle M_1 \dot\cup M_2, \leq\rangle$ such that $\leq$ is the linear order $\leq_1 \cup \leq_2 \cup \{(a_1, a_2) \mid a_1 \in M_1, a_2 \in M_2\}$.

Remind that the operation of concatenation is associative, i.e. satisfies $(\mathcal{M}_1 + \mathcal{M}_2) + \mathcal{M}_3 = \mathcal{M}_1 + (\mathcal{M}_2 + \mathcal{M}_3)$ for any linear orderings $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, and therefore the notation $\mathcal{M}_1 + \mathcal{M}_2 + \ldots + \mathcal{M}_n$ is correct for any linear orderings $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$. Besides, the empty "ordering" Λ satisfies $\mathcal{M}+\Lambda = \Lambda+\mathcal{M} = \mathcal{M}$ for any linear ordering $\mathcal{M}$. Thus the closure under concatenations of each family of linear orderings, containing Λ, forms a monoid.

Remark 2. Admitting infinite concatenations we observe that any linear ordering is represented as a concatenation of finite ones, moreover, as a concatenation of singletons. Here, of cause, both absolute rigidity and almost rigidity can be violated till nohow finitely rigid structures.

2. Variations of rigidity for concatenations of linear orderings

The following obvious proposition asserts that any concatenation of absolutely finitely nonrigid linear ordering with an arbitrary linear ordering is again absolutely finitely nonrigid.

Proposition 1. *Let $\mathcal{M}_1 = \langle M_1, <\rangle$ be a linear ordering with* $\deg_4(\mathcal{M}_1) = (\infty, \infty, \infty, \infty)$. *Then* $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = \deg_4(\mathcal{M}_2 + \mathcal{M}_1) = (\infty, \infty, \infty, \infty)$ *for any linear ordering $\mathcal{M}_2 = \langle M_2, <\rangle$.*

Example 1. Let's consider $\langle \omega \cdot \omega^*, <\rangle$. This is countably many copies of ω ordered by the type ω^*. Consider the following formulas:

$$S(x, y) := x < y \wedge \forall t[x \leq t \leq y \rightarrow t = x \vee t = y],$$

i.e. x is the immediate predecessor for y (or y is the immediate successor for x), $\theta(x) := \exists y[x < y \wedge S(x, y)] \wedge \forall z[z < x \rightarrow \neg S(z, x)]$, i.e. x has the immediate successor and does not have an immediate predecessor.

Further,

$$\phi_1(x) := \theta(x) \wedge \forall t[x < t \rightarrow \exists u_1 \exists u_2 (S(u_1, t) \wedge S(t, u_2))],$$

$$\phi_n(x) := \theta(x) \wedge \exists t[x < t \wedge \phi_{n-1}(t) \wedge \forall u(x < u < t \rightarrow$$
$$\rightarrow \exists u_1 \exists u_2 [S(u_1, u) \wedge S(u, u_2)])], n \geq 2.$$

Here, $\phi_1(x)$ distinguishes the first element of the rightmost copy of ω, $\phi_n(x)$ distinguishes the first element of the n-th copy of ω from the right. Thus, $\deg_4(\langle \omega \cdot \omega^*, <\rangle) = (0, 0, 0, 0)$.

Let's consider now $\langle \omega \cdot \omega^* + \omega \cdot \omega^*, <\rangle$. Clearly, the first elements of copies of ω in the rightmost copy of $\omega \cdot \omega^*$ are distinguished by $\phi_n(x)$ for $n \geq 1$.

Consider the following formulas:

$$\phi_1^2(x) := \theta(x) \wedge \exists t[t < x \wedge \theta(t) \wedge \forall u(t < u < x \to \exists u_1 \exists u_2[S(u_1, u) \wedge S(u, u_2)])]$$

$$\wedge \forall y[x < y \wedge \theta(y) \to \exists w(x < w < y \wedge \theta(w))],$$

$$\phi_n^2(x) := \theta(x) \wedge \exists t[x < t \wedge \phi_{n-1}^2(t) \wedge \forall u(x < u < t \to$$

$$\to \exists u_1 \exists u_2[S(u_1, u) \wedge S(u, u_2)])], n \geq 2.$$

Here, $\phi_1^2(x)$ distinguishes the first element of the rightmost copy of ω in the leftmost copy of $\omega \cdot \omega^*$, $\phi_n^2(x)$ distinguishes the first element of the n-th copy of ω from the right in the leftmost copy of $\omega \cdot \omega^*$. Thus, $\deg_4(\langle \omega \cdot \omega^* + \omega \cdot \omega^*, <\rangle) = (0, 0, 0, 0)$.

Similarly, we can consider $\langle \omega \cdot \omega^* \cdot \omega^*, <\rangle$ and establish that $\deg_4(\langle \omega \cdot \omega^* \cdot \omega^*, <\rangle) = (0, 0, 0, 0)$.

Thus the linear orderings $\langle \omega \cdot \omega^* + \omega \cdot \omega^*, <\rangle$ and $\langle \omega \cdot \omega^* \cdot \omega^*, <\rangle$ are absolutely rigid.

Observe that $\deg_4(\langle M \cdot \omega, <\rangle) = (0, 0, 0, 0)$ is not true for each infinite countable linear ordering $\mathcal{M} = \langle M, <\rangle$ with $\deg_4(\mathcal{M}) = (0, 0, 0, 0)$. Indeed, let $\mathcal{M} = \langle \omega + m + \omega^*, <\rangle$, where $m \geq 0$. Clearly, $\deg_4(\mathcal{M}) = (0, 0, 0, 0)$. However, $\deg_4(\langle M \cdot \omega, <\rangle) = (\infty, \infty, \infty, \infty)$, since this ordering has infinitely many copies of $\mathbb{Z}$.

The following two lemmas extend the list of absolutely rigid linear orderings.

Lemma 1. (1) $\deg_4(\langle \omega^k, <\rangle) = (0, 0, 0, 0)$ *for every natural* $k \geq 0$*;*
(2) $\deg_4(\langle (\omega^*)^k, <\rangle) = (0, 0, 0, 0)$ *for every natural* $k \geq 0$.

Proof. Prove (1). Obviously, the lemma is true for $k \leq 1$.

Step $k = 2$. Consider $\langle \omega^2, <\rangle$. It is countably many copies of ω ordered by the type ω. Let $S(x, y)$ mean that x is the immediate predecessor for y, and $\theta_1(x)$ mean that x has an immediate successor and has no immediate predecessor.

The first element of the leftmost copy of ω is determined by $\phi_1^1(x) := \forall y[x \leq y]$. The first element of the n-th copy of ω, where $n \geq 2$, is determined by the following formula:

$$\phi_n^1(x) := \theta_1(x) \wedge \exists t[t < x \wedge \phi_{n-1}^1(t) \wedge \forall u(t < u < x \to$$

$$\exists u_1 \exists u_2[t < u_1 < u < u_2 < x \wedge S(u_1, u) \wedge S(u, u_2)])].$$

Thus, $\mathrm{dcl}(\emptyset) = \omega^2$, whence $\deg_4(\langle \omega^2, <\rangle) = (0, 0, 0, 0)$.

Step $k = 3$. Consider $\langle \omega^3, <\rangle$. It is countably many copies of ω^2 ordered by the type ω. Let

$$S_\theta^1(x, y) := x < y \wedge \theta_1(x) \wedge \theta_1(y) \wedge \forall t(x \leq t \leq y \wedge \theta_1(t) \to t = x \vee t = y),$$

i.e. x is the immediate predecessor for y among elements of $\theta_1(\mathcal{M})$, and

$$\theta_2(x) := \theta_1(x) \wedge \exists t[x < t \wedge \theta_1(t) \wedge S^1_\theta(x,t)] \wedge \forall z[z < x \wedge \theta_1(z) \to$$

$$\to \exists y(z < y < x \wedge \theta_1(y))],$$

i.e. x has an immediate successor and has no immediate predecessor among elements of $\theta_1(\mathcal{M})$.

The first element of the leftmost copy of ω^2 is determined by $\phi^2_1(x) := \phi^1_1(x)$. The first element of the n-th copy of ω^2, where $n \geq 2$, is determined by the following formula:

$$\phi^2_n(x) := \theta_2(x) \wedge \exists t[t < x \wedge \phi^2_{n-1}(t) \wedge \forall u(t < u < x \wedge \theta_1(u) \to$$

$$\exists u_1 \exists u_2[t < u_1 < u < u_2 < x \wedge S^1_\theta(u_1,u) \wedge S^1_\theta(u,u_2)])].$$

Thus, $\mathrm{dcl}(\emptyset) = \omega^3$, whence $\deg_4(\langle \omega^3, < \rangle) = (0,0,0,0)$.

Suppose that we proved the lemma for $k \leq m$, and prove it for $k = m+1$.

Step $k = m+1$. Consider the following formulas:

$$S^{m-1}_\theta(x,y) := x < y \wedge \theta_{m-1}(x) \wedge \theta_{m-1}(y) \wedge \forall t(x \leq t \leq y \wedge \theta_{m-1}(t) \to$$

$$\to t = x \vee t = y),$$

i.e. x is the immediate predecessor for y among elements of $\theta_{m-1}(\mathcal{M})$, and

$$\theta_m(x) := \theta_{m-1}(x) \wedge \exists t[x < t \wedge \theta_{m-1}(t) \wedge S^{m-1}_\theta(x,t)] \wedge$$

$$\forall z[z < x \wedge \theta_{m-1}(z) \to \exists y(z < y < x \wedge \theta_{m-1}(y))],$$

i.e. x has an immediate successor and has no immediate predecessor among elements of $\theta_{m-1}(\mathcal{M})$.

The first element of the leftmost copy of ω^m is determined by $\phi^m_1(x) := \phi^{m-1}_1(x)$. The first element of the n-th copy of ω^m, where $n \geq 2$, is determined by the following formula:

$$\phi^m_n(x) := \theta_m(x) \wedge \exists t[t < x \wedge \phi^m_{n-1}(t) \wedge \forall u(t < u < x \wedge \theta_{m-1}(u) \to$$

$$\exists u_1 \exists u_2[t < u_1 < u < u_2 < x \wedge S^{m-1}_\theta(u_1,u) \wedge S^{m-1}_\theta(u,u_2)])].$$

Thus, $\mathrm{dcl}(\emptyset) = \omega^{m+1}$, whence $\deg_4(\langle \omega^{m+1}, < \rangle) = (0,0,0,0)$, and we proved (1).

We can prove (2) similarly. □

Lemma 2. (1) $\deg_4(\langle \omega^k \cdot (\omega^*)^s, < \rangle) = (0,0,0,0)$ *for any natural* $k, s \geq 0$*;*

(2) $\deg_4(\langle (\omega^*)^k \cdot \omega^s, < \rangle) = (0,0,0,0)$ *for any natural* $k, s \geq 0$.

Proof. If $k = 0$ or $s = 0$ then the conclusion follows by Lemma 1. Therefore suppose that $k \geq 1$ and $s \geq 1$. Let $S(x, y)$ mean that x is the immediate predecessor for y, and $\theta^{11}(x)$ mean that x has an immediate successor and has no immediate predecessor.

Step $k = 1 \wedge s = 1$. Consider $\langle \omega \cdot \omega^*, < \rangle$. This is countably many copies of ω ordered by the type ω^*. The first element of the rightmost copy of ω is determined by the following formula:

$$\phi_1^{11}(x) := \theta^{11}(x) \wedge \forall t[x < t \to \exists u_1 \exists u_2 (x < u_1 < t < u_2 \wedge S(u_1, t) \wedge S(t, u_2))].$$

The first element of the n-th copy of ω (from the right) is determined by the following formula:

$$\phi_n^{11}(x) := \theta^{11}(x) \wedge \exists t[x < t \wedge \phi_{n-1}^{11}(t) \wedge \forall u(x < u < t \to$$
$$\exists u_1 \exists u_2 [x < u_1 < u < u_2 < t \wedge S(u_1, u) \wedge S(u, u_2)]])], n \geq 2.$$

Thus, $\mathrm{dcl}(\emptyset) = \omega \cdot \omega^*$, whence $\deg_4(\langle \omega \cdot \omega^*, < \rangle) = (0, 0, 0, 0)$.

Step $k = 1 \wedge s = 2$. Consider $\langle \omega \cdot (\omega^*)^2, < \rangle$. This is countably many copies of $\omega \cdot \omega^*$ ordered by the type ω^*. Let

$$S_\theta^{11}(x, y) := x < y \wedge \theta^{11}(x) \wedge \theta^{11}(y) \wedge \forall t(x \leq t \leq y \wedge \theta^{11}(t) \to t = x \vee t = y),$$

$$\theta^{12}(x) := \theta^{11}(x) \wedge \exists t[t < x \wedge \theta^{11}(t) \wedge S_\theta^{11}(t, x)] \wedge \forall y[x < y \wedge \theta^{11}(y) \to$$
$$\to \exists z(x < z < y \wedge \theta^{11}(z))],$$

i.e. x has an immediate predecessor and has no immediate successor on $\theta^{11}(\mathcal{M})$.

The first element of the rightmost copy of ω in the rightmost copy of $\omega \cdot \omega^*$ is determined by the formula: $\phi_1^{12}(x) := \phi_1^{11}(x)$. The first element of the rightmost copy of ω in the n-th copy of $\omega \cdot \omega^*$ (from the right) is determined by the following formula:

$$\phi_n^{12}(x) := \theta^{12}(x) \wedge \exists t[x < t \wedge \phi_{n-1}^{12}(t) \wedge \forall y(x < y < t \wedge \theta^{11}(y) \to$$
$$[\exists z(y < z < t \wedge \theta^{11}(z)) \to \exists z'(y < z' < t \wedge \theta^{11}(z') \wedge S_\theta^{11}(y, z'))]])], n \geq 2.$$

Thus, $\mathrm{dcl}(\emptyset) = \omega \cdot (\omega^*)^2$, whence $\deg_4(\langle \omega \cdot (\omega^*)^2, < \rangle) = (0, 0, 0, 0)$.

Suppose that the lemma has been established for $k = 1$ and $s \leq m$.

Step $k = 1 \wedge s = m + 1$. Consider the following formulas:

$$S_\theta^{1m}(x, y) := x < y \wedge \theta^{1m}(x) \wedge \theta^{1m}(y) \wedge \forall t(x \leq t \leq y \wedge \theta^{1m}(t) \to t = x \vee t = y),$$

$$\theta^{1,m+1}(x) := \theta^{1m}(x) \wedge \exists t[t < x \wedge \theta^{1m}(t) \wedge S_\theta^{1m}(t, x)] \wedge$$
$$\forall y[x < y \wedge \theta^{1m}(y) \to \exists z(x < z < y \wedge \theta^{1m}(z))],$$

i.e. x has an immediate predecessor and has no immediate successor on $\theta^{1m}(\mathcal{M})$.

The first element of the rightmost copy of ω in the rightmost copy of $\omega\cdot(\omega^*)^m$ is determined by the formula: $\phi_1^{1,m+1}(x):=\phi_1^{1m}(x)$. The first element of the rightmost copy of ω in the n-th copy of $\omega\cdot(\omega^*)^m$ (from the right) is determined by the following formula:

$$\phi_n^{1,m+1}(x):=\theta^{1,m+1}(x)\wedge\exists t[x<t\wedge\phi_{n-1}^{1,m+1}(t)\wedge\forall y(x<y<t\wedge\theta^{1m}(y)\to$$

$$[\exists z(y<z<t\wedge\theta^{1m}(z))\to\exists z'(y<z'<t\wedge\theta^{1m}(z')\wedge S_\theta^{1m}(y,z'))])],n\geq 2.$$

Thus, $\mathrm{dcl}(\emptyset)=\omega\cdot(\omega^*)^{m+1}$, whence $\deg_4(\langle\omega\cdot(\omega^*)^{m+1},<\rangle)=(0,0,0,0)$.

Similarly, we can prove that the lemma is true for any $k\geq 1$ and $s=1$.

Suppose that the lemma has been established for each $k\geq 1$ and $s\leq l$.

Step $k=m\wedge s=l+1$. Consider $\langle\omega^m\cdot(\omega^*)^{l+1},<\rangle$. This is countably many copies of $\omega^m\cdot(\omega^*)^l$ ordered by the type ω^*.

The rightmost first element of the copy of ω satisfying $\theta^{ml}(x)$ in the rightmost copy of $\omega^m\cdot(\omega^*)^l$ is determined by $\phi_1^{m,l+1}(x):=\phi_1^{ml}(x)$. The rightmost first element of the copy of ω satisfying $\theta^{ml}(x)$ in the n-th copy of $\omega^m\cdot(\omega^*)^l$ (from the right) is determined by the following formula:

$$\phi_n^{m,l+1}(x):=\theta^{m,l+1}(x)\wedge\exists t[x<t\wedge\phi_{n-1}^{m,l+1}(t)\wedge\forall y(x<y<t\wedge\theta^{ml}(y)\to$$

$$\exists u_1\exists u_2[x\leq u_1<y<u_2\wedge\theta^{ml}(u_1)\wedge\theta^{ml}(u_2)\wedge S_\theta^{ml}(u_1,y)\wedge S_\theta^{ml}(y,u_2)])].$$

Thus, $\mathrm{dcl}(\emptyset)=\omega^m\cdot(\omega^*)^{l+1}$, whence $\deg_4(\langle\omega^m\cdot(\omega^*)^{l+1},<\rangle)=(0,0,0,0)$. □

Corollary 1. *For any natural $n\geq 1$, $k_1,\ldots,k_n\geq 0$ the following holds:*
(1) $\deg_4(\langle\omega^{k_1}\cdot(\omega^*)^{k_2}\cdot\ldots\cdot\omega^{k_{n-1}}\cdot(\omega^*)^{k_n},<\rangle)=(0,0,0,0)$*;*
(2) $\deg_4(\langle(\omega^*)^{k_1}\cdot\omega^{k_2}\cdot\ldots\cdot(\omega^*)^{k_{n-1}}\cdot\omega^{k_n},<\rangle)=(0,0,0,0)$*.*

The following theorem gives a description of absolutely rigid linear orderings.

Theorem 1. *Let $\mathcal{M}=\langle M,<\rangle$ be a linear ordering. Then* $\deg_4(\mathcal{M})=(0,0,0,0)$ *iff $\mathcal{M}$ is finite or it is isomorphic to*

$$\langle m_0+\omega^{k_{11}}\cdot(\omega^*)^{k_{12}}\cdot\ldots\cdot\omega^{k_{1,n_1-1}}\cdot(\omega^*)^{k_{1,n_1}}\cdot s_1+m_1+\ldots$$

$$+m_{l-1}+\omega^{k_{l1}}\cdot(\omega^*)^{k_{l2}}\cdot\ldots\cdot\omega^{k_{l,n_l-1}}\cdot(\omega^*)^{k_{l,n_l}}\cdot s_l+m_l,<\rangle$$

for some integers $l\geq 1$, $m_0,\ldots,m_l\geq 0, n_1,\ldots,n_l\geq 1; s_1,\ldots,s_l\geq 1$; k_{11}, $\ldots$, k_{1,n_1}; $\ldots$; $k_{l1},\ldots,k_{l,n_l}\geq 0$ so that for every $1\leq i\leq l$ $\omega^{k_{i1}}\cdot(\omega^*)^{k_{i2}}\cdot\ldots\cdot\omega^{k_{i,n_i-1}}\cdot(\omega^*)^{k_{i,n_i}}\neq\emptyset$ and the following holds:

(1) *for every $1\leq i\leq l$ $k_{i1}\neq 0$ or $k_{i2}\neq 0$ and for every $2\leq j\leq n_i$ if $k_{ij}=0$ then $k_{ir}=0$ for every $j+1\leq r\leq n_r$;*

(2) *for every $1\leq i\leq l-1$ if $k_{i1}=0$ and $k_{i2}\neq 0$ then $k_{i3}\neq 0$ or $k_{i+1,2}\neq 0$.*

Proof. Since all finite linear orderings are absolutely rigid, below we consider infinite ones. Besides, since uncountable linear orderings are not syntactically rigid as $\mathrm{dcl}(\emptyset)$ is at most countable for each at most countable signature, the description of absolutely rigid linear orderings is reduced to countable ones.

($\Leftarrow$) Let $\mathcal{M}$ is isomorphic to

$$\langle m_0 + \omega^{k_{11}} \cdot (\omega^*)^{k_{12}} \cdot \ldots \cdot \omega^{k_{1,n_1-1}} \cdot (\omega^*)^{k_{1,n_1}} \cdot s_1 + m_1 + \ldots$$
$$+ m_{l-1} + \omega^{k_{l1}} \cdot (\omega^*)^{k_{l2}} \cdot \ldots \cdot \omega^{k_{l,n_l-1}} \cdot (\omega^*)^{k_{l,n_l}} \cdot s_l + m_l, <\rangle$$

for some integers $l \geq 1$, $m_0, \ldots, m_l \geq 0, n_1, \ldots, n_l \geq 1; s_1, \ldots, s_l \geq 1;$ $k_{11}, \ldots, k_{1,n_1}; \ldots; k_{l1}, \ldots, k_{l,n_l} \geq 0$. If (2) doesn't hold, i.e. there exists $1 \leq i \leq l-1$ such that $k_{i1} = 0$, $k_{i2} \neq 0$, $k_{i3} = 0$ and $k_{i+1,2} = 0$ then by (1) we obtain that $\omega^{k_{i1}} \cdot (\omega^*)^{k_{i2}} \cdot \ldots \cdot \omega^{k_{i,n_i-1}} \cdot (\omega^*)^{k_{i,n_i}} = (\omega^*)^{k_{i2}}$ and $\omega^{k_{i+1,1}} \cdot (\omega^*)^{k_{i+1,2}} \cdot \ldots \cdot \omega^{k_{i+1,n_{i+1}-1}} \cdot (\omega^*)^{k_{i+1,n_{i+1}}} = \omega^{k_{i+1,1}}$, whence $\deg_{\mathrm{rig}}^{\exists-\mathrm{sem}}(\mathcal{M}) \geq 1$ and $\deg_{\mathrm{rig}}^{\exists-\mathrm{synt}}(\mathcal{M}) \geq 1$. Thus, the condition (2) is necessary. By Lemmas 1, 2 and Corollary 1 we conclude that $\deg_4(\mathcal{M}) = (0,0,0,0)$.

($\Rightarrow$) Let $\deg_4(\mathcal{M}) = (0,0,0,0)$. Clearly, M does not contain any copy of $\mathbb{Z}$, $\mathbb{Q}$ and copies of the form $\mathbb{Q}_n(t_1, \ldots, t_n)$, where $t_i, 1 \leq i \leq n$, are arbitrary linear orderings. Consequently, the universe M can contain only copies of ω, ω^* and finite linear orderings. By Corollary 1 $\deg_4(\langle \omega^{k_1} \cdot (\omega^*)^{k_2} \cdot \ldots \cdot \omega^{k_{n-1}} \cdot (\omega^*)^{k_n}, <\rangle) = (0,0,0,0)$ for any natural $n \geq 1$, $k_1, \ldots, k_n \geq 0$. Each of such orderings can not be ordered by the type $\mathbb{Z}$ or $\mathbb{Q}$ since otherwise we obtain that $\deg_4(\mathcal{M}) \neq (0,0,0,0)$. If we order such an ordering by ω^k or $(\omega^*)^k$ for some $k \geq 1$ then by Corollary 1 we preserve $\deg_4(\mathcal{M})$. Obviously, if we concatenate such orderings finitely many times then we also preserve the tuple $\deg_4(\mathcal{M})$. $\square$

Example 2. Consider the following linear orderings: $\mathcal{M}_1 = \langle \omega, <\rangle$ and $\mathcal{M}_2 = \langle \omega^*, <\rangle$. Obviously, $\deg_4(\mathcal{M}_1) = (0,0,0,0)$ and $\deg_4(\mathcal{M}_2) = (0,0,0,0)$. We also have that $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = (0,0,0,0)$, but $\deg_4(\mathcal{M}_2 + \mathcal{M}_1) = (1,1,1,1)$. The latter equality holds since $\omega^* + \omega$ becomes rigid after selecting an arbitrary element.

Lemma 3. *Let $\mathcal{M}_1 = \langle M_1, <\rangle$ and $\mathcal{M}_2 = \langle M_2, <\rangle$ be infinite countable linear orderings with* $\deg_4(\mathcal{M}_1) = \deg_4(\mathcal{M}_2) = (0,0,0,0)$*. Then* $\deg_4(\mathcal{M}_1 + \mathcal{M}_2)$ *equals* $(0,0,0,0)$, $(1,1,m,m)$ *for some natural* $m \geq 1$*, or* $(1,1,\infty,\infty)$.

Proof. In view of Example 2 we have infinite linear orderings $\mathcal{M}_1 = \langle M_1, <\rangle$ and $\mathcal{M}_2 = \langle M_2, <\rangle$ such that $\deg_4(\mathcal{M}_1) = \deg_4(\mathcal{M}_2) = (0,0,0,0)$, $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = (0,0,0,0)$ and $\deg_4(\mathcal{M}_2 + \mathcal{M}_1) = (1,1,1,1)$.

Let $\mathcal{M}_1 = \langle \omega^*, <\rangle$, $\mathcal{M}_2 = \langle \omega + m, <\rangle$ for some natural $m \geq 1$. Obviously, $\deg_4(\mathcal{M}_1) = \deg_4(\mathcal{M}_2) = (0,0,0,0)$. We have that $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = (1,1,m+1,m+1)$.

Let now $\mathcal{M}_1 = \langle\omega^*, <\rangle$ and $\mathcal{M}_2 = \langle\omega + \omega, <\rangle$. Then we have:

$$\deg_4(\mathcal{M}_1) = \deg_4(\mathcal{M}_2) = (0, 0, 0, 0)$$

and $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = (1, 1, \infty, \infty)$.

Further, if $\mathcal{M} = \langle M, <\rangle$ is an infinite countable linear ordering with $\deg_4(\mathcal{M}) = (0, 0, 0, 0)$ then by Theorem 1 $\mathcal{M}$ is isomorphic to $\langle m_0 + \omega^{k_{11}} \cdot (\omega^*)^{k_{12}} \cdot \ldots \cdot \omega^{k_{1,n_1-1}} \cdot (\omega^*)^{k_{1,n_1}} \cdot s_1 + m_1 + \ldots \ldots + m_{l-1} + \omega^{k_{l1}} \cdot (\omega^*)^{k_{l2}} \cdot \ldots \cdot \omega^{k_{l,n_l-1}} \cdot (\omega^*)^{k_{l,n_l}} \cdot s_l + m_l, <\rangle$, whence when such orderings are concatenated, only one copy of $\mathbb{Z}$ can appear, which completes our proof. □

Theorem 2. *Let $\mathcal{M} = \langle M, <\rangle$ be an infinite countable linear ordering. Then $\deg_4(\mathcal{M}) = (1, 1, 1, 1)$ iff $\mathcal{M}$ is isomorphic to $\langle\lambda \cdot \mathbb{Z}, <\rangle$, where λ is a linear ordering with $\deg_4(\langle\lambda, <\rangle) = (0, 0, 0, 0)$.*

Proof. ($\Leftarrow$) If $\mathcal{M}$ is isomorphic to $\langle\lambda \cdot \mathbb{Z}, <\rangle$, where λ is a linear ordering with $\deg_4(\langle\lambda, <\rangle) = (0, 0, 0, 0)$ then obviously $\deg_4(\mathcal{M}) = (1, 1, 1, 1)$.

($\Rightarrow$) Let $\deg_4(\mathcal{M}) = (1, 1, 1, 1)$. Clearly, M does not contain any copy of $\mathbb{Q}$ and copies of the form $\mathbb{Q}_n(t_1, \ldots, t_n)$, where $t_i, 1 \leq i \leq n$, are arbitrary linear orderings. Consequently, the universe M can contain only copies of $\mathbb{Z}$, ω, ω^* or finite linear orderings.

First suppose that M contains at least one copy of $\mathbb{Z}$. Then

$$\deg_{rig}^{\exists-sem}(\mathcal{M}) \geq 1$$

and $\deg_{rig}^{\exists-synt}(\mathcal{M}) \geq 1$. If $M \setminus \mathbb{Z} \neq \emptyset$ then $\deg_{rig}^{\forall-sem}(\mathcal{M}) > 1$ and

$$\deg_{rig}^{\forall-synt}(\mathcal{M}) > 1.$$

Consequently, if M contains at least one copy of $\mathbb{Z}$ then M must consist exactly one copy of $\mathbb{Z}$, i.e. $M = \mathbb{Z}$. Thus, in this case $\mathcal{M} = \langle 1 \cdot \mathbb{Z}, <\rangle$.

Suppose now that M does not contain any copy of $\mathbb{Z}$. Consider the case when M consists only of copies of ω. If there are only finitely many such copies then $\deg_4(\mathcal{M}) = (0, 0, 0, 0)$. Consequently, M contains infinitely many such copies.

Let $\mathcal{M}$ is isomorphic to

$$\langle\omega^{k_{11}} \cdot (\omega^*)^{k_{12}} \cdot \ldots \cdot (\omega^*)^{k_{1,n_1}} \cdot s_1 + \ldots + \omega^{k_{l1}} \cdot (\omega^*)^{k_{l2}} \cdot \ldots \cdot (\omega^*)^{k_{l,n_l}} \cdot s_l, <\rangle,$$

where $k_{i1} \neq 0$ for every $1 \leq i \leq l$. By Corollary 1 and Theorem 1 $\deg_4(\mathcal{M}) = (0, 0, 0, 0)$. If at least some part of these copies are densely ordered then $\deg_4(\mathcal{M}) = (\infty, \infty, \infty, \infty)$.

Let λ be an at most countable linear ordering with $\deg_4(\langle\lambda, <\rangle) = (0, 0, 0, 0)$, Then if $M = \omega + \lambda \cdot \mathbb{Z}$, we have $\deg_4(\mathcal{M}) = (1, 1, \infty, \infty)$. If $M = m + \lambda \cdot \mathbb{Z}$ for some $m \geq 0$ then $\deg_4(\mathcal{M}) = (1, 1, m + 1, m + 1)$.

Thus, if M contains only countably many copies of ω then $\deg_4(\mathcal{M}) = (1,1,1,1)$ iff $\mathcal{M} = \langle \lambda \cdot \mathbb{Z}, < \rangle$, where λ is isomorphic to

$$\langle \omega^{k_{11}} \cdot (\omega^*)^{k_{12}} \cdot \ldots \cdot (\omega^*)^{k_{1,n_1}} \cdot s_1 + \ldots + \omega^{k_{l1}} \cdot (\omega^*)^{k_{l2}} \cdot \ldots \cdot (\omega^*)^{k_{l,n_l}} \cdot s_l, < \rangle,$$

where $k_{i1} \neq 0$ for every $1 \leq i \leq l$.

Similarly, if M consists only of copies of ω^* then $\deg_4(\mathcal{M}) = (1,1,1,1)$ iff $\mathcal{M} = \langle \lambda \cdot \mathbb{Z}, < \rangle$, where λ is isomorphic to

$$\langle (\omega^*)^{k_{11}} \cdot \omega^{k_{12}} \cdot \ldots \cdot \omega^{k_{1,n_1}} \cdot s_1 + \ldots + (\omega^*)^{k_{l1}} \cdot \omega^{k_{l2}} \cdot \ldots \cdot \omega^{k_{l,n_l}} \cdot s_l, < \rangle,$$

where $k_{i1} \neq 0$ for every $1 \leq i \leq l$.

Suppose now that M consists only of copies of ω, ω^* and finite linear orderings. For example, if $\mathcal{M} = \langle \omega \cdot \mathbb{Z} + \omega^* \cdot \mathbb{Z} + m \cdot \mathbb{Z}, < \rangle$ for some natural $m \geq 1$ then $\deg_4(\mathcal{M}) = (3,3,\infty,\infty)$.

Here we also can establish that $\deg_4(\mathcal{M}) = (1,1,1,1)$ iff $\mathcal{M} = \langle \lambda \cdot \mathbb{Z}, < \rangle$, where λ is isomorphic to

$$\langle m_0 + \omega^{k_{11}} \cdot (\omega^*)^{k_{12}} \cdot \ldots \cdot \omega^{k_{1,n_1-1}} \cdot (\omega^*)^{k_{1,n_1}} \cdot s_1 + m_1 + \ldots$$

$$+ m_{l-1} + \omega^{k_{l1}} \cdot (\omega^*)^{k_{l2}} \cdot \ldots \cdot \omega^{k_{l,n_l-1}} \cdot (\omega^*)^{k_{l,n_l}} \cdot s_l + m_l, < \rangle$$

as in Theorem 1. □

Theorem 2 immediately implies:

Proposition 2. *Let $\mathcal{M} = \langle M, < \rangle$ be an infinite countable linear ordering. Then $\deg_4(\mathcal{M}) = (1,1,m.m)$ for some natural $m > 1$ iff $\mathcal{M}$ is isomorphic to $\langle m_1 + \lambda \cdot \mathbb{Z} + m_2, < \rangle$ for some nonnegative integers m_1, m_2 with $m_1 + m_2 \geq 1$ and $m = m_1 + m_2 + 1$, where λ is an at most countable linear ordering with $\deg_4(\langle \lambda, < \rangle) = (0,0,0,0)$.*

The following corollary follows by Theorems 1, 2 and Proposition 2:

Corollary 2. *Let $\mathcal{M}_1 = \langle M_1, < \rangle$, $\mathcal{M}_2 = \langle M_2, < \rangle$ be infinite countable linear orderings with $\deg_4(\mathcal{M}_1) = \deg_4(\mathcal{M}_2) = (0,0,0,0)$. Then*

(1) $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = (1,1,1,1)$ iff $\mathcal{M}_1 \cong \langle \omega^*, < \rangle$ and $\mathcal{M}_2 \cong \langle \omega, < \rangle$.

(2) $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = (1,1,m,m)$ for some natural $m > 1$ iff $\mathcal{M}_1 \cong \langle m_1 + \omega^*, < \rangle$ and $\mathcal{M}_2 \cong \langle \omega + m_2, < \rangle$ for some natural m_1, m_2 with $m_1 + m_2 \geq 1$ and $m = m_1 + m_2 + 1$.

Lemma 4. *Let $\mathcal{M}_1 = \langle M_1, < \rangle$, $\mathcal{M}_2 = \langle M_2, < \rangle$ be infinite countable linear orderings with $\deg_4(\mathcal{M}_1) = (1,1,m_1,m_1)$ and $\deg_4(\mathcal{M}_2) = (1,1,m_2,m_2)$ for some natural $m_1, m_2 \geq 1$. Then $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = \deg_4(\mathcal{M}_2 + \mathcal{M}_1) = (2,2,\infty,\infty)$.*

Proof. By Theorem 2 and Corollary 2 we have that $M_1 = m'_1 + \lambda_1 \cdot \mathbb{Z} + m'_2$ and $M_2 = m''_1 + \lambda_2 \cdot \mathbb{Z} + m''_2$ for some natural m'_1, m'_2, m''_1, m''_2 with $m'_1 + m'_2 \geq 0$ and $m''_1 + m''_2 \geq 0$, and $\deg_4(\langle \lambda_i, < \rangle) = (0,0,0,0)$ for $1 \leq i \leq 2$. Then $M_1 + M_2 = m'_1 + \lambda_1 \cdot \mathbb{Z} + m'_2 + m''_1 + \lambda_2 \cdot \mathbb{Z} + m''_2$ and obviously $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = (2,2,\infty,\infty)$. Similarly, $\deg_4(\mathcal{M}_2 + \mathcal{M}_1) = (2,2,\infty,\infty)$. □

Lemma 5. *Let $\mathcal{M}_1 = \langle M_1, < \rangle$, $\mathcal{M}_2 = \langle M_2, < \rangle$ be infinite countable linear orderings with $\deg_4(\mathcal{M}_1) = (0,0,0,0)$ and $\deg_4(\mathcal{M}_2) = (1,1,m,m)$ for some natural $m \geq 1$. Then $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = \deg_4(\mathcal{M}_2 + \mathcal{M}_1) = (1,1,\infty,\infty)$.*

Proof. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are such infinite linear orderings then both $\mathcal{M}_1 + \mathcal{M}_2$ and $\mathcal{M}_2 + \mathcal{M}_1$ contains exactly one copy of $\lambda \cdot \mathbb{Z}$ for some linear ordering λ having $\deg_4(\langle \lambda, < \rangle) = (0,0,0,0)$, and any of $\mathcal{M}_1 + \mathcal{M}_2$ and $\mathcal{M}_2 + \mathcal{M}_1$ without $\lambda \cdot \mathbb{Z}$ is infinite. Therefore, $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = \deg_4(\mathcal{M}_2 + \mathcal{M}_1) = (1,1,\infty,\infty)$. □

Analyzing possibilities in Lemmas 4 and 5 we obtain:

Proposition 3. *Let $\mathcal{M} = \langle M, < \rangle$ be an infinite countable linear ordering. Then $\deg_4(\mathcal{M}) = (1,1,\infty,\infty)$ iff $\mathcal{M}$ is isomorphic to $\langle \lambda_1 + \lambda_2 \cdot \mathbb{Z} + \lambda_3, < \rangle$, where λ_i is a linear ordering with $\deg_4(\langle \lambda_i, < \rangle) = (0,0,0,0)$ for each $1 \leq i \leq 3$, and at least one of $\{\lambda_1, \lambda_3\}$ is infinite.*

Proposition 4. *Let $\mathcal{M} = \langle M, < \rangle$ be an infinite countable linear ordering. Then $\deg_4(\mathcal{M}) = (m,m,\infty,\infty)$ for some natural $m > 1$ iff $\mathcal{M}$ is isomorphic to $\langle \lambda^0_1 + \lambda_1 \cdot \mathbb{Z} + \lambda^0_2 + \ldots + \lambda^0_m + \lambda_m \cdot \mathbb{Z} + \lambda^0_{m+1}, < \rangle$, where λ^0_i, λ_j are at most countable linear orderings with $\deg_4(\langle \lambda^0_i, < \rangle) = \deg_4(\langle \lambda_j, < \rangle) = (0,0,0,0)$ for every $1 \leq i \leq m+1$ and $1 \leq j \leq m$.*

Proof. ($\Leftarrow$) If $\mathcal{M}$ is isomorphic to $\langle \lambda^0_1 + \lambda_1 \cdot \mathbb{Z} + \lambda^0_2 + \ldots + \lambda^0_m + \lambda_m \cdot \mathbb{Z} + \lambda^0_{m+1}, < \rangle$ for some natural $m > 1$ then obviously $\deg_4(\mathcal{M}) = (m,m,\infty,\infty)$.

($\Rightarrow$) Let $\deg_4(\mathcal{M}) = (m,m,\infty,\infty)$ for some natural $m > 1$. Clearly, M does not contain any copy of $\mathbb{Q}$ and copies of the form $\mathbb{Q}_n(t_1, \ldots, t_n)$, where $t_i, 1 \leq i \leq n$, are arbitrary linear orderings. Consequently, the universe M can contain only copies of $\mathbb{Z}$, ω, ω^* or finite linear orderings.

If M contains only copies of $\mathbb{Z}$ then $M = \mathbb{Z} \cdot m$. If M contains only copies of $\mathbb{Z}$ and finite linear orderings then $M = l_0 + \mathbb{Z} + l_1 + \ldots + l_{m-1} + \mathbb{Z} + l_m$, where l_i, $0 \leq i \leq m$ are finite linear orderings or empty sets.

Similarly, we can consider the remaining cases. □

Rigidity characteristics obtained above for countable linear orderings imply the following:

Proposition 5. *Let $\mathcal{M}_1 = \langle M_1, < \rangle$, $\mathcal{M}_2 = \langle M_2, < \rangle$ be infinite countable linear orderings with $\deg_4(\mathcal{M}_1) = (0,0,0,0)$ and $\deg_4(\mathcal{M}_2) = (m,m,\infty,\infty)$ for some natural $m \geq 1$. Then $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = (k,k,\infty,\infty)$, where $m \leq k \leq m+1$.*

Obviously, there exist infinite countable linear orderings $\mathcal{M}_1 = \langle M_1, < \rangle$, $\mathcal{M}_2 = \langle M_2, < \rangle$ with $\deg_4(\mathcal{M}_1) = (0,0,0,0)$ and $\deg_4(\mathcal{M}_2) = (m, m, \infty, \infty)$ for some natural $m \geq 1$ such that $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) \neq \deg_4(\mathcal{M}_2 + \mathcal{M}_1)$.

Proposition 6. *Let $\mathcal{M}_1 = \langle M_1, < \rangle$, $\mathcal{M}_2 = \langle M_2, < \rangle$ be infinite countable linear orderings with $\deg_4(\mathcal{M}_1) = (1, 1, m_1.m_1)$ and $\deg_4(\mathcal{M}_2) = (m_2, m_2, \infty, \infty)$ for some natural $m_1, m_2 \geq 1$. Then $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = \deg_4(\mathcal{M}_2 + \mathcal{M}_1) = (m_2 + 1, m_2 + 1, \infty, \infty)$.*

Proposition 7. *Let $\mathcal{M}_1 = \langle M_1, < \rangle$, $\mathcal{M}_2 = \langle M_2, < \rangle$ be infinite countable linear orderings with $\deg_4(\mathcal{M}_1) = (m_1, m_1, \infty, \infty)$ and $\deg_4(\mathcal{M}_2) = (m_2, m_2, \infty, \infty)$ for some natural $m_1, m_2 \geq 1$. Then $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = (m, m, \infty, \infty)$, where $m_1 + m_2 \leq m \leq m_1 + m_2 + 1$. The values m with $m = m_1 + m_2$ and $m = m_1 + m_2 + 1$ are realized by appropriate $\mathcal{M}_1 + \mathcal{M}_2$.*

Proof. Consider the following infinite linear orderings for any natural m_1, $m_2 \geq 1$:

$$\mathcal{M}_1 = \langle \underbrace{\mathbb{Z} + \ldots + \mathbb{Z}}_{m_1 \text{ times}}, < \rangle, \mathcal{M}_2 = \langle \underbrace{\mathbb{Z} + \ldots + \mathbb{Z}}_{m_2 \text{ times}}, < \rangle.$$

for some natural $m_1, m_2 \geq 1$. Obviously, $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = (m_1 + m_2, m_1 + m_2, \infty, \infty)$.

Consider now the following infinite linear orderings for any natural m_1, $m_2 \geq 1$:

$$\mathcal{M}_1 = \langle \underbrace{\mathbb{Z} + \ldots + \mathbb{Z}}_{m_1 \text{ times}} + \omega^*, < \rangle, \mathcal{M}_2 = \langle \omega + \underbrace{\mathbb{Z} + \ldots + \mathbb{Z}}_{m_2 \text{ times}}, < \rangle.$$

Then, obviously, $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = (m_1 + m_2 + 1, m_1 + m_2 + 1, \infty, \infty)$.

In general, let $\deg_4(\mathcal{M}_1) = (m_1, m_1, \infty, \infty)$ and $\deg_4(\mathcal{M}_2) = (m_2, m_2, \infty, \infty)$ for some natural $m_1, m_2 \geq 1$. Then by Propositions 3 and 4

$$\mathcal{M}_1 \equiv \langle \lambda_1^0 + \lambda_1 \cdot \mathbb{Z} + \lambda_2^0 + \lambda_2 \cdot \mathbb{Z} + \ldots + \lambda_{m_1}^0 + \lambda_{m_1} \cdot \mathbb{Z} + \lambda_{m_1+1}^0, < \rangle,$$

$$\mathcal{M}_2 \equiv \langle (\lambda_1^0)' + (\lambda_1)' \cdot \mathbb{Z} + (\lambda_2^0)' + (\lambda_2)' \cdot \mathbb{Z} + \ldots + (\lambda_{m_2}^0)' + (\lambda_{m_2})' \cdot \mathbb{Z} + (\lambda_{m_2+1}^0)', < \rangle,$$

where $\deg_4(\langle \lambda_i^0, < \rangle) = \deg_4(\langle \lambda_j, < \rangle) = \deg_4(\langle (\lambda_r^0)', < \rangle) = \deg_4(\langle (\lambda_s)', < \rangle) = (0,0,0,0)$ for every $1 \leq i \leq m_1 + 1$, $1 \leq j \leq m_1$, $1 \leq r \leq m_2 + 1$, $1 \leq s \leq m_2$. Then

$$M_1 + M_2 = \lambda_1^0 + \lambda_1 \cdot \mathbb{Z} + \ldots + \lambda_{m_1} \cdot \mathbb{Z} + \lambda_{m_1+1}^0 + (\lambda_1^0)' + (\lambda_1)' \cdot \mathbb{Z} + \ldots$$
$$\ldots + (\lambda_{m_2})' \cdot \mathbb{Z} + (\lambda_{m_2+1}^0)'.$$

By Lemma 3, $\deg_4(\langle \lambda_{m_1+1}^0 + (\lambda_1^0)', < \rangle)$ can be $(0,0,0,0)$, $(1,1,m,m)$ for some $m \geq 1$ or $(1,1,\infty,\infty)$, whence $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = (m, m, \infty, \infty)$, where $m_1 + m_2 \leq m \leq m_1 + m_2 + 1$. □

Theorem 3. *Let $\mathcal{N} := \mathcal{M}_1 + \ldots + \mathcal{M}_n$ for some $n \geq 2$, where $\mathcal{M}_i$ is an infinite countable linear ordering for each $1 \leq i \leq n$. Then the following holds:*

(1) *If $\deg_4(\mathcal{M}_i) = (0,0,0,0)$ for each $1 \leq i \leq n$ then only the following values for $\deg_4(\mathcal{N})$ are possible: $(0,0,0,0)$, $(1,1,m,m)$ for some natural $m \geq 1$ or (l,l,∞,∞) for some $1 \leq l \leq n-1$.*

(2) *If $\deg_4(\mathcal{M}_i) = (1,1,m_i,m_i)$ with $m_i \geq 1$ for each $1 \leq i \leq n$ then $\deg_4(\mathcal{N}) = (n,n,\infty,\infty)$.*

(3) *If $\deg_4(\mathcal{M}_i) = (m_i,m_i,\infty,\infty)$ with $m_i \geq 1$ for each $1 \leq i \leq n$ then $\deg_4(\mathcal{N}) = (k,k,\infty,\infty)$ for some $\Sigma_{i=1}^n m_i \leq k \leq \Sigma_{i=1}^n m_i + n - 1$.*

(4) *If $\deg_4(\mathcal{M}_i) = (\infty,\infty,\infty,\infty)$ for some $1 \leq i \leq n$ then $\deg_4(\mathcal{N}) = (\infty,\infty,\infty,\infty)$.*

Proof. (1) If $\mathcal{M}_i = \langle \omega, < \rangle$ for each $1 \leq i \leq n$ then obviously $\deg_4(\mathcal{N}) = (0,0,0,0)$. If $n = 2$, $\mathcal{M}_1 = \langle \omega^*, < \rangle$ and $\mathcal{M}_2 = \langle \omega, < \rangle$ then $\deg_4(\mathcal{N}) = (1,1,1,1)$. If $n = 2$, $\mathcal{M}_1 = \langle \omega^*, < \rangle$ and $\mathcal{M}_2 = \langle \omega + m, < \rangle$ for some natural $m \geq 1$ then $\deg_4(\mathcal{N}) = (1,1,m+1,m+1)$.

Take an arbitrary natural l with $1 \leq l \leq n-1$. If $\mathcal{M}_1 = \langle \omega^*, < \rangle$, $\mathcal{M}_i = \langle \omega + \omega^*, < \rangle$ for each $2 \leq i \leq l+1$ and $\mathcal{M}_j = \langle \omega^*, < \rangle$ for each $l+2 \leq j \leq n$ then $\deg_4(\mathcal{N}) = (l,l,\infty,\infty)$.

Since $\deg_4(\mathcal{M}_i) = (0,0,0,0)$ for each $1 \leq i \leq n$, we can form at most $n-1$ copies of $\mathbb{Z}$.

(2) By Theorem 2 and Proposition 2 for each $1 \leq i \leq n$ $\mathcal{M}_i$ is isomorphic to $\langle m_i^1 + \lambda_i \cdot \mathbb{Z} + m_i^2, < \rangle$ for some nonnegative integers m_i^1, m_i^2 with $m_i = m_i^1 + m_i^2 + 1$, where λ_i is at most countable linear ordering with $\deg_4(\langle \lambda_i, < \rangle) = (0,0,0,0)$. Then obviously $\deg_4(\mathcal{N}) = (n,n,\infty,\infty)$.

(3) If $\mathcal{M}_i = \langle \underbrace{\mathbb{Z} + \ldots + \mathbb{Z}}_{m_i}, < \rangle$ with $m_i \geq 1$ for each $1 \leq i \leq n$ then obviously $\deg_4(\mathcal{N}) = (\Sigma_{i=1}^n m_i, \Sigma_{i=1}^n m_i, \infty, \infty)$.
If $\mathcal{M}_i = \langle \omega + \underbrace{\mathbb{Z} + \ldots + \mathbb{Z}}_{m_i} + \omega^*, < \rangle$ with $m_i \geq 1$ for each $1 \leq i \leq n$ then obviously $\deg_4(\mathcal{N}) = (\Sigma_{i=1}^n m_i + n - 1, \Sigma_{i=1}^n m_i + n - 1, \infty, \infty)$.

By Propositions 3 and 4 $\deg_4(\mathcal{M}_i) = (m_i, m_i, \infty, \infty)$ for some $m_i \geq 1$ iff $\mathcal{M}_i$ is isomorphic to $\langle \lambda_1^0 + \lambda_1 \cdot \mathbb{Z} + \ldots + \lambda_{m_i}^0 + \lambda_{m_i} \cdot \mathbb{Z} + \lambda_{m_i+1}^0, < \rangle$, where λ_l^0, λ_j are at most countable linear orderings with $\deg_4(\langle \lambda_l^0, < \rangle) = \deg_4(\langle \lambda_j, < \rangle) = (0,0,0,0)$ for every $1 \leq l \leq m_i + 1$ and $1 \leq j \leq m_i$. Then obviously $\deg_4(\mathcal{N}) = (k,k,\infty,\infty)$ for some $\Sigma_{i=1}^n m_i \leq k \leq \Sigma_{i=1}^n m_i + n - 1$. □

3. Conclusion

We studied possibilities of rigidity characteristics for linear orderings, including absolute rigidity, the value $(1,1,1,1)$, and obtained a description of countable linear orderings with respect to these characteristics in general case. It would be interesting to describe possibilities of rigidity characteristics for natural expansions and extensions of linear orderings including partial orderings, ordered groups, ordered fields, etc.

References

1. Birget J.-C. Intersection and union of regular languages and state complexity. *Information Processing Letters*, 1992, vol. 43, no. 4, pp. 185–190. https://doi.org/10.1016/0020-0190(92)90198-5
2. Kleene S.C. Representation of events in nerve nets and finite automata. *RAND Research Memorandum RM-704*, 1951, 98 p.
3. Kulpeshov B.Sh., Sudoplatov S.V. Variations of rigidity for ordered theories. *Bulletin of Irkutsk State University, Series Mathematics*, 2024, vol. 48, pp. 129–144. https://doi.org/10.26516/1997-7670.2024.48.129
4. Kulpeshov B.Sh., Sudoplatov S.V. Variations of rigidity for models of weakly o-minimal theories. *Lobachevskii Journal of Mathematics*, 2024, vol. 45, no. 11, pp. 5837-–5844. https://doi.org/10.1134/S1995080224606970
5. Lupanov O.B. On the comparison of two types of finite sources. *Problems of cybernetics. Issue* 9. Moscow, Fizmatgiz Publ., 1963, pp. 321–326. (in Russian)
6. Pavlyuk I.I., Sudoplatov S.V. Variations of rigidity for abelian groups. *Mathematics and Statistics*, 2024, vol. 12, no. 2, pp. 204–210. https://doi.org/10.13189/ms.2024.120210
7. Plotkin B.I, Gringlaz L.Ya., Gvaramiya A.A. *Elements of algebraic theory of automata*. Moscow, High School Publ., 1994, 192 p. (in Russian)
8. Rabin M.O., Scott D. Finite automata and their decision problems. *IBM Journal of Research and Development*, 1959, vol. 3, no. 2, pp. 114–125. https://doi.org/10.1147/rd.32.0114
9. Sakharov I.A., Stepanova A.A. Semantic and syntactic degrees of rigidity of unars. *Siberian Electronic Mathematical Reports*, 2026, vol. 23, no. 1, pp. 517–531. https://doi.org/110.33048/semi.2026.23.032
10. Sudoplatov S.V. Variations of rigidity. *Bulletin of Irkutsk State University, Series Mathematics*, 2024, vol. 47, pp. 119–136. https://doi.org/10.26516/1997-7670.2024.47.119

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